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EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. We investigate the existence of positive solutions to the nonlinear third-order three-point integral boundary value problem

$$u'''(t) + a(t) f(t, u(t)) = 0, \qquad 0 < t < T,$$
$$u(0) = u''(0) = 0, \quad u(T) = \alpha \int_0^\eta u(s)\,\mathrm{d}s,$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$ are given constants. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying Krasnoselskii's fixed point theorem in cones.

1. Introduction

Boundary value problems (BVP) of ordinary differential equations arise in kinds of different areas of applied mathematics and physics. Many authors have studied extensively two-point, three-point and multi-point boundary value problems for differential equations, see [**1, 6, 9, 13, 15, 17, 18, 21, 22**] and the references therein. Problems with integral boundary conditions have been used in the description of many phenomena in the applied sciences, for example, heat conduction, chemical engineering, underground water flow, and plasma physics. Benaicha and Haddouchi [**2**] studied the existence of positive solutions for a nonlinear fourth-order two-point boundary value problem. For second order nonlinear three-point integral boundary-value problem the interested reader can consult [**4, 5, 10, 11, 12, 20**]. Third-order boundary value problems BVP with integral conditions have attracted a lot of attention [**3, 7, 16, 19, 23**]. Yanping Guo and Fei Yang [**8**] considered the following problem:

$$u'''(t) + f(t, u(t), u'(t)) = 0, \qquad t \in [0,1], \tag{1}$$

$$u(0) = u''(0) = 0, \quad u(1) = \alpha \int_0^1 g(t)\, u(t)\,\mathrm{d}t, \tag{2}$$

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where f is a nonnegative continuous function depend on the first-order derivatives, and $g \in L[0,1]$.

In this paper, we extend the results obtained in [**5, 20**] to third-order problems. Unlike the reference mentioned above concerning the third-order nonlinear boundary value problems with integral conditions, we study the existence of positive solutions on $[0,T]$, for the BVP

$$u'''(t) + a(t) f(t, u(t)) = 0, \qquad 0 < t < T, \tag{3}$$

$$u(0) = u''(0) = 0, \qquad u(T) = \alpha \int_0^\eta u(s)\,\mathrm{d}s, \tag{4}$$

where $0 < \eta < T$. The aim of this paper is to give some results for existence of positive solutions to (3)–(4), assuming that $0 < \alpha < \frac{2T}{\eta^2}$ and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(t,u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(t,u)}{u}. \tag{5}$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By the positive solution of (3)–(4), we mean that a function $u(t)$ is positive on $0 < t < T$ and satisfies the problem (3)–(4).

Throughout this paper, we suppose the following conditions hold:
(H_1) $f \in C([0,T] \times [0,+\infty), [0,+\infty))$,
(H_2) $a \in C([0,T], [0,+\infty))$
and there exists $t_0 \in [\eta, T]$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1 ([**14**]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A \colon K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to K$$

be a completely continuous operator such that
(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

2. Auxiliary results on a linear BVP

In order to prove our main result, we need some auxiliary lemmas. These lemmas are based on the linear boundary value problem.

Lemma 2.1. *Let $2T \ne \alpha\eta^2$. Then for $y \in C([0,T], [0,\infty))$, the problem*

$$u'''(t) + y(t) = 0, \tag{6}$$

$$u(0) = u''(0) = 0, \quad u(T) = \alpha \int_0^\eta u(s)\,\mathrm{d}s, \quad \eta \in (0,T), \quad \alpha > 0, \tag{7}$$

has a unique solution given by

$$u(t) = \frac{t}{2T - \alpha\eta^2}\int_0^T (T-s)^2 y(s)\,\mathrm{d}s - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta - s)^3 y(s)\,\mathrm{d}s$$
$$- \frac{1}{2}\int_0^t (t-s)^2 y(s)\,\mathrm{d}s.$$

Proof. From equation (6), we have $u'''(t) = -y(t)$. Then, integrating from 0 to t, we obtain

$$u''(t) = -\int_0^t y(s)\,\mathrm{d}s.$$

By integrating $t \in [0, T]$ and using integration by parts, we have

$$u'(t) = u'(0) - \int_0^t \left(\int_0^x y(s)\,\mathrm{d}s\right)\mathrm{d}x = u'(0) - \int_0^t (t-s)\,y(s)\,\mathrm{d}s$$

(8)
$$u(t) = u'(0)\,t - \int_0^t \left(\int_0^x (x-s)\,y(s)\,\mathrm{d}s\right)\mathrm{d}x = u'(0)\,t - \frac{1}{2}\int_0^t (t-s)^2 y(s)\,\mathrm{d}s.$$

Thus, for $t = T$, we find

$$u(T) = u'(0)\,T - \frac{1}{2}\int_0^T (T-s)^2 y(s)\,\mathrm{d}s. \tag{9}$$

Integrating the expression (8), again from 0 to η where $\eta \in (0, T)$, we obtain

$$\begin{aligned} \int_0^\eta u(s)\,\mathrm{d}s &= \frac{1}{2}u'(0)\,\eta^2 - \frac{1}{2}\int_0^\eta \left(\int_0^x (x-s)^2 y(s)\,\mathrm{d}s\right)\mathrm{d}x \\ &= \frac{1}{2}u'(0)\,\eta^2 - \frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)\,\mathrm{d}s. \end{aligned} \tag{10}$$

From (7) and (9), we have

$$\int_0^\eta u(s)\,\mathrm{d}s = \frac{1}{\alpha}u(T) = u'(0)\,\frac{T}{\alpha} - \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,\mathrm{d}s.$$

Then, using (10), we see that

$$u'(0)\,\frac{T}{\alpha} - \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,\mathrm{d}s = \frac{1}{2}u'(0)\,\eta^2 - \frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)\,\mathrm{d}s.$$

Thus,

$$u'(0)\left(\frac{2T - \alpha\eta^2}{2\alpha}\right) = \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,\mathrm{d}s - \frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)\,\mathrm{d}s$$

or

$$u'(0) = \frac{1}{(2T-\alpha\eta^2)}\int_0^t (T-s)^2 y(s)\,\mathrm{d}s - \frac{\alpha}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 y(s)\,\mathrm{d}s.$$

Therefore, the boundary value problem (6)–(7) has a unique solution

$$u(t)=\frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2\, y(s)\,\mathrm{d}s-\frac{\alpha t}{3\left(2T-\alpha\eta^2\right)}\int_0^\eta (\eta-s)^3\, y(s)\,\mathrm{d}s$$
$$-\frac{1}{2}\int_0^t (t-s)^2\, y(s)\,\mathrm{d}s.$$ □

The existence of positive solutions of the BVP (6)–(7) is given in the next result.

Lemma 2.2. *Let* $0<\alpha<\frac{2T}{\eta^2}$. *If* $y\in C([0,T],[0,+\infty))$, *then the unique solution of the problem* (6)–(7) *satisfies* $u(t)\geq 0$ *for* $t\in[0,T]$.

Proof. From $u'''(t)=-y(t)$, $t\in[0,T]$, we get that $u''(t)$ is decreasing on $[0,T]$. Then, the condition $u''(0)=0$ ensures that have $u''(t)\leq 0$ for $t\in[0,T]$, which implies $u(t)$ is concave. Observe also that if $u(T)\geq 0$, the concavity of u and the fact that $u(0)=0$ imply that $u(t)\geq 0$ for $t\in[0,T]$.

Since the graph of u is concave down $(0,T)$, we get

$$\int_0^\eta u(s)\,\mathrm{d}s\geq \frac{1}{2}\eta u(\eta), \tag{11}$$

where $\frac{1}{2}\eta u(\eta)$ is the area of triangle under the curve $u(t)$ from $t=0$ to $t=\eta$ for $\eta\in(0,T)$.
If we assume that $u(T)<0$, then from (7) we have

$$\int_0^\eta u(s)\,\mathrm{d}s<0. \tag{12}$$

By concavity of u and $\int_0^\eta u(s)\,\mathrm{d}s<0$, it implies that $u(\eta)<0$.

Hence

$$u(T)=\alpha\int_0^\eta u(s)\,\mathrm{d}s>\frac{2T}{\eta^2}\times\frac{1}{2}\eta u(\eta)=\frac{T}{\eta}u(\eta),$$

which contradicts the concavity of u. □

Lemma 2.3. *Let* $\alpha>\frac{2T}{\eta^2}$. *If* $y\in C([0,T],[0,+\infty))$, *then the problem* (6)–(7) *has no positive solution.*

Proof. Suppose that the problem (6)–(7) has a positive solution u.
If $u(T)>0$, then $\int_0^\eta u(s)\,\mathrm{d}s>0$. It implies that $u(\eta)>0$ and

$$\frac{u(T)}{T}=\frac{\alpha}{T}\int_0^\eta u(s)\,\mathrm{d}s>\frac{2}{\eta^2}\left(\frac{1}{2}\eta u(\eta)\right)=\frac{u(\eta)}{\eta}.$$

This contradicts the concavity of u.

If $u(T)=0$, then $\int_0^\eta u(s)\,\mathrm{d}s=0$, this is $u(t)\equiv 0$ for all $t\in[0,\eta]$. If there exists $t_0\in(\eta,T)$ such that $u(t_0)>0$, then $u(0)=u(\eta)<u(t_0)$, which contradicts the concavity of u. Therefore, no positive solutions exist. □

Lemma 2.4. *Let* $0<\alpha<\frac{2T}{\eta^2}$. *If* $y\in C([0,T],[0,+\infty))$, *then the unique solution of the problem* (6)–(7) *satisfies*

$$\min_{t\in[\eta,T]} u(t)\geq\gamma\|u\|,\qquad \|u(t)\|=\max_{t\in[0,T]}|u(t)|, \tag{13}$$

where

$$\gamma := \min\left\{\frac{\eta}{T}, \frac{\alpha\eta^2}{2T}, \frac{\alpha\eta\,(T-\eta)}{2T-\alpha\eta^2}\right\}. \tag{14}$$

Proof. Set $u(\tau) = \|u\|$. We consider three cases.

Case 1. If $\eta \leq \tau \leq T$ and $\min_{t\in[\eta,T]} u(t) = u(\eta)$, then the concavity of u implies that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\tau)}{\tau} \geq \frac{u(\tau)}{T}.$$

Thus,

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\eta}{T}\|u\|.$$

Case 2. If $\eta \leq \tau \leq T$ and $\min_{t\in[\eta,T]} u(t) = u(T)$, then (7), (11) and the concavity of u implies

$$u(T) = \alpha\int_0^\eta u(s)\,\mathrm{d}s \geq \alpha\frac{\eta^2}{2}\left[\frac{u(\eta)}{\eta}\right] \geq \alpha\frac{\eta^2}{2}\left[\frac{u(\tau)}{\tau}\right] \geq \frac{\alpha\eta^2}{2T}u(\tau).$$

Therefore,

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\alpha\eta^2}{2T}\|u\|.$$

Case 3. If $\tau \leq \eta \leq T$, then $\min_{t\in[\eta,T]} u(t) = u(T)$. Using the concavity of u and (7), (11), we have

$$\frac{u(\tau)-u(T)}{\tau-T} \geq \frac{u(T)-u(\eta)}{T-\eta},$$

$$u(\tau) \leq u(T) + \frac{u(T)-u(\eta)}{T-\eta}(\tau-T),$$

$$\begin{aligned} u(\tau) &\leq u(T) + \frac{u(T)-u(\eta)}{T-\eta}(0-T) \\ &\leq u(T)\left[1 - T\frac{1-\frac{2}{\alpha\eta}}{T-\eta}\right] = u(T)\left[\frac{2T-\alpha\eta^2}{\alpha\eta\,(T-\eta)}\right]. \end{aligned} \tag{15}$$

This implies that

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\alpha\eta\,(T-\eta)}{2T-\alpha\eta^2}\|u\|.$$

This completes the proof. □

3. Existence of positive solutions for the nonlinear BVP

In this section, we state and prove our main results.

Theorem 3.1. *Assume* (H_1) *and* (H_2) *hold, and* $0 < \alpha < \frac{2T}{\eta^2}$. *Then the problem* (3)–(4) *has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear), or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. From Lemma 2.1, u is a solution to the boundary value problem (3)–(4) if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} Au(t) = {} & \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ & - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,\mathrm{d}s \\ & - \frac{1}{2}\int_0^t (t-s)^2 a(s) f(s,u(s))\,\mathrm{d}s. \end{aligned} \tag{16}$$

Denote that

$$K = \left\{u \;:\; u\in C([0,T],\mathbb{R}),\ u\geq 0,\ \min_{t\in[\eta,T]} u(t) \geq \gamma\|u\|\right\},$$

where γ is defined in (14).

It is obvious that K is a cone in $C([0,T],\mathbb{R})$. Moreover, from Lemma 2.2 and Lemma 2.4, $AK \subset K$. It is also easy to check that $A\colon K \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $\rho_1 > 0$ so that $f(t,u) \leq \epsilon u$ for $0 < u < \rho_1$, where $\epsilon > 0$ satisfies

$$\varepsilon\frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,\mathrm{d}s \leq 1. \tag{17}$$

Thus, if we let

$$\Omega_1 = \{u\in C([0,T],\mathbb{R}) \;:\; \|u\| < \rho_1\}, \tag{18}$$

then for $u \in K\cap\partial\Omega_1$, we get

$$\begin{aligned} Au(t) = {} & \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ & - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,\mathrm{d}s \\ & - \frac{1}{2}\int_0^t (t-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ \leq {} & \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ \leq {} & \varepsilon\frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) u(s)\,\mathrm{d}s \\ \leq {} & \varepsilon\frac{T}{2T-\alpha\eta^2}\|u\|\int_0^T (T-s)^2 a(s)\,\mathrm{d}s \leq \|u\|. \end{aligned}$$

Thus $\|Au\| \leq \|u\|$ for $u \in K\cap\partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{\rho_2} > 0$ such that $f(t,u) \geq Mu$, for $u \geq \widehat{\rho_2}$ where $M > 0$ is chosen so that

$$\frac{M\gamma\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s)\,\mathrm{d}s \geq 1. \tag{19}$$

Let $\rho_2 = \max\left\{2\rho_1, \frac{\widehat{\rho_2}}{\gamma}\right\}$ and $\Omega_2 = \{u \in C([0,T],\mathbb{R}) : \|u\| < \rho_2\}$. Then $u \in K \cap \partial\Omega_2$ implies that

$$\min_{t\in[\eta,T]} u(t) \geq \gamma \|u\| = \gamma\rho_2 \geq \widehat{\rho_2}, \tag{20}$$

and so,

$$\begin{aligned}
Au(\eta) = {} & \frac{\eta}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
& - \frac{\alpha\eta}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,\mathrm{d}s \\
& - \frac{1}{2(2T-\alpha\eta^2)}\int_0^\eta \left(2T-\alpha\eta^2\right)(\eta-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
= {} & \frac{\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
& + \frac{1}{6(2T-\alpha\eta^2)}\int_0^\eta \Big[6\eta(T-s)^2 - 2\alpha\eta(\eta-s)^3 - 6T(\eta-s)^2 + 3\alpha\eta^2(\eta-s)^2\Big] \\
& \times a(s) f(s,u(s))\,\mathrm{d}s \\
\geq {} & \frac{\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
& + \frac{1}{6(2T-\alpha\eta^2)}\int_0^\eta \Big[6T(\eta-s)^2 - 2\alpha\eta(\eta-s)^3 - 6T(\eta-s)^2 + 3\alpha\eta^2(\eta-s)^2\Big] \\
& \times a(s) f(s,u(s))\,\mathrm{d}s \\
= {} & \frac{\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
& + \frac{1}{6(2T-\alpha\eta^2)}\int_0^\eta \alpha\eta(\eta-s)^2(\eta+2s) a(s) f(s,u(s))\,\mathrm{d}s \\
\geq {} & \frac{\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
\geq {} & \frac{\eta M\gamma}{2T-\alpha\eta^2}\|u\|\int_\eta^T (T-s)^2 a(s)\,\mathrm{d}s \geq \|u\|.
\end{aligned}$$

Hence, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem 1.1, operator A has a fixed point u in $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$ such that $\rho_1 \leq \|u\| \leq \rho_2$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, we may choose $\rho_3 > 0$ so that $f(t,u) \geq Nu$ for $0 < u \leq \rho_3$, where $N > 0$ satisfies

$$\frac{N\gamma\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s)\,\mathrm{d}s \geq 1. \tag{21}$$

Let

$$\Omega_3 = \{u \in C([0,T],\mathbb{R}) : \|u\| < \rho_3\}, \tag{22}$$

then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned} Au(\eta) &= \frac{\eta}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ &\quad - \frac{\alpha\eta}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^\eta (\eta-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ &\geq \frac{\eta}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ &\geq \frac{\eta N\gamma}{2T-\alpha\eta^2}\int_\eta^T (T-s)^2 a(s)\,\mathrm{d}s\,\|u\| \geq \|u\|. \end{aligned}$$

Thus, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $\widehat{\rho}_4 > 0$ so that $f(t,u) \leq \lambda u$ for $u \geq \widehat{\rho}_4$, where $\lambda > 0$ satisfies

$$\frac{\lambda T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,\mathrm{d}s \leq 1. \tag{23}$$

We consider two cases:

<u>Case 1</u>. Suppose f is bounded. Then, there exists $L > 0$ such that $f(t,u) \leq L$ for all $(t,u) \in [0,T] \times [0,\infty)$. Choosing $\rho_4 = \max\left\{2\rho_3, \frac{L}{\lambda}\right\}$. For $u \in K$ with $\|u\| = \rho_4$, we have

$$\begin{aligned} Au(t) &= \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ &\quad - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^t (t-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\ &\leq \frac{LT}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,\mathrm{d}s \\ &\leq \rho_4\frac{\lambda T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,\mathrm{d}s \leq \rho_4, \end{aligned}$$

and consequently, $\|Au\| \leq \|u\|$.

<u>Case 2.</u> Suppose f is unbounded, then from condition (H_1), there is $\rho_4 \geq \max\left\{2\rho_3, \frac{\hat{\rho}_4}{\gamma}\right\}$ such that

$$f(t,u) \leq f(t,\rho_4) \qquad \text{for } u \in [0,\rho_4].$$

Then for $u \in K$,

$$\begin{aligned}
Au(t) &= \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
&\quad - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,\mathrm{d}s \\
&\quad - \frac{1}{2}\int_0^t (t-s)^2 a(s) f(s,u(s))\,\mathrm{d}s \\
&\le \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 f(s,u(s))\, a(s)\,\mathrm{d}s \\
&\le \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 f(s,\rho_4)\, a(s)\,\mathrm{d}s \\
&\le \rho_4 \frac{\lambda T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,\mathrm{d}s \le \rho_4 = \|u\|.
\end{aligned}$$

Therefore, in either case we may set

$$\Omega_4 = \{u \in C([0,T],\mathbb{R}) \,:\, \|u\| < \rho_4\},$$

and for $u \in K \cap \partial\Omega_4$, we may have $\|Au\| \le \|u\|$. By the second part of Theorem 1.1, operator A has a fixed point u in $K \cap (\overline{\Omega_4} \setminus \Omega_3)$ such that $\rho_3 \le \|u\| \le \rho_4$. This completes the sublinear part of the theorem. Therefore, the problem (3)–(4) has at least one positive solution. □

4. Examples

Example 4.1. Consider the boundary value problem

$$u'''(t) + t^3 u^2 \sinh(u) = 0, \qquad 0 < t < \frac{3}{4}, \tag{24}$$

$$u(0) = 0, \quad u''(0) = 0, \quad u\Big(\frac{3}{4}\Big) = 20\int_0^{\frac{1}{4}} u(s)\,\mathrm{d}s. \tag{25}$$

Set $\alpha = 20$, $\eta = \frac{1}{4}$, $T = \frac{3}{4}$, $a(t) = t^2$, $f(t,u) = tu^2\sinh(u)$. Clearly, the conditions (H_1) and (H_2) are satisfied. We can show that $0 < \alpha = 20 < 24 = \frac{2T}{\eta^2}$. Through a simple calculation we can get $f_0 = 0$ and $f_\infty = \infty$. Thus, by the first part of Theorem 3.1, we can get that the problem (24)–(25) has at least one positive solution.

Example 4.2. Consider the boundary value problem

$$u'''(t) + \mathrm{e}^t \frac{\left(\sqrt{1+u}+u\right)t}{u^2} = 0, \qquad 0 < t < 1, \tag{26}$$

$$u(0) = 0, \quad u''(0) = 0, \quad u(1) = 4\int_0^{\frac{2}{3}} u(s)\,\mathrm{d}s. \tag{27}$$

Set $\alpha = 4$, $\eta = \frac{2}{3}$, $T = 1$, $a(t) = \mathrm{e}^t$, $f(t,u) = \frac{(\sqrt{1+u}+u)t}{u^2} = 0$. Clearly, the conditions (H_1) and (H_2) are satisfied. We can show that $0 < \alpha = 4 < \frac{9}{2} = \frac{2T}{\eta^2}$.

Through a simple calculation we can get $f_0 = \infty$ and $f_\infty = 0$. Thus, by the second part of Theorem 3.1, we can get that the problem (26)–(27) has at least one positive solution.

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