Acta Math. Univ. Comenianae
Vol. LXXXVIII, 1 (2019), pp. 145–156

MERSENNE, JACOBSTHAL, AND JACOBSTHAL-LUCAS NUMBERS WITH NEGATIVE SUBSCRIPTS

A. DAŞDEMIR

Abstract. In this paper, we extend the usual Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers to their terms with negative subscripts. Many identities for new forms of these numbers, including Gelin-Cesáro identity, d'Ocagne's identity, and some sum formulas are presented. Furthermore, we give certain generating matrices and show how the sums of the presented number sequences can be computed by employing matrix technique.

1. Introduction

In the current literature, there are many integer sequences that are defined by a recurrence relation. Fibonacci numbers are one of the most famous sequences. The recursive sequences have very extensive applications in many sciences, such as mathematics and psychics. This has recently contributed intense research and drawn considerable attention to the subject. Today, we have many papers devoted to investigating such integer sequences, see the monographs in [**13, 11, 7**] for more detailed information on the subject.

In this paper, we consider the usual Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers. In recent days, they have investigated extensively due to their many applications. For example, the Mersenne numbers $\{M_n\}_{n=0}^{\infty}$ play a key role in an investigation on the prime numbers. In the references [**10**]–[**8**], certain important studies listed. Note that the Mersenne numbers are defined by the recurrence relation

$$M_0 = 0 \quad \text{and} \quad M_{n+1} = 2M_n + 1 \qquad \text{for } n \geqslant 0. \tag{1}$$

However, this recurrence relation is inhomogeneous. Equation (1) can be homogenized by subtracting two consecutive terms of this sequence as

$$M_0 = 0, \quad M_1 = 1 \quad \text{and} \quad M_{n+1} = 3M_n - 2M_{n-1}. \tag{2}$$

Similarly, the usual Jacobsthal $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal-Lucas $\{j_n\}_{n=0}^{\infty}$ numbers are defined by recursively

$$J_0 = 0, \quad J_1 = 1 \quad \text{and} \quad J_{n+1} = J_n + 2J_{n-1} \qquad \text{for } n \geqslant 0 \tag{3}$$

Received April 11, 2018; revised August 13, 2018.
2010 *Mathematics Subject Classification.* Primary 11B37, 15B36.
Key words and phrases. Mersenne number; Jacobsthal number; negative subscript; Catalan's identity; matrix method; sum.

and

$$j_0 = 2, \quad j_1 = 1 \quad \text{and} \quad j_{n+1} = j_n + 2j_{n-1} \qquad \text{for } n \geqslant 0, \tag{4}$$

respectively. Another way to obtain the members of the mentioned integer sequences is to use their Binet's formulas. The explicit forms of these formulas are

$$M_n = 2^n - 1, \tag{5}$$

$$J_n = \frac{2^n - (-1)^n}{3} \tag{6}$$

$$j_n = 2^n + (-1)^n. \tag{7}$$

In [**2**], Daşdemir presented the identities

$$J_n^2 = \frac{1}{3}(J_{2n} + 2(-1)^{n+1}J_n) \tag{8}$$

$$j_n^2 = j_{2n} + 2(-1)^n j_n - 2. \tag{9}$$

In [**1**], Catarino et al. gave certain correlations between the Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences as

$$M_k = \begin{cases} 3J_k & \text{if } k \text{ is even} \\ j_k & \text{if } k \text{ is odd} \end{cases} \tag{10}$$

$${M_n}^2 = 4^n - M_{n+1}, \tag{11}$$

$$\sum_{i=1}^{n} M_i = M_{n+1} - n - 1. \tag{12}$$

Certain sequences such as Fibonacci, Lucas or Pell numbers are extended to their terms with negative subscripts. Further, the correlations between both cases presented. For more details, one can investigate the references in [**9, 5**]. Note that in each case, the obtained sequence consists of integers. But, similar investigations have yet to be made for the Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers. Hence, there is currently no mathematical model can characterize their terms with negative subscripts of sequences given in (2)–(4). In this paper, the first attempt is displayed to fill this gap. In addition, we discover many identities such as Gelin-Cesáro identity and d'Ocagne's identity and present sum formulas and generating matrices for new cases.

2. Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers with negative subscripts

Now we consider a kth order linear recurrence sequence $\{u_n\}_{n=0}^{\infty}$ defined in the form

$$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k} \qquad \text{for } n > 0$$

with initial terms $u_0 = r_0, u_{-1} = r_1, \ldots, u_{1-k} = r_{k-1}$ under the two fundamental assumptions such

(i) c_i's and r_i's are any rational numbers and

(ii) the polynomial $f(x) = x^k - c_1 x^{k-1} - \cdots - c_{k-1} x - c_k$ has exactly k distinct roots which we denote by $\lambda_1, \lambda_2, \ldots, \lambda_k$.

In this case, we can write any term of the sequence u_n as

$$u_n = \sum_{i=1}^{k} a_i {\lambda_i}^n, \tag{13}$$

where a_i's are the components of the field $Q(\lambda_1, \lambda_2, \ldots, \lambda_k)$ defined as $a_i = \frac{p}{\sqrt{D(f)}}$, where p is algebraic integer and $D(f)$ denotes the discriminant of the polynomial $f(x)$. Hence, Equation (13) allows us to extend the sequence u_n to that with negative subscripts. Putting $v_n = u_{-n}$ for $n = 0, 1, 2, \ldots$, we have

$$v_{n+k} = -\Big(\frac{c_{k-1}}{c_k}\Big) v_{n+k-1} - \cdots - \Big(\frac{c_1}{c_k}\Big) v_{n+1} + \frac{1}{c_k} v_n. \tag{14}$$

Inspired by the above-stated, we introduce negatively subscripted terms of second order recurrence sequences given in (2)–(4) now. For combinatorial simplicity, we introduce new representations

$$G_n = M_{-n}, \quad H_n = J_{-n} \quad \text{and} \quad I_n = j_{-n}. \tag{15}$$

For convenience, we recommend to call these the backward Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences, respectively.

As can be seen from Equations (5)–(7), the usual Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences provide the above-stated assumptions. Hence, these sequences can be extended to negative n. Consequently, applying the same approach in (14) to the sequences considered, we obtain the following recurrence relations:

$$G_{-1} = 1, \quad G_0 = 0 \quad \text{and} \quad G_{n+1} = \frac{3}{2} G_n - \frac{1}{2} G_{n-1}, \tag{16}$$

$$H_{-1} = 1, \quad H_0 = 0 \quad \text{and} \quad H_{n+1} = \frac{1}{2} H_n - \frac{1}{2} H_{n-1}, \tag{17}$$

$$I_{-1} = 1, \quad I_0 = 2 \quad \text{and} \quad I_{n+1} = \frac{1}{2} I_n - \frac{1}{2} I_{n-1}. \tag{18}$$

As an example what could happen, we can write

$$\begin{aligned}
G_0 &= 0,\\
G_1 &= -\frac{1}{2} = -\frac{M_1}{2},\\
G_2 &= \frac{3G_1 - G_0}{2} = -\frac{3}{4} = -\frac{M_2}{4},\\
G_3 &= \frac{3G_2 - G_1}{2} = -\frac{7}{8} = -\frac{M_3}{8},\\
G_4 &= \frac{3G_3 - G_2}{2} = -\frac{15}{16} = -\frac{M_3}{16},
\end{aligned}$$

$$G_5 = \frac{3G_4 - G_3}{2} = -\frac{31}{32} = -\frac{M_5}{32}.$$

This process continues the same as the above with small changes as regularly for increasing n. Hence, we can write this observation as one of our main goals in the following theorem.

Theorem 2.1. *Let n be any positive integer. Then we have*

$$G_n = -\frac{M_n}{2^n}, \qquad H_n = \frac{(-1)^{n+1}}{2^n} J_n \qquad \text{and} \qquad I_n = \frac{(-1)^n}{2^n} j_n. \tag{19}$$

Proof. By the Binet's formula in (5), we can write

$$G_n = M_{-n} = 2^{-n} - 1 = \frac{1}{2^n} - 1 = -\frac{2^n - 1}{2^n} = -\frac{M_n}{2^n},$$

which is the first equation. Other equations can be proved similarly. □

Equations in (19) indicate that although G_n is a sequence that converge to -1, H_n and I_n are divergent sequences. However, these new sequences have a lower and upper bounds.

Now we consider certain properties concerning the backward Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences. Note that these properties play key roles in the proof procedure of some important theorems. Hence, we list them in the following theorem.

Lemma 2.2. *For the backward Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences, we have*

$$G_n = \begin{cases} 3H_n & \text{if } n \text{ is even} \\ I_n & \text{if } n \text{ is odd} \end{cases} \tag{20}$$

$${G_n}^2 = 1 + 2^{1-n} G_{n+1}, \tag{21}$$

$${H_n}^2 = \frac{1}{3}\left(-H_{2n} + 2^{1-n} H_n\right), \tag{22}$$

$${I_n}^2 = I_{2n} + I_{2n-1} + 2^{1-n} I_n + 1, \tag{23}$$

$$H_n + H_{n-1} = \frac{1}{2^n}, \tag{24}$$

$$I_n + I_{n-1} = \frac{3}{2^n}, \tag{25}$$

$$I_n + H_n = 2H_{n-1}, \tag{26}$$

$$I_n - H_n = -4H_{n+1} \tag{27}$$

$$H_n = 2H_{n+1} + (-1)^{n+1}. \tag{28}$$

Proof. By employing (10) after considering the definitions of G_n, H_n and I_n, we obtain (20). From the definition of G_n and (11), we can write

$$G_n{}^2 = \frac{M_n^2}{2^{2n}} = \frac{4^n - M_{n+1}}{2^{2n}} = 1 - \frac{M_{n+1}}{2^{n+1}2^{n-1}}.$$

The last equation gives the proof of (21). Repeating the same steps, (22) and (23) can be proved. On the other hand, we get

$$H_n + H_{n-1} = \frac{(-1)^{n+1}}{2^n} J_n + \frac{(-1)^n}{2^{n-1}} J_{n-1} = \frac{(-1)^{n+1}}{2^n} (J_n - 2J_{n-1}),$$

and using identity $J_n - 2J_{n-1} = (-1)^{n-1}$, we obtain (24). The remaining equations can be proved by using the same procedure. □

Theorem 2.3. *For any integer n, we have*

$$G_n H_{n+1} - G_{n+1} H_n = 2 \times \begin{cases} -H_n & \text{if } n \text{ is even} \\ H_{n+1} & \text{if } n \text{ is odd} \end{cases} \tag{29}$$

and

$$G_n I_{n+1} - G_{n+1} I_n = \begin{cases} \frac{1}{2} I_{n-1} & \text{if } n \text{ is even} \\ 2I_n & \text{if } n \text{ is odd.} \end{cases} \tag{30}$$

Proof. To reduce the size of the paper, we only prove (29). Hence, we can write

$$\begin{aligned} G_n H_{n+1} - G_{n+1} H_n &= \Big(-\frac{M_n}{2^n}\Big) \frac{(-1)^{n+2} J_{n+1}}{2^{n+1}} - \Big(-\frac{M_{n+1}}{2^{n+1}}\Big) \frac{(-1)^{n+1}}{2^n} J_n \\ &= \frac{(-1)^{n+1}}{2^{2n+1}} (M_n J_{n+1} + M_{n+1} J_n) \\ &= \frac{(-1)^{n+1}}{2^{2n+1}} \Big((2^n - 1) \frac{2^{n+1} - (-1)^{n+1}}{3} + (2^{n+1} - 1) \frac{2^n - (-1)^n}{3} \Big) \\ &= \frac{(-1)^{n+1}}{3 2^{2n+1}} \Big(2^{2n+1} - 2^n (-1)^{n+1} - 2^{n+1} + (-1)^{n+1} \\ &\quad + 2^{2n+1} - 2^{n+1}(-1)^n - 2^n + (-1)^n \Big) \\ &= \frac{(-1)^{n+1}}{2^{n+1} 3} \Big(2^{n+2} - 3 + (-1)^{n+1} \Big) \end{aligned}$$

The last equation gives the desired result. □

The next theorems inform us certain special identities such as Catalan's identity, Cassini's identity, d'Ocagne's identity and Gelin-Cesáro identity for the considered sequences. Note that some of these identities for the usual Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences were given by Daşdemir [**3, 2**], Catarino et al. [**1**] and Horadam [**6**] before.

Theorem 2.4 (*Catalan's Identities*). *For any integers n and r, we have*

$$G_{n+r}G_{n-r} - {G_n}^2 = \frac{G_r M_r}{2^n}, \tag{31}$$

$$H_{n+r}H_{n-r} - {H_n}^2 = \frac{(-1)^{n+r+1}}{2^{n-r}}{H_r}^2 \tag{32}$$

$$I_{n+r}I_{n-r} - {I_n}^2 = \frac{9(-1)^{n+r}}{2^{n-r}}{H_r}^2. \tag{33}$$

Proof. By the definition of G_n, we can write

$$\begin{aligned}
G_{n+r}G_{n-r} - {G_n}^2 &= \Big(-\frac{M_{n+r}}{2^{n+r}}\Big)\Big(-\frac{M_{n-r}}{2^{n-r}}\Big) - \Big(-\frac{M_n}{2^n}\Big)^2 \\
&= \frac{1}{2^{2n}}\left(M_{n+r}M_{n-r} - {M_n}^2\right) \\
&= \frac{1}{2^{2n}}\left[\left(2^{n+r}-1\right)\left(2^{n-r}-1\right) - \left(2^n-1\right)^2\right] \\
&= \frac{1}{2^{2n}}\left(2^{n+1} - 2^{n+r} - 2^{n-r}\right) \\
&= -\frac{1}{2^{n+r}}\left(2^r-1\right)^2.
\end{aligned}$$

Hence, first equation is obtained. Equations (32) and (33) can be proved in a similar way. □

When $r = 1$ in Theorem 2.4, we have Cassini's identities. They are presented in the following conclusion.

Corollary 2.5 (*Cassini's Identities*). *The following equations hold for any n:*

$$G_{n+1}G_{n-1} - {G_n}^2 = -\frac{1}{2^{n+1}}, \tag{34}$$

$$H_{n+1}H_{n-1} - {H_n}^2 = \frac{(-1)^n}{2^{n+1}}, \tag{35}$$

$$I_{n+1}I_{n-1} - {I_n}^2 = \frac{9(-1)^{n+1}}{2^{n+1}}. \tag{36}$$

The next theorem gives d'Ocagne's identity for the backward Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences.

Theorem 2.6 (*d'Ocagne's Identities*). *For any integers m and n, we have*

$$G_m G_{n+1} - G_{m+1}G_n = -\frac{G_{m-n}}{2^{n+1}}, \tag{37}$$

$$H_m H_{n+1} - H_{m+1}H_n = \frac{(-1)^n}{2^{n+1}}H_{m-n} \tag{38}$$

$$I_m I_{n+1} - I_{m+1}I_n = \frac{9(-1)^m}{2^{m+1}}H_{n-m}. \tag{39}$$

Proof. From Theorem 2.1, we have

$$G_m G_{n+1} - G_{m+1}G_n = \Big(-\frac{M_m}{2^m}\Big)\Big(-\frac{M_{n+1}}{2^{n+1}}\Big) - \Big(-\frac{M_{m+1}}{2^{m+1}}\Big)\Big(-\frac{M_n}{2^n}\Big)$$

$$\begin{aligned}
&= \frac{1}{2^{m+n+1}}\left(M_m M_{n+1} - M_{m+1} M_n\right)\\
&= \frac{1}{2^{m+n+1}}\left[\left(2^m - 1\right)\left(2^{n+1} - 1\right) - \left(2^{m+1} - 1\right)\left(2^n - 1\right)\right]\\
&= \frac{1}{2^{m+n+1}}\left(2^{m+1} - 2^m - 2^{n+1} + 2^n\right)\\
&= \frac{1}{2^{m+n+1}}\left(2^m - 2^n\right)\\
&= \frac{1}{2^{m+1}}\left(2^{m-n} - 1\right)
\end{aligned}$$

and the result follows. Similarly, the others can be obtained. □

Theorem 2.7 (*Gelin-Cesáro Identities*)**.** *For any integers m and n, we have*

$$G_n{}^4 - G_{n-2}G_{n-1}G_{n+1}G_{n+2} = \frac{G_{n+1}}{2^{n+2}}\left(\frac{11}{2^{n-1}} - 9\right) + \frac{1}{2^{n+1}}, \tag{40}$$

$$H_n{}^4 - H_{n-2}H_{n-1}H_{n+1}H_{n+2} = (-1)^{n+1}\frac{H_n{}^2}{2^{n+2}} + \frac{1}{2^{2n+3}} \tag{41}$$

$$I_n{}^4 - I_{n-1}I_{n-2}I_{n+1}I_{n+2} = (-1)^n\frac{9I_n{}^2}{2^{n+2}} + \frac{81}{2^{2n+3}}. \tag{42}$$

Proof. To prove the theorem, we consider the cases $r = 1$ and $r = 2$ in Theorem 2.4, separately. Clearly, we have

$$G_{n+1}G_{n-1} - G_n{}^2 = -\frac{1}{2^{n+1}} \tag{43}$$

$$G_{n+2}G_{n-2} - G_n{}^2 = \frac{G_2 M_2}{2^n}, \tag{44}$$

respectively. Hence, from Equations (43) and (44), we obtain

$$\begin{aligned}
G_{n+1}G_{n-1}G_{n+2}G_{n-2} &= \Big(G_n{}^2 - \frac{1}{2^{n+1}}\Big)\Big(G_n{}^2 - \frac{9}{2^{n+2}}Big)\\
&= G_n{}^4 - G_n{}^2\frac{11}{2^{n+2}} + \frac{9}{2}^{2n+3}
\end{aligned}$$

and from (21)

$$\begin{aligned}
G_n{}^4 - G_{n+1}G_{n-1}G_{n+2}G_{n-2} &= \left(1 + 2^{1-n}G_{n+1}\right)\frac{11}{2^{n+2}} - \frac{9}{2^{2n+3}}\\
&= 9\frac{2^{n+1} - 1}{2^{2n+3}} + \frac{11G_{n+1}}{2^{2n+1}} + \frac{1}{2^{n+1}}\\
&= -9\frac{G_{n+1}}{2^{n+2}} + \frac{11G_{n+1}}{2^{2n+1}} + \frac{1}{2^{n+1}},
\end{aligned}$$

which gives (40). Using the same procedure, we can obtain Equations (41) and (42). □

One of our main aims is that we give formulas to calculate the sum of terms of the backward Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences. For this purpose, we present the next theorem. However, we note that the infinite sum of these sequences is divergent. This can be seen by employing the divergence test. Consequently, we consider their finite sums. First of all, we give the following lemma without the proof.

Lemma 2.8. *Let n and r be any integers such that $r \leq n$. Then we have*

$$2^{n-r}M_r = M_n - M_{n-r}. \tag{45}$$

Theorem 2.9. *Let n be a positive integer. Then, we have*

$$C_n = \sum_{i=1}^{n} G_i = -G_n - n, \tag{46}$$

$$D_n = \sum_{i=1}^{n} H_i = \begin{cases} -H_n & \text{if } n \text{ is even} \\ 1 - H_n & \text{if } n \text{ is odd} \end{cases} \tag{47}$$

and

$$E_n = \sum_{i=1}^{n} I_i = \begin{cases} I_n & \text{if } n \text{ is even} \\ 1 - I_n & \text{if } n \text{ is odd.} \end{cases} \tag{48}$$

Proof. We consider (46). By (5), (12) and (45), we can write

$$\begin{aligned}
\sum_{i=1}^{n} G_i &= -\frac{M_1}{2} - \frac{M_2}{2^2} - \cdots - \frac{M_n}{2^n} \\
&= -\frac{2^{n-1}M_1 + 2^{n-2}M_2 + \cdots + M_n}{2^n} \\
&= -\frac{M_n - M_{n-1} + M_n - M_{n-2} + \cdots + M_n}{2^n} \\
&= -\frac{nM_n - \sum\limits_{i=1}^{n-1} M_i}{2^n} \\
&= -\frac{nM_n - M_n + n}{2^n} \\
&= -\frac{n\left(2^n - 1\right) - M_n + n}{2^n} \\
&= -\frac{n2^n - M_n}{2^n} \\
&= \frac{M_n}{2^n} - n
\end{aligned}$$

and the result follows. To prove (47), we use the induction method. For $n = 1$, the validity of that is clear. Assume that this holds for any integer. If we can show that the case holds for $n + 1$, then we complete the proof. Depending on whether

n is an odd integer or an even integer, we have two cases. From our assumption and (28), we can write

$$\begin{aligned}\sum_{i=1}^{n+1} H_i = H_{n+1} + \sum_{i=1}^{n} H_i &= H_{n+1} + \begin{cases} -H_n & \text{if } n \text{ is even} \\ 1 - H_n & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} H_{n+1} - H_n & \text{if } n \text{ is even} \\ 1 + H_{n+1} - H_n & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 1 - H_{n+1} & \text{if } n \text{ is, even} \\ -H_{n+1} & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} -H_{n+1} & \text{if } n+1 \text{ is even} \\ 1 - H_{n+1} & \text{if } n+1 \text{ is odd} \end{cases}\end{aligned}$$

So, the desired result is achieved. Applying the same method, the last equation can be proved. □

3. Matrix Approach

In this section, we investigate generating matrices of the backward Mersenne, Jacobsthal, and Jacobsthal-Lucas sequences. Now, we consider the recurrence relation given in (16). This equation may also be expressed by the matrix recurrence relation as

$$\mathbf{G_{n+1}} = \mathbf{A}\mathbf{G_n}, \tag{49}$$

where

$$\mathbf{G_n} = \begin{bmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix}. \tag{50}$$

If we extend (49) to the left-hand side to deal with terms with large subscripts simultaneously, then we have

$$\mathbf{G_n} = \mathbf{A}^{n-1}\mathbf{G}_1 \tag{51}$$

by an inductive argument. In addition, an auxiliary matrix P is defined as

$$P = \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{52}$$

It should be noted that $\mathbf{G}_1 = \mathbf{A}P$. Consequently, we can write

$$\mathbf{G_n} = \mathbf{A}^n P \tag{53}$$

This process can be repeated in the backward Jacobsthal and Jacobsthal-Lucas sequences. First of all, we introduce the matrices

$$\mathbf{H_n} = \begin{bmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{I_n} = \begin{bmatrix} I_{n+1} & I_n \\ I_n & I_{n-1} \end{bmatrix}. \tag{54}$$

Hence, we have

$$\mathbf{H_n} = \mathbf{B}^n R \quad \text{and} \quad \mathbf{I_n} = \mathbf{B}^n S, \tag{55}$$

where

$$\mathbf{B} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} -1/2 & 2 \\ 2 & 1 \end{bmatrix}. \tag{56}$$

Note that the matrix equations given in (53) and (55) have somewhat unfavorable structures because they are computed by auxiliary matrices. Hence, let us define new matrices as

$$\mathbf{W_n} = \begin{bmatrix} -G_{n+1} & \frac{G_n}{2} \\ -G_n & \frac{G_{n-1}}{2} \end{bmatrix}, \tag{57}$$

$$\mathbf{U_n} = \begin{bmatrix} 2H_{n+1} & H_n \\ 2H_n & H_{n-1} \end{bmatrix} \tag{58}$$

and

$$\mathbf{V_n} = \begin{bmatrix} 2I_{n+1} & I_n \\ 2I_n & I_{n-1} \end{bmatrix}. \tag{59}$$

For simplicity, these matrices can be called the backward Mersenne, Jacobsthal and, Jacobsthal-Lucas matrices, respectively. Then, we can give the next theorem.

Theorem 3.1. *For any positive integer n, we have*

$$\mathbf{W_n} = \mathbf{A}^n \quad \text{and} \quad \mathbf{U_n} = \mathbf{B}^n. \tag{60}$$

Proof. We only show the validity of the first equation in (60). The other is analogous. This is obviously true when $n = 1$. Let it be true for $n = 1, 2, \ldots, k$. Hence, we must show that $\mathbf{G_{k+1}} = \mathbf{A}^{k+1}$, namely that

$$\mathbf{A}^{k+1} = \mathbf{A}\mathbf{A}^k = \mathbf{A}\mathbf{G_k},$$

and by the recurrence relation and the product of two matrices, $\mathbf{A}^{k+1} = \mathbf{G_{k+1}}$, which completes the proof. □

It should be noted that to obtain the matrix $\mathbf{V_n}$ in (59) consecutively, we cannot find a generating matrix. However, it is possible to obtain it by using an auxiliary matrix in the form

$$T = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}. \tag{61}$$

Hence, we give the following theorem.

Theorem 3.2. *For all positive integers n, we have*

$$\mathbf{V_n} = \mathbf{B}^n T = \mathbf{U_n} T. \tag{62}$$

Proof. By employing the rule of matrix multiplication and from (26) and (27), the proof is completed. □

The matrix method has played an important and effective role stemming from Number Theory. As an example of the usage of the matrix approach, we can give to obtain determinantal identities. Now, we focus on computing (46) and (47). To do this, we define two matrices in the forms

$$K=\begin{bmatrix}1&0&0\\1&\frac{3}{2}&\frac{-1}{2}\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&&\\0&&\mathbf{A}\end{bmatrix} \tag{63}$$

and

$$L=\begin{bmatrix}1&0&0\\1&\frac{-1}{2}&\frac{1}{2}\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&&\\0&&\mathbf{B}\end{bmatrix}. \tag{64}$$

Hence, we give the next theorem.

Theorem 3.3. *For all positive integers n, we have*

$$K^n=\begin{bmatrix}1&0&0\\-C_n&&\\-C_{n-1}&&\mathbf{W_n}\end{bmatrix} \tag{65}$$

and

$$L^n=\begin{bmatrix}1&0&0\\2D_n&&\\2D_{n-1}&&\mathbf{U_n}\end{bmatrix}. \tag{66}$$

Proof. We consider (65). For $n=1$, it can be seen that this holds. Assume that this holds for $n=1,2,\ldots,k$. Hence, for $n=k+1$, we can write

$$\begin{aligned}K^{k+1}=K^kK&=\begin{bmatrix}1&0&0\\-C_k&-G_{k+1}&\frac{G_k}{2}\\-C_{k-1}&-G_k&\frac{G_{k-1}}{2}\end{bmatrix}\begin{bmatrix}1&0&0\\1&\frac{3}{2}&\frac{-1}{2}\\0&1&0\end{bmatrix}\\&=\begin{bmatrix}1&0&0\\-C_k-G_{k+1}&-\frac{3}{2}G_{k+1}+\frac{G_k}{2}&\frac{G_{k+1}}{2}\\-C_{k-1}-G_k&-\frac{3}{2}G_k+\frac{G_{k-1}}{2}&\frac{G_k}{2}\end{bmatrix}.\end{aligned}$$

This completes the proof. The other can also be proved in a similar way. □

In Conclusion 2.5, we give the Cassini's formulas for the considered sequences. Note that these formulas can be obtained by computing the determinants of the matrix equations in (60) and (62).

Acknowledgment. The author is a member of a research project supported by the Research Fund of Kastamonu University via project number KÜ-BAP-01/2017-1. The author is grateful to the anonymous referees for their time, effort, and extensive comments, which have greatly improved the quality of this paper.

References

1. Catarino P., Campos H. and Vasco P., *On the Mersenne sequence*, Ann. Math. Inform. **46** (2016), 37–53.
2. Daşdemir A., *A Study on the Jacobsthal and Jacobsthal-Lucas Numbers*, Dicle Uni. J Inst. Nat. Appl. Sci. **3** (2014), 13–18.
3. Daşdemir A., *On the Jacobsthal numbers by matrix method*, SDU J. Sci. **7** (2012), 69–76.
4. Erdos P., Kiss P. and Pomerance C., *On prime divisors of Mersenne numbers*, Acta Arith. **60** (1991), 267–281.
5. Horadam A. F., *Unique minimal representation of integers by negatively subscripted Pell numbers*, Fibonacci Quart. **32** (1994), 202–206.
6. Horadam A. F., *Jacobsthal Representation Numbers*, Fibonacci Quart. **34** (1996), 40–54.
7. Koshy T., *Elementary Number Theory with Applications*, Academic Press, 2007.
8. Koshy T. and Gao Z., *Catalan numbers with Mersenne subscripts*, Math. Scientist **38** (2013), 86–91.
9. Koshy T., *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons: New York, 2001.
10. Pomerance C., *On primitive divisors of Mersenne numbers*, Acta Arith. **47** (1986), 355–367.
11. Vajda S., *Fibonacci and Lucas numbers, and the Golden Section, Theory and applications*, Ellis Horwood Limited: Chichester, 1989.
12. Wagstaff S. S., *Divisors of Mersenne numbers*, Math. Comp. **40** (1983), 385–397.
13. Wright G. H. and Hardy E. M., *An Introduction to the Theory of Numbers*, Oxford University Press: New York, 1975.

A. Daşdemir, Department of Mathematics, Faculty of Arts and Sciences, Kastamonu University, Kastamonu 37150, Turkey, *e-mail*: ahmetdasdemir37@gmail.com