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DIHEDRAL COVERS OF THE COMPLETE GRAPH K_5

M. GHASEMI

Abstract. A regular cover of a connected graph is called dihedral if its transformation group is dihedral. In this paper, the author classifies all dihedral coverings of the complete graph K_5 whose fibre-preserving automorphism subgroups act arc-transitively.

1. Introduction

Throughout this paper, we consider finite connected graphs without loops or multiple edges. For a graph X, every edge of X gives rise to a pair of opposite arcs. By $\mathrm{V}(X)$, $\mathrm{E}(X)$, $\mathrm{A}(X)$ and $\mathrm{Aut}(X)$, we denote the vertex set, the edge set, the arc set and the automorphism group of the graph X, respectively. The *neighborhood* of a vertex $v \in \mathrm{V}(X)$ denoted by $N(v)$ is the set of vertices adjacent to v in X. Let a group G act on a set Ω and let $\alpha \in \Omega$. We denote by G_α the *stabilizer* of α in G, that is, the subgroup of G fixing α. The group G is said to be *semiregular* if $G_\alpha = 1$ for each $\alpha \in \Omega$, and *regular* if G is semiregular and transitive on Ω. A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p\colon \widetilde{X} \to X$ if there is a surjection $p\colon V(\widetilde{X}) \to V(X)$ such that $p|_{N(\tilde{v})}\colon N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in \mathrm{V}(X)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{X}$ is called the *covering graph* and X is the *base graph*. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic, elementary abelian or dihedral then $\widetilde{X}$ is called a *cyclic*, *elementary abelian* or *dihedral covering* of X, respectively. If $\widetilde{X}$ is connected, K is the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre while an element of the covering transformation group fixes each fibre setwise. All of fibre-preserving automorphisms form a group called the *fibre-preserving group*.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$; in other

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words, a directed walk of length s which never includes a backtracking. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. An s-arc-transitive graph is said to be *s-transitive* if it is not $(s+1)$-arc-transitive. In particular, a subgroup of the automorphism group of a graph X is said to be *s-regular* if it acts regularly on the set of s-arcs of X. Also if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of X, then X is said to be *s-regular*. Thus, if a graph X is s-regular, then $\mathrm{Aut}(X)$ is transitive on the set of s-arcs and the only automorphism fixing an s-arc is the identity automorphism of X.

Regular coverings of a graph have received considerable attention. For example, for a graph X which is the complete graph K_4, the complete bipartite graph $K_{3,3}$, hypercube Q_3 or Petersen graph O_3, the s-regular cyclic or elementary abelian coverings of X, whose fibre-preserving groups are arc-transitive, classified for each $1 \leq s \leq 5$ [**3, 4, 6, 7**]. As an application of these classifications, all s-regular cubic graphs of order $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, and $10p^2$ constructed for each $1 \leq s \leq 5$ and each prime p [**3, 4, 6**]. In [**14**], it was shown that all cubic graphs admitting a solvable edge-transitive group of automorphisms arise as regular covers of one of the following basic graphs: the complete graph K_4, the dipole Dip3 with two vertices and three parallel edges, the complete bipartite graph $K_{3,3}$, the Pappus graph of order 18, and the Gray graph of order 54. Also all dihedral coverings of the complete graph K_4 and cubic symmetric graphs of order $2p$ were classified in [**5, 8**]. But apart from the octahedron graph [**11**], graphs of higher valencies have not received much attention. For more results see [**1, 2, 13, 15**]. In a series of reductions of this kind, the final, irreducible graph is often a complete graph. Thus studying K_5 is the obvious next choice in order to establish a base of examples for further investigation. All pairwise non-isomorphic connected arc-transitive p-elementary abelian covers of the complete graph K_5 are constructed in [**10**]. In this paper all dihedral coverings of the complete graph K_5 whose fibre-preserving automorphism subgroups act arc-transitively are determined. Also we give a family of 2-arc-transitive graphs.

Let n be a non-negative integer. Let $\mathbb{Z}_n$ denote the cyclic group of order n and D_{2n} the dihedral group of order $2n$. Set

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

By $\{0,1,2,3,4\}$ denote the vertex set of K_5. For $n \geq 3$, the graph $DK(2n)$ is defined to have vertex set

$$V(DK(2n)) = \{0,1,2,3,4\} \times D_{2n}$$

and edge set

$$\begin{aligned} E(DK(2n)) = \{&(0,c)(3,c), (1,c)(3,c), (1,c)(4,c), (2,c)(4,c), (0,c)(1,bc),\\ &(0,c)(2,a^{-1}bc), (0,c)(4,ac), (1,c)(2,bc), (2,c)(3,ac),\\ &(3,c)(4,a^{-2}bc), (4,c)(0,a^{-1}c) \mid c \in D_{2n}\}. \end{aligned}$$

Note that the first four edges in the edge set $E(DK(2n))$ correspond with the tree edges in the spanning tree T as depicted by the dashed lines in Fig. 1 and these four edges have the common c as the second coordinates. In fact, the graph $DK(2n)$ is the covering graph derived from a T-reduced voltage assignment $\phi\colon A(K_5) \to D_{2n}$ which assigns the six values b, $a^{-1}b$, a, b, $a^{-2}b$, a^{-1} to the six cotree edges in K_5.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $\widetilde{X}$ be a connected D_{2n}-covering ($n \geq 3$) of the complete graph K_5 whose fibre-preserving subgroup is arc-transitive. Then $\widetilde{X}$ is arc-transitive if and only if $\widetilde{X}$ is isomorphic to $DK(2n)$ for $n \geq 3$.*

2. Preliminaries related to coverings

Let X be a graph and K a finite group. By a^{-1}, we mean the reverse arc to an arc a. A *voltage assignment* (or *K-voltage assignment*) of X is a function $\phi\colon A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called *voltages* and K is the *voltage group*. The *graph* $X \times_\phi K$ derived from a voltage assignment $\phi\colon A(X) \to K$ has a vertex set $V(X) \times K$ and an edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p\colon X \times_\phi K \to X$ which is called the *natural projection*. By defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a *K-covering*. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv, where $a = (u, v)$. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be

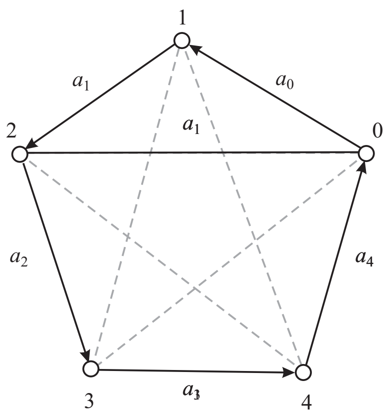

Figure 1. A choice of the six cotree arcs in K_5.

T-reduced if the voltages on the tree arcs are the identity. Gross and Tucker [**9**] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of α, and α the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the trivial group, that is $K = \{\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X}) \colon\ p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of α, then $K\tilde{\alpha}$ consists of all the lifts of α.

Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. The problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\phi(C))^{\bar{\alpha}} = \phi(C^{\alpha}),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^{\alpha})$ are the voltages on C and C^{α}, respectively. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X.

The next proposition is a special case of [**12**, Theorem 3.5].

Proposition 2.1. *Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then, an automorphism α of X lifts if and only if $\bar{\alpha}$ extends to an automorphism of K.*

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projections p_1 and p_2, respectively, are said to be *equivalent* if there exists a graph isomorphism $\tilde{\alpha} \colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\tilde{\alpha}p_2 = p_1$. We quote the following proposition.

Proposition 2.2 ([**16**])**.** *Two connected regular coverings $X \times_\phi K$ and $X \times_\psi K$, where ϕ and ψ are T-reduced, are equivalent if and only if there exists an automorphism $\sigma \in$ Aut(K) such that $\phi(u,v)^{\sigma} = \psi(u,v)$ for any cotree* arc(u,v) *of X.*

3. Proof of Theorem 1.1

Suppose that $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. If $n = 2$, then $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now since elementary abelian coverings of the complete graph K_5 were classified by Kuzman [**10**], we only consider $n \geq 3$.

By K_5, we denote the complete graph with vertex set $\{0, 1, 2, 3, 4\}$. Let T be a spanning tree of K_5 as shown by dashed lines in Figure 2. Let ϕ be such a voltage assignment defined by $\phi = 1$ on T and $\phi = a_0, a_1, a_2, a_3$, a_4, and b_0 on the cotree arcs $(0,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(4,0)$, and $(0,2)$, respectively. Let $\rho = (01234)$, $\tau = (0132)$ and $\sigma = (024)$. Then ρ, τ, and σ are automorphisms of K_5.

By $i_1 i_2 \ldots i_s$ denote a directed cycle which has vertices i_1, i_2, $\ldots$, i_s in a consecutive order. There are six fundamental cycles 130, 124, 1423, 134, 1403, and 13024 in K_5 which are generated by the six cotree arcs $(0,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(4,0)$ and $(0,2)$, respectively. Each cycle is mapped to a cycle of the same length under the actions of ρ, τ, σ. We list all these cycles and their voltages in Table 1 in which C denotes a fundamental cycle of K_5 and $\phi(C)$ denotes the voltage of C.

Let $\widetilde{X} = K_5 \times_\phi D_{2n}$ be a covering graph of the graph K_5 satisfying the hypotheses in the theorem, where $\phi = 1$ on the spanning tree T which is depicted by the dashed lines in Figure 2. Note that the vertices of K_5 are labeled by 0, 1, 2, 3, and 4. By the hypotheses, the fibre-preserving group, say $\widetilde{L}$, of the covering graph $K_5 \times_\phi D_{2n}$ acts arc-transitively on $K_5 \times_\phi D_{2n}$. Hence, the projection of $\widetilde{L}$, say L, is arc-transitive on the base graph K_5. Thus L is isomorphic to $\mathrm{AGL}(1,5) = \langle \rho, \tau \rangle$, $A_5 = \langle \rho, \sigma \rangle$, or $S_5 = \langle \rho, \sigma, \tau \rangle$. Consider the mapping $\bar{\rho}$ from the set $\{a_0, a_1, a_2, a_3, a_4, b_0\}$ of the voltages of the six fundamental cycles of K_5 to the group D_{2n}, defined by $(\phi(C))^{\bar{\rho}} = \phi(C^\rho)$, where C ranges over the six fundamental cycles. From Table 1, one can see that $a_0^{\bar{\rho}} = a_1$, $a_1^{\bar{\rho}} = a_2 b_0$, $a_2^{\bar{\rho}} = b_0^{-1} a_3$, $a_3^{\bar{\rho}} = a_4 b_0$, $a_4^{\bar{\rho}} = b_0^{-1} a_0$ and $b_0^{\bar{\rho}} = b_0$. Similarly, we can define $\bar{\sigma}$ and $\bar{\tau}$.

C	$\phi(C)$	C^ρ	$\phi(C^\rho)$
130	a_0	241	a_1
124	a_1	230	$a_2 b_0$
1423	a_2	2034	$b_0^{-1} a_3$
134	a_3	240	$a_4 b_0$
1403	a_4	2014	$b_0^{-1} a_0$
13024	b_0	24130	b_0
C^σ	$\phi(C^\sigma)$	C^τ	$\phi(C^\tau)$
132	$a_2^{-1} a_1^{-1}$	321	$a_2^{-1} a_1^{-1}$
140	$a_4 a_0$	304	$a_4^{-1} a_3^{-1}$
1043	$a_0^{-1} a_4^{-1} a_3^{-1}$	3402	$a_3 a_4 b_0 a_2$
130	a_0	324	$a_2^{-1} a_3^{-1}$
1023	$a_0^{-1} b_0 a_2$	3412	$a_3 a_1 a_2$
13240	$a_2^{-1} a_4 a_0$	32104	$a_2^{-1} a_1^{-1} a_0^{-1} a_4^{-1} a_3^{-1}$

Table 1. Fundamental cycles and their images with corresponding voltages.

Here we make the following general assumption.

(I) Let $\widetilde{X}$ be a connected D_{2n}-covering ($n \geq 3$) of the complete graph K_5 whose fibre-preserving subgroup is arc-transitive.

For the three following lemmas we suppose that n is an odd number.

Lemma 3.1. *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* ρ *and* σ*, say* L*, lifts. Under the assumption* (I)*,* $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(6)$*.*

Proof. Since $\rho, \sigma \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\sigma}$ can be extended to automorphisms of D_{2n}. We denote by ρ^* and σ^* these extended automorphisms, respectively. In this case $o(a_0) = o(a_1) = o(a_3)$. Now we consider the following two subcases:

Subcase I. $o(a_0) = o(a_1) = o(a_3) = 2$.

By considering $a_1^{\sigma^*} = a_4a_0$, we have $o(a_4a_0) = 2$. It follows that $o(a_4) \neq 2$. Since $a_4^{\rho^*} = b_0^{-1}a_0$, we have $o(b_0^{-1}a_0) \neq 2$. So $o(b_0^{-1}) = 2$, and hence $o(a_2) \neq 2$, by $a_2^{\rho^*} = b_0^{-1}a_3$. Now we may assume that $a_0 = a^ib$, $a_1 = a^jb$, $a_3 = a^kb$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^lb$, where $0 \leq i,j,k,l \leq n-1$ and $0 < r,s \leq n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2 we may assume that $a_0 = b$, $a_1 = a^ib$, $a_3 = a^jb$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^kb$, where $0 \leq i,j,k \leq n-1$ and $0 < r,s \leq n-1$. Also since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0\rangle$. Thus we may assume that $(t,n) = 1$, where $t \in \{i,j,k,r,s\}$. Without loss of generality, we may assume that $(i,n) = 1$ or $(r,n) = 1$. In fact, with the same arguments as in other cases we get the same results. First suppose that $(i,n) = 1$. Since $\sigma\colon a \mapsto a^i, b \mapsto b$ is an automorphism of D_{2n}, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^ib$, $a_2 = a^r$, $a_4 = a^s$, and $b_0 = a^jb$, where $0 \leq i,j \leq n-1$ and $0 < r,s \leq n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = ab$, $a_1^{\rho^*} = (ab)^{\rho^*} = a^{\rho^*}b^{\rho^*} = a^{r+j}b$. Thus $a^{\rho^*} = a^{r+j-1}$. By considering the image of $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^jb$ under ρ^*, we conclude that $a^{r(r+j-1)} = a^{j-i}$, $a^{s(r+j-1)} = a^j$ and $a^{j(r+j-1)}ab = a^jb$. Also $a_0^{\sigma^*} = b^{\sigma^*} = a^{-r+1}b$ and $a_1^{\sigma^*} = (ab)^{\sigma^*} = a^{\sigma^*}b^{\sigma^*} = a^sb$. Thus $a^{\sigma^*} = a^{s+r-1}$.

Now by considering the image of $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^jb$ under σ^*, we conclude that $a^{r(r+s-1)} = a^{s-i}$, $a^{s(r+s-1)} = a^{-j+r}$ and $a^{j(s+r-1)}a^{-r+1}b = a^{s-r}b$.

Therefore, we have the following:

(1) $r(r+j-1) = j-i$,	(2) $s(r+j-1) = j$,
(3) $j(r+j-1) = j-1$,	(4) $j(s+r-1) = s-1$,
(5) $r(s+r-1) = s-i$	(6) $s(s+r-1) = -j+r$.

By (1) and (3), $rj(r+j-1) = j^2-ij$ and $rj(r+j-1) = rj-r$. Thus $j^2-ji = rj-r$. Also by (4) and (5), $rj(s+r-1) = sr-r$ and $rj(s+r-1) = sj-ij$. Thus $sj-ij = sr-r$. So $j^2-rj = sj-sr$, and hence $(j-r)(j-s) = 0$. Also by (2) and (3), $sj(r+j-1) = j^2$ and $sj(r+j-1) = sj-s$. Thus $j^2 = sj-s$. By $(j-r)(j-s) = 0$, we have $j = r$ or $j = s$. If $j = r$, then $s^2+sr-s = 0$, by (6). Thus $s = 0$ or $s = -r+1$. If $s = 0$, then $j = 0$ by (2). Thus $r = 0$, a contradiction. If $s = -r+1$, then $s = 1$ by $j(s+r-1) = s-1$. So $r = 0$, a contradiction. If $j = s$, then by $j^2 = sj-s$, we have $s = 0$, a contradiction.

Now suppose that $(r,n)=1$. Since $\sigma\colon a\mapsto a^r, b\mapsto b$ is an automorphism of D_{2n}, by Proposition 2.2, we may assume that $a_0=b$, $a_1=a^ib$, $a_3=a^jb$, $a_2=a$, $a_4=a^r$ and $b_0=a^kb$, where $0\le i,j,k\le n-1$ and $0<r\le n-1$. From Table 1, one can see that $a_0^{\rho^*}=b^{\rho^*}=a^ib$, $a_2^{\rho^*}=(a)^{\rho^*}=a^{k-j}$. By considering the image of $a_1=a^ib$, $a_3=a^jb$, $a_4=a^r$ and $b_0=a^kb$ under ρ^*, we conclude that $a^{i(k-j)}a^ib=a^{k+1}b$, $a^{j(k-j)}a^ib=a^{r+k}b$, $a^{r(k-j)}=a^k$ and $a^{k(k-j)}a^ib=a^kb$. Also $a_0^{\sigma^*}=b^{\sigma^*}=a^{i-1}b$ and $a_2^{\sigma^*}=(a)^{\sigma^*}=a^{r-j}$. Now by considering the image of $a_1=a^ib$, $a_3=a^jb$, $a_4=a^r$ and $b_0=a^kb$ under σ^*, we conclude that $a^{i(r-j)}a^{i-1}b=a^rb$, $a^{j(r-j)}a^{i-1}b=b$, $a^{r(r-j)}=a^{-k+1}$ and $a^{k(r-j)}a^{i-1}b=a^{r-1}b$.

Therefore, we have the following:

(1)	$i(k-j)+i=k+1,$	(2)	$j(k-j)+i=r+k,$
(3)	$r(k-j)=k,$	(4)	$k(k-j)+i=k,$
(5)	$i(r-j)+i-1=r,$	(6)	$j(r-j)+i-1=0,$
(7)	$r(r-j)=-k+1,$	(8)	$k(r-j)=r-i.$

By (2) and (3), $rj(k-j)=r^2+rk-ir$ and $rj(k-j)=kj$. Thus $r^2+rk-ir=kj$. Also by (7) and (8), $rk(r-j)=-k^2+k$ and $rk(r-j)=r^2-ir$. Thus $-k^2+k=r^2-ir$. So $kj-rk=-k^2+k$, and hence $k(j-r+k-1)=0$. Thus $k=0$ or $j=r-k+1$. If $k=0$, then $i=0$ by (4). Thus by $-k^2+k=r^2-ir$, we have $r=0$, a contradiction. If $j=r-k+1$, then $(k-1)(r+1)=0$ by (7). Hence $k=1$ or $r=-1$. If $k=1$, then $j=r$. Now by (6), $i=1$, and so by (8), we have $r=1$. So by (5), $1=0$, a contradiction. If $r=-1$, then $j=-k$. Also by (5), $i(r-j+1)=0$, and so $i=0$ or $r=j-1$. If $i=0$, then by (1), $k=-1$. Thus $j=1$, and so by (3), $2=-1$. Therefore, $n=3$ and

$$a_0=b,\quad a_1=b,\quad a_3=ab,\quad a_2=a,\quad a_4=a^{-1},\quad b_0=a^{-1}b.$$

From Table 1, it is easy to check that $\bar\rho$, $\bar\sigma$ and $\bar\tau$ can be extended to automorphisms of D_{2n}. Thus by Proposition 2.1, ρ, σ and τ lift. Since $S_5=\langle\rho,\sigma,\tau\rangle$ is 2-arc-transitive, it follows that $\mathrm{Aut}(\widetilde{X})$ contains a 2-arc-transitive subgroup lifted by $\langle\rho,\sigma,\tau\rangle$. Therefore, $\widetilde{X}$ is 2-arc-transitive.

Finally, if $r=j-1$, then by $r=-1$, we have $j=0$. So by (6), $i=1$. Also by (7), $k=0$. Now by (2), $1=-1$, and so $n=2$, a contradiction.

Subcase II. $o(a_0)=o(a_1)=o(a_3)\ne 2$.

By considering $a_1^{\sigma^*}=a_4a_0$, we have $o(a_4a_0)\ne 2$. It follows that $o(a_4)\ne 2$. Since $a_4^{\rho^*}=b_0^{-1}a_0$, we have $o(b_0^{-1}a_0)\ne 2$. So $o(b_0^{-1})\ne 2$, and hence $o(a_2)\ne 2$ by $a_2^{\rho^*}=b_0^{-1}a_3$. Now we may assume that $a_0=a^i$, $a_1=a^j$, $a_2=a^k$, $a_3=a^l$, $a_4=a^m$ and $b_0=a^n$, where $0\le i,j,k,l,m,n\le n-1$. Since $K_5\times_\phi D_{2n}$ is connected, we have a contradiction. □

Lemma 3.2. *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* ρ *and* τ*, say* L*, lifts. Under the assumption* (I), $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(2n)$ *for* $n>3$.

Proof. Since $\rho,\tau\in L$, Proposition 2.1 implies that $\bar\rho$ and $\bar\tau$ can be extended to automorphisms of D_{2n}. We denote these extended automorphisms by ρ^* and

τ^*, respectively. In this case $o(a_0) = o(a_1)$. Now we consider the following two subcases:

Subcase I. $o(a_0) = o(a_1) = 2$.

By considering $a_0^{\tau^*} = a_2^{-1}a_1^{-1}$, we have $o(a_2^{-1}a_1^{-1}) = 2$. It follows that $o(a_2) \neq 2$. Since $a_2^{\rho^*} = b_0^{-1}a_3$, we have either $o(b_0) = o(a_3) = 2$ or $o(b_0) \neq 2$ and $o(a_3) \neq 2$. First suppose that $o(b_0) \neq 2$ and $o(a_3) \neq 2$. Since $a_3^{\rho^*} = a_4 b_0$, we have $o(a_4) \neq 2$. Also since $a_4^{\rho^*} = b_0^{-1}a_0$, it follows that $o(a_0) \neq 2$, a contradiction.

Now suppose that $o(b_0) = o(a_3) = 2$. Since $a_3^{\rho^*} = a_4 b_0$, it implies that $o(a_4) \neq 2$. Now we may assume that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n-1$ and $0 < r, s \leq n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$ and $0 < r, s \leq n-1$. Since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, r, s\}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get the same results. First suppose that $(i, n) = 1$. Since $\sigma\colon a \mapsto a^i, b \mapsto b$ is an automorphism of D_{2n}, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^j b$, where $0 \leq i, j \leq n-1$ and $0 < r, s \leq n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = ab$, $a_1^{\rho^*} = (ab)^{\rho^*} = a^{\rho^*} b^{\rho^*} = a^{r+j} b$. Thus $a^{\rho^*} = a^{r+j-1}$. By considering the image of $a_2 = a^r$, $a_3 = a^i b$ and $b_0 = a^j b$ under ρ^*, we conclude that $a^{r(r+j-1)} = a^{j-i}$, $a^{i(r+j-1)}ab = a^{s+j}b$ and $a^{j(r+j-1)}ab = a^j b$. Also $a_0^{\tau^*} = b^{\tau^*} = a^{-r+1}b$, $a_1^{\tau^*} = (ab)^{\tau^*} = a^{\tau^*} b^{\tau^*} = a^{i-s}b$. Thus $a^{\tau^*} = a^{i-s+r-1}$. By considering the image of $a_2 = a^r$ and $b_0 = a^j b$ under τ^*, we conclude that $a^{r(i-s+r-1)} = a^{i-s-j+r}$ and $a^{j(i-s+r-1)}a^{-r+1} = a^{-r+1-s+i}$.

Therefore, we have the following:

(1) $r(r+j-1) = j - i$, (2) $i(r+j-1) + 1 = s + j$,
(3) $j(r+j-1) + 1 = j$, (4) $r(i-s+r-1) = i - s - j + r$,
(5) $j(i-s+r-1) = i - s$.

By (4) and (5), $(j-r)(i-s+r-2) = 0$. Thus $j = r$ or $i - s + r = 2$. If $i - s + r = 2$, then by (4) $j = i - s$. Now by (1), $r(r+i-s-1) = -s$. So by considering (4) $i + r = j$. Thus $r = -s$ by $j = i - s$. So $i = 2s + 2$, and hence $j = s + 2$. Now by (2), $1 = 0$, a contradiction. If $j = r$, then $r(2r-1) = r - i$ by (1). Also by (3), $r(2r-1) = r - 1$. So $i = 1$, and hence by (2), $s = r$. Now by (5), $s = r = j = 1$. Thus by (1), $1 = 0$, a contradiction.

Now suppose that $(r, n) = 1$. Since $\sigma\colon a \mapsto a^r, b \mapsto b$ is an automorphism of D_{2n}, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a$, $a_4 = a^r$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$ and $0 < r \leq n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = a^i b$, $a_2^{\rho^*} = (a)^{\rho^*} = a^{k-j}$. By considering the image of $a_1 = a^i b$, $a_3 = a^j b$, $a_4 = a^r$ and $b_0 = a^k b$ under ρ^*, we conclude that $a^{i(k-j)}a^i b = a^{k+1}b$, $a^{j(k-j)}a^i b = a^{k+r}b$, $a^{r(k-j)} = a^k$ and $a^{k(k-j)}a^i b = a^k b$. Also $a_0^{\tau^*} = b^{\tau^*} = a^{i-1}b$, $a_2^{\tau^*} = a^{\tau^*} = a^{j-r-k+1}$. By considering the image of $a_1 = a^i b$, $a_3 = a^j b$ and $b_0 = a^k b$ under τ^*, we conclude that $a^{i(j-r-k+1)}a^{i-1}b = a^{j-r}b$, $a^{j(j-r-k+1)}a^{i-1}b = a^{j-1}b$ and $a^{k(j-r-k+1)}a^{i-1}b = a^{i-1-r+j}b$.

Therefore, we have the following:

(1)	$ik - ij + i = k + 1,$	(2)	$jk - j^2 + i = r + k,$
(3)	$rk - rj = k,$	(4)	$k^2 - kj + i = k,$
(5)	$i(j - r - k + 1) + i - 1 = -r + j,$	(6)	$j(j - r - k + 1) = j - i,$
(7)	$k(j - r - k + 1) = j - r.$		

By (6), $j^2-jr-jk+i=0$. Also by (6) and (7), we have $kj(j-r-k+1)=kj-ki$ and $kj(j-r-k+1)=j^2-rj$. Thus $j^2-jr=kj-ki$. Thus $i(k-1)=0$, and so $i=0$ or $k=1$. If $i=0$, then by (1), we have $k=-1$. Also by (4), $j=-2$. Now by (2), $r=-1$. Therefore,

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2}b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of D_{2n}. By Proposition 2.1, ρ and τ lift. Clearly, $AGL(1,5)=\langle\rho,\tau\rangle$ is 1-regular. Thus $\mathrm{Aut}(\widetilde{X})$ contains a 1-regular subgroup lifted by $\langle\rho,\tau\rangle$.

Now if $k=1$, then by (3) and (4), $r-rj=1$ and $i-j=0$. Since $i=j$, it follows that $i(i-r)=-r+1$ by (5). So $i^2-ir=-r+1=-1-rj+1$. Thus $i=j=0$, and so $r=1$. Now by (2), $2=0$, a contradiction.

Subcase II. $o(a_0)=o(a_1)\neq 2$.

By considering $a_0^{\tau^*}=a_2^{-1}a_1^{-1}$, we have $o(a_2^{-1}a_1^{-1})\neq 2$. It follows that $o(a_2)\neq 2$. Since $a_2^{\rho^*}=b_0^{-1}a_3$, we have either $o(b_0)=o(a_3)=2$ or $o(b_0)\neq 2$ and $o(a_3)\neq 2$. First suppose that $o(b_0)=o(a_3)=2$. Since $a_3^{\rho^*}=a_4b_0$, it follows that $o(a_4)\neq 2$. Now by considering $a_4^{\rho^*}=b_0^{-1}a_0$, we have $o(b_0^{-1}a_0)\neq 2$ a contradiction.

Now suppose that $o(b_0)\neq 2$ and $o(a_3)\neq 2$. Since $a_3^{\rho^*}=a_4b_0$, we have $o(a_4)\neq 2$. Therefore, $K_5\times_\phi D_{2n}$ is not connected, a contradiction. □

Lemma 3.3. *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* ρ, σ *and* τ, *say* L, *lifts. Under the assumption* (I), $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(2n)$ *for* $n\geq 3$.

Proof. ρ and σ lift. With the same arguments as in Cubcase I, we have $n=3$ and

$$a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\rho}$ $\bar{\sigma}$ and $\bar{\tau}$ can be extended to automorphisms of D_{2n}. By Proposition 2.1, ρ, σ and τ lift. Also $S_5=\langle\rho,\sigma,\tau\rangle$ is 2-arc-transitive. Thus $\mathrm{Aut}(\widetilde{X})$ contains a 2-arc-transitive subgroup lifted by $\langle\rho,\sigma,\tau\rangle$. Thus $\widetilde{X}$ is 2-arc-transitive. Moreover, ρ and τ lift. With the same arguments as in Subcase II, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2}b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\sigma}$ can be extended to automorphisms of D_{2n} whenever $n=3$. Now if $n=3$, then by Proposition 2.1, σ lift. Now with the same arguments as above, $\widetilde{X}$ is 2-arc-transitive. □

Now suppose that n is even.

Lemma 3.4. *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* ρ *and* σ*, say* L*, lifts. Then there is no connected regular covering of the complete graph* K_5 *whose fibre-preserving group is arc-transitive.*

Proof. Since $\rho, \sigma \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\sigma}$ can be extended to automorphisms of D_{2n}. We denote these extended automorphisms by ρ^* and σ^*, respectively. In this case $o(a_0) = o(a_1) = o(a_3)$. Now we consider the following two subcases:

Subcase I. $o(a_0) = o(a_1) = o(a_3) = 2$.

Since $o(a_0) = 2$, we may assume that $a_0 = a^{n/2}$ or $a_0 \neq a^{n/2}$ and $a_0 = a^i b$ $(0 \leq i < n)$. If $a_0 = a^{n/2}$, then $a_1 = a_3 = a^{n/2}$. By Table 1, $a_1^{\sigma^*} = a_4 a_0$ and $a_3^{\sigma^*} = a_0$. Thus $a_4 = 1$ and so by $a_4^{\rho^*} = b_0^{-1} a_0$, we have $b_0 = a^{n/2}$. Also by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $a_2 = 1$. Therefore $K_5 \times_\phi D_{2n}$ is not connected, a contradiction.

Thus we may assume that $a_0 \neq a^{n/2}$. So $a_1 \neq a^{n/2}$ and $a_3 \neq a^{n/2}$. Thus we may assume that $a_0 = a^i b$, $a_1 = a^j b$ and $a_3 = a^k b$, where $0 \leq i, j, k < n$. By considering $a_1^{\rho^*} = a_2 b_0$, we have one of the following cases:

i) $a_2 = a^l b$, $b_0 = a^t$ $\quad (0 \leq l < n,\ 0 < t < n)$;
ii) $a_2 = a^l$, $b_0 = a^t b$ $\quad (0 < l < n,\ 0 \leq t < n)$.

First suppose that $a_2 = a^l b$, $b_0 = a^t$ $(0 \leq l < n,\ 0 < t < n)$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we may suppose that $a_4 = a^s b$, where $0 \leq s < n$. Now since $b_0^{\sigma^*} = a_2^{-1} a_4 a_0$, we have a contradiction. Now suppose that $a_2 = a^l$, $b_0 = a^t b$ $(0 < l < n,\ 0 \leq t < n)$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. First suppose that $o(a_4) \neq 2$. Now by Proposition 2.2, we may assume that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^l$, $a_4 = a^k$ and $b_0 = a^t b$, where $0 \leq i, j, k, t \leq n-1$ and $0 < l, k \leq n-1$. Now with the same arguments as in Subcase I, when n is odd, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1} b.$$

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\sigma}$ can be extended to automorphisms of D_{2n} when $n = 3$, a contradiction.

Now suppose that $a_4 = a^{n/2}$. Now we may assume that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^r$, $a_4 = a^{n/2}$, and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n-1$ and $0 < r \leq n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^{n/2}$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$ and $0 < r \leq n-1$. Since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, r\}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get same results. First suppose that $(i, n) = 1$. Since $\sigma\colon a \mapsto a^i, b \mapsto b$ is an automorphism of D_{2n}, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^r$, $a_4 = a^{(n/2)}$ and $b_0 = a^j b$, where $0 \leq i, j \leq n-1$ and $0 < r \leq n-1$. Now with the same arguments as in Subcase I, when n is odd (by replacing s with $(n/2)$), we have a contradiction.

Now suppose that $(r, n) = 1$. Since $\sigma\colon a \mapsto a^r, b \mapsto b$ is an automorphism of D_{2n}, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a$, $a_4 = a^{(n/2)}$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$. Now by replacing r with $(n/2)$ in Case I, when n is odd, we have $(n/2)(k-j) = k$ and $(n/2)((n/2)-j) = -k+1$ (see Equations (3) and (7) in Subcase I). So $n = 2$, a contradiction.

Subcase II. $o(a_0) = o(a_1) = o(a_3) \neq 2$.

By considering $a_1^{\sigma^*} = a_4 a_0$, we have $o(a_4 a_0) \neq 2$. So we have $o(a_4) \neq 2$ or $o(a_4) = 2$ and $a_4 = a^{n/2}$. If $o(a_4) \neq 2$, then $o(b_0^{-1} a_0) \neq 2$ by $a_4^{\rho^*} = b_0^{-1} a_0$. Now we have $o(b_0) \neq 2$ or $o(b_0) = 2$ and $b_0 = a^{n/2}$. If $b_0 = a^{n/2}$, then $o(a_2) \neq 2$ by $a_2^{\rho^*} = b_0^{-1} a_3$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. If $o(b_0) \neq 2$, then by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_2) \neq 2$ or $o(a_2) = 2$ and $a_2 = a^{n/2}$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. Finally, if $a_4 = a^{n/2}$, then by considering $a_3^{\rho^*} = a_4 b_0$, we have $o(b_0) \neq 2$ or $o(b_0) = 2$ and $b_0 = a^{n/2}$. Clearly, $b_0 \neq a^{n/2}$ by $a_3^{\rho^*} = a_4 b_0$. Thus $o(b_0) \neq 2$, and so by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_2) \neq 2$ or $o(a_2) = 2$ and $a_2 = a^{n/2}$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. □

Lemma 3.5. *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* ρ *and* τ*, say* L*, lifts. Under the assumption* (I)*,* $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(2n)$ *for* $n > 3$*.*

Proof. Since $\rho, \tau \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of D_{2n}. We denote these extended automorphisms by ρ^* and τ^*, respectively. In this case $o(a_0) = o(a_1)$. Now we consider the following two subcases:

Subcase I. $o(a_0) = o(a_1) = 2$.

Since $o(a_0) = 2$, we may assume that $a_0 = a^{n/2}$ or $a_0 \neq a^{n/2}$ and $a_0 = a^i b$ $(0 \le i < n)$. If $a_0 = a^{n/2}$, then $a_1 = a^{n/2}$. By Table 1, we have $a_0^{\tau^*} = a_2^{-1} a_1^{-1}$ and $a_1^{\rho^*} = a_2 b_0$. Therefore, $a_2 = 1$ and $b_0 = a^{n/2}$. Also by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $a_3 = a^{n/2}$. Now by $a_3^{\rho^*} = a_4 b_0$, we have $a_4 = 1$. Thus $\widetilde{X}$ is not connected, a contradiction. Thus we may assume that $a_0 \neq a^{n/2}$ and $a_0 = a^i b$. So $a_1 \neq a^{n/2}$ and so we may assume that $a_0 = a^i b$, $a_1 = a^j b$, where $0 \le i, j < n$. By considering $a_0^{\tau^*} = a_2^{-1} a_1^{-1}$, we have $o(a_2) \neq 2$ or $a_2 = a^{n/2}$. First assume that $o(a_2) \neq 2$. Thus $b_0 = a^k b$ $(0 \le k < n)$ by $a_1^{\rho^*} = a_2 b_0$. Also since $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_3) = 2$ and $a_3 = a^l b$ $(0 \le l < n)$. Finally, since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. First suppose that $a_4 = a^{n/2}$. We have $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^r$, $a_4 = a^{n/2}$ and $b_0 = a^l b$, where $0 \le i, j, k, l \le n-1$ and $0 < r \le n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^{n/2}$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$ and $0 < r \le n-1$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $k = n/2$. Now $a_4 a_0 = b_0$, and so $(a_4 a_0)^{\rho^*} = b_0^{\rho^*}$. Thus $a_0 = a_1$, and so $i = 0$. We have $a_0^{\rho^*} = a_1^{\rho^*}$. So $a_1 = a_2 b_0$, and hence $r = n/2$. Now $a_2 = a_4$, and so $a_2^{\rho^*} = a_4^{\rho^*}$. Therefore, $a_0 = a_3$, and hence $a_3 = b$. Now $K_5 \times_\phi D_{2n}$ is not connected a contradiction.

Now suppose that $o(a_4) \neq 2$. With the same arguments as in Subcase II, when n is odd, we have

$$a_0 = b, a_1 = b, a_3 = a^{-2}b, a_2 = a, a_4 = a^{-1}, b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of D_{2n}. By Proposition 2.1, ρ and τ lift. Also $AGL(1,5) = \langle \rho, \tau \rangle$ is 1-regular. Thus $\mathrm{Aut}(\widetilde{X})$ contains a 1-regular subgroup lifted by $\langle \rho, \tau \rangle$.

Now assume that $a_2 = a^{n/2}$. Thus $b_0 = a^k b$ $(0 \leq k < n)$ by $a_1^{\rho^*} = a_2 b_0$. Also since $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_3) = 2$ and $a_3 = a^l b$ $(0 \leq l < n)$. Finally, since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. First suppose that $a_4 = a^{n/2}$. We have $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a_4 = a^{n/2}$ and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n-1$. Since $a_4^{\tau^*} = a_3 a_1 a_2$, we have $k = j$. Also since $a_2^{\tau^*} = a_3 a_4 b_0 a_2$, we have $l = k = j$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^i b$, $a_2 = a_4 = a^{n/2}$, and $b_0 = a^i b$, where $0 \leq i, j, k \leq n-1$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $i = n/2$, a contradiction.

Now suppose that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^{n/2}$, $a_4 = a^s$ and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n-1$ and $0 < s \leq n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^{n/2}$, $a_4 = a^s$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$ and $0 < s \leq n-1$. Since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, s\}$. Without loss of generality we may assume that $(i, n) = 1$ or $(s, n) = 1$. In fact, with the same arguments the in other cases we get the same results. First suppose that $(i, n) = 1$. Therefore, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^{(n/2)}$, $a_4 = a^s$, and $b_0 = a^j b$, where $0 \leq i, j \leq n-1$ and $0 < s \leq n-1$. Now with the same arguments as in Case II, when n is odd we get a contradiction. Now suppose that $(s, n) = 1$. Therefore, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^{n/2}$, $a_4 = a$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = a^i b$, $a_4^{\rho^*} = (a)^{\rho^*} = a^k$. By considering the image of $a_1 = a^i b$, $a_3 = a^j b$ and $a_2 = a^{n/2}$ under ρ^*, we conclude that $a^{ik+i} b = a^{(n/2)+k} b$, $a^{jk+i} b = a^{k+1} b$ and $a^{(n/2)k} = a^{k-j}$. Thus, we have $ik + i = n/2 + k$, $jk + i = k + 1$ and $(n/2)k = k - j$. By $(n/2)k = k - j$, we have $nk = 2k - 2j$. It follows that $2j = 2k$. Also $a^{\tau^*} = a^j b a^i b a^{(n/2)} = a^{j-i+(n/2)}$. Thus $a_2^{\tau^*} = a^{n/2(j-i+(n/2))} = a^j b a a^k b a^{(n/2)} = a^{j-1-k+(n/2)}$. So, $2j - 2k - 2 = 0$ and so $2 = 0$, a contradiction.

<u>*Subcase II*</u>. $o(a_0) = o(a_1) \neq 2$.

By considering $a_0^{\tau^*} = a_2^{-1} a_1^{-1}$, we have $o(a_2^{-1} a_1^{-1}) \neq 2$. Thus $o(a_2) \neq 2$ or $a_2 = a^{n/2}$. First suppose that $o(a_2) \neq 2$. By considering $a_2^{\rho^*} = b_0^{-1} a_3$, we have one of the following cases:

i) $a_3 = a^i b$, $b_0 = a^j b$ $\quad (0 \leq i, j < n)$;
ii) $a_3 = a^i$, $b_0 = a^{n/2}$ $\quad (0 < i < n)$;
iii) $a_3 = a^{n/2}$, $b_0 = a^i$ $\quad (0 < i < n)$.

By $a_1^{\rho^*} = a_2b_0$, we have a contradiction in the first case. Now consider the second case. Since $a_3^{\rho^*} = a_4b_0$, we have $o(a_4) \neq 2$. Now $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. Now consider the last case. Since $a_3^{\rho^*} = a_4b_0$, we have $o(a_4) \neq 2$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction.

Now suppose that $a_2 = a^{n/2}$. By $a_1^{\rho^*} = a_2b_0$, we have $o(b_0) \neq 2$. Also since $a_2^{\rho^*} = b_0^{-1}a_3$, we have $o(a_3) \neq 2$. Finally, since $a_3^{\rho^*} = a_4b_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. □

Lemma 3.6. *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* ρ, σ *and* τ, *say* L, *lifts. Then there is no connected regular covering of the complete graph* K_5 *whose fibre-preserving group is arc-transitive.*

Proof. ρ and σ lift. With the same arguments as in Case I, we have a contradiction. Also ρ and τ lift. With the same arguments as in Subcase II, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2}b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\sigma}$ can be extended to automorphisms of D_{2n} whenever $n = 3$, a contradiction. □

Proof of Theorem 1.1. This follows from Lemmas 3.1, 3.2 , 3.3, 3.4, 3.5 and 3.6. □

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M. Ghasemi, Department of Mathematics, University of Urmia, Urmia 57135, Iran, *e-mail*: m.ghasemi@urmia.ac.ir