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SOME RESULTS OF f-BIHARMONIC MAPS

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Abstract. In this paper, we introduce the stress f-bienergy tensor $S_{2,f}$ of maps between Riemannian manifols. By using the stress f-bienergy tensor, we obtain some nonexistence results of proper f-biharmonic maps under the assumption $S_{2,f} = 0$.

1. Introduction

There are two ways to formalize the link between bi-harmonic maps and f-harmonic maps. The first formalization is that by mimicking the theory of bi-harmonic maps, we can extend the bi-energy functional to f-bi-energy functional and obtain a new type of harmonic maps called bi-f-harmonic maps. This idea was already considered by S. Ouakkas, R. Nasri, and M. Djaa[**4**].

The second formalization is that by following the concept of f-harmonic map, we can extend the f-energy functional to f-bi-energy functional and obtain another type of harmonic maps called f-bi-harmonic maps as critical points of the f-bi-energy functional.[**7**]

f-harmonic maps and their equations: f-Harmonic maps are critical points of the f-energy functional for maps $\varphi\colon (M,g) \to (N,h)$ between Riemannian manifolds

$$E_f(\varphi) = \frac{1}{2}\int_K f|\mathrm{d}\varphi|^2 v_g,$$

where K is a compact domain of M. The Euler-Lagrange equation gives the f-harmonic map equation ([**1**], [**4**])

$$\tau_f(\varphi) = f\tau(\varphi) + \mathrm{d}\varphi(\operatorname{grad} f) = 0, \tag{1}$$

where $\tau(\varphi) = \operatorname{Tr}_g \nabla \mathrm{d}\varphi$ is the tension field of φ vanishing of which means φ is a harmonic map.

***f*-biharmonic maps and their equations:** f-Biharmonic maps are critical points of the f-bienergy functional for maps $\varphi\colon (M,g) \to (N,h)$ between Riemannian manifolds

$$E_{2,f}(\varphi) = \frac{1}{2}\int_K f|\tau(\varphi)|^2 v_g,$$

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where K is a compact domain of M. The Euler-Lagrange equation gives the f-biharmonic map equation ([**6**])

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + (\triangle f)\tau(\varphi) + 2\nabla^{\varphi}_{\operatorname{grad} f}\tau(\varphi) = 0, \tag{2}$$

where $\tau(\varphi)$ and $\tau_2(\varphi)$ are the tension and bitension fields of φ, respectively.

f-biharmonic maps was introduced in [**6**], where the author calculated the first variation to obtain the f-biharmonic map equation and the equation for the f-biharmonic conformal maps between the same dimensional manifolds. In this paper, we introduce the stress f-bienergy tensor $S_{2,f}$

$$S_{2,f} = fS_2 + \langle \tau(\varphi), \mathrm{d}\varphi(\operatorname{grad} f)\rangle g - 2\mathrm{d}f \odot (\mathrm{d}\varphi \cdot \tau(\varphi)), \tag{3}$$

where

$$S_2 = \frac{1}{2} f|\tau(\varphi)|^2 + f\langle \mathrm{d}\varphi, \nabla^{\varphi}\tau(\varphi)\rangle g - 2\nabla\tau(\varphi) \odot \mathrm{d}\varphi$$

is the stress bi-energy tensor of φ.

Then, by using the stress f-bienergy tensor $S_{2,f}$, we obtain some nonexistence results of proper f-biharmonic maps under the assumption $S_{2,f} = 0$.

2. The Euler-Lagrange equation and applications

For a smooth maps $\varphi\colon (M,g) \to (N,h)$ between Riemannian manifolds, M compact and orientable, and $f \in \mathcal{C}^{\infty}(M)$ positive. Consider the f-bienergy functional $E_{2,f}$

$$E_{2,f}\colon \mathcal{C}^{\infty}(M,N) \to \mathbb{R}, \quad E_{2,f}(\varphi) = \frac{1}{2}\int_M f|\tau(\varphi)|^2 v_g,$$

and a map φ is f-biharmonic if it is a critical point of $E_{2,f}$, that is, for any variation $\{\varphi_t\}$ of φ, $\frac{\mathrm{d}}{\mathrm{d}t}E_{2,f}(\varphi_t)|_{t=0} = 0$.

Given $\varphi\colon M \to (N,h)$, consider the functional

$$F\colon G \to \mathbb{R}, \quad F(g) = E_{2,f}(\varphi),$$

where G is the set of Riemannian metrics on M. As G is an infinite dimensional manifold [**3**], it admits a tangent space at g, the set of symmetric (0,2)-tensors on M, i.e.,

$$T_gG = C(\odot^2 T^*M).$$

For a curve $t \mapsto g_t$ in G with $g_0 = g$, denote by

$$\omega = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\{g_t\} \in T_gG,$$

the corresponding variational tensor field which in local coordinates, can be written

$$\omega = \frac{\partial g_{ij}}{\partial t}(x,0)\mathrm{d}x^i\mathrm{d}x^j = \omega_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where $g_t = g_{ij}(x,t)\mathrm{d}x^i\mathrm{d}x^j$, and write $\delta = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}$ for the first variation.

For a one parameter variation $\{g_t\}$ of g, we have

$$F(g_t) = \frac{1}{2}\int_M f|\tau_t(\varphi)|^2 v_{g_t}.$$

We now compute $\delta(F(g_t))$. Differentiating $F(g_t)$ leads to

$$\delta(F(g_t)) = \frac{1}{2}\int_M f\delta(|\tau_t(\varphi)|^2)v_{g_t} + \frac{1}{2}\int_M f|\tau_t(\varphi)|^2\delta(v_{g_t}). \tag{4}$$

The calculation of the first term breaks in two Lemmas.

Lemma 2.1 ([**3**])**.** *The vector field* $\xi = (\operatorname{div}\omega)^\sharp - \frac{1}{2}\operatorname{grad}(\operatorname{trace}_g\omega)$ *satisfies*

$$\delta(|\tau_t(\varphi)|^2) = -2\langle\tau(\varphi)\cdot\nabla \mathrm{d}\varphi, \omega\rangle - 2\langle\tau(\varphi), \mathrm{d}\varphi(\xi)\rangle,$$

where $\tau(\varphi)\cdot\nabla\mathrm{d}\varphi \in C(\odot^2 T^*M)$ *is defined by*

$$(\tau(\varphi)\cdot\nabla\mathrm{d}\varphi)(X,Y) = \langle\tau(\varphi), \nabla\mathrm{d}\varphi(X,Y)\rangle.$$

Lemma 2.2 ([**3**])**.** *Consider the one-form* $\mathrm{d}\varphi\cdot\tau(\varphi)\in\Lambda^1(M)$ *defined by* $\mathrm{d}\varphi\cdot\tau(\varphi)(X) = \langle\mathrm{d}\varphi(X), \tau(\varphi)\rangle$*, then*

$$\int_M f\langle\tau(\varphi), \mathrm{d}\varphi(\xi))\rangle v_g = \int_M \Big\langle -(\operatorname{sym}\nabla\eta) + \frac{1}{2}\operatorname{div}(\eta)^\sharp g, \omega\Big\rangle v_g,$$

where $\eta = f\mathrm{d}\varphi\cdot\tau(\varphi)$*.*

Proof. First observe that

$$\langle\tau(\varphi), \mathrm{d}\varphi(Z)\rangle = \langle\mathrm{d}\varphi\cdot\tau(\varphi), Z^\flat\rangle \qquad \text{for all } Z \in C(TM), \tag{5}$$

then $\eta(Z) = \langle\eta, Z^\flat\rangle$.

By the definition of ξ, we have

$$\begin{aligned}\int_M f\langle\tau(\varphi), \mathrm{d}\varphi(\xi)\rangle v_g &= \int_M f\langle\tau(\varphi), \mathrm{d}\varphi(\mathrm{div}\omega)^\sharp\rangle v_g \\ &\quad -\frac{1}{2}\int_M f\langle\tau(\varphi), \mathrm{d}\varphi(\operatorname{grad}(\operatorname{trace}_g\omega))\rangle v_g \\ &= \int_M \eta((\mathrm{div}\omega)^\sharp)v_g - \frac{1}{2}\int_M \eta(\operatorname{grad}(\operatorname{trace}_g\omega))v_g,\end{aligned} \tag{6}$$

so, the first term of the right-hand side of (6) becomes

$$\begin{aligned}\int_M \eta((\mathrm{div}\omega)^\sharp)v_g &= \int_M \langle\eta, \mathrm{div}\ (\omega)\rangle v_g \\ &= -\int_M \langle\mathrm{sym}\ \nabla\eta, \omega\rangle v_g,\end{aligned} \tag{7}$$

the second term of the right-hand side of (6) can then be written

$$\begin{aligned}&-\frac{1}{2}\int_M \eta(\operatorname{grad}(\operatorname{trace}_g\omega))v_g = -\frac{1}{2}\int_M\langle\eta, d(\mathrm{trace}_g\omega)\rangle v_g \\ &= -\frac{1}{2}\int_M\langle\eta^\sharp, \operatorname{grad}(\mathrm{trace}_g\omega)\rangle v_g = \frac{1}{2}\int_M \mathrm{trace}_g(\omega)\mathrm{div}(\eta^\sharp)v_g \\ &= \frac{1}{2}\int_M\langle\mathrm{div}(\eta^\sharp)g, \omega\rangle v_g.\end{aligned} \tag{8}$$

The lemma follows from (6), (7), and (8). □

This preparation is the key to theorem.

Theorem 2.3. *Let $\varphi\colon (M,g)\to(N,h)$ be a smooth map, M compact and orientable, $f\in\mathcal{C}^\infty(M)$ and $\{g_t\}$ a one-parameter variation of g through Riemannian metrics. Then*

$$\delta(F(g_t)) = -\frac{1}{2}\int_M \langle S_{2,f},\omega\rangle v_g,$$

where $S_{2,f}\in C(\odot^2T^*M)$ is the stress bi-energy tensor given by

$$S_{2,f} = fS_2 + \langle\tau(\varphi),\mathrm{d}\varphi(\operatorname{grad} f)\rangle g - 2df\odot(\mathrm{d}\varphi\cdot\tau(\varphi)) \tag{9}$$

where

$$S_2 = \frac{1}{2}|\tau(\varphi)|^2 + \langle\mathrm{d}\varphi,\nabla^\varphi\tau(\varphi)\rangle g - 2\nabla\tau(\varphi)\odot\mathrm{d}\varphi,$$

is the stress bi-energy tensor of φ.

Proof. Recall that $\delta(v_{g_t})=\langle\frac{1}{2}g,\omega\rangle v_g$ (see, for example, [**5**]). Then, by Lemma 2.1, we can rewrite (4)

$$\begin{aligned}\delta(F(v_{g_t})) = {} & -\int_M\langle f(\tau(\varphi)\cdot\nabla\mathrm{d}\varphi),\omega\rangle v_g - \int_M f\langle\tau(\varphi),\mathrm{d}\varphi(\xi)\rangle v_g\\ & +\frac{1}{2}\int_M\Big\langle\frac{1}{2}f|\tau_t(\varphi)|^2g,\omega\Big\rangle v_g.\end{aligned}\tag{10}$$

By the Lemma 2.2, we obtain

$$\begin{aligned}\delta(F(v_{g_t})) &= \int_M\langle\operatorname{sym}(\nabla\eta)-\frac{1}{2}\operatorname{div}(\eta^\sharp)g - f(\tau(\varphi)\cdot\nabla\mathrm{d}\varphi),\omega\rangle v_g\\ &\quad+\frac{1}{2}\int_M\Big\langle\frac{1}{2}f|\tau(\varphi)|^2g,\omega\Big\rangle v_g\\ &= \int_M\Big\langle\frac{1}{4}f|\tau(\varphi)|^2g - f(\tau(\varphi)\cdot\nabla\mathrm{d}\varphi)+\operatorname{sym}(\nabla\eta)-\frac{1}{2}\operatorname{div}(\eta^\sharp)g,\omega\Big\rangle v_g.\end{aligned}\tag{11}$$

In another hind, let $p\in M$ and $\{e_i\}_{i=1}^m$ a geodesic frame centered on p, then we get

$$\begin{aligned}\operatorname{div}(\eta^\sharp) &= \sum_{i=1}^m\langle\nabla^\varphi_{e_i}\eta^\sharp,e_i\rangle = \sum_{i=1}^m\langle(\nabla^\varphi_{e_i}\eta)^\sharp,e_i\rangle = \sum_{i=1}^m(\nabla^\varphi_{e_i}\eta)(e_i)\\ &= \sum_{i=1}^m[e_i(\eta(e_i)) - \eta(\nabla^M_{e_i}e_i)] = \sum_{i=1}^m[e_i\big(f\langle\tau(\varphi),\mathrm{d}\varphi(e_i)\big)\\ &= \sum_{i=1}^m[e_i(f)\langle\tau(\varphi),\mathrm{d}\varphi(e_i)\rangle + fe_i\langle\tau(\varphi),\mathrm{d}\varphi(e_i)\rangle\\ &= \langle\tau(\varphi),\mathrm{d}\varphi(\operatorname{grad} f)\rangle + \sum_{i=1}^m f[\langle\nabla^\varphi_{e_i}\tau(\varphi),\mathrm{d}\varphi(e_i)\rangle + \langle\tau(\varphi),\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_i)\rangle]\\ &= \langle\tau(\varphi),\mathrm{d}\varphi(\operatorname{grad} f)\rangle + f|\tau(\varphi)|^2 + f\langle\mathrm{d}\varphi,\nabla^\varphi\tau(\varphi)\rangle\end{aligned}\tag{12}$$

and for the expression $\mathrm{sym}(\nabla\eta)$,

$$
\begin{aligned}
\mathrm{sym}(\nabla\eta)(e_i,e_j) &= \frac{1}{2}[(\nabla\eta)(e_i,e_j)+(\nabla\eta)(e_j,e_i)]\\
&= \frac{1}{2}[e_i(\eta(e_j))-\eta(\nabla^M_{e_i}e_j)+e_j(\eta(e_i))-\eta(\nabla^M_{e_j}e_i)]\\
&= \frac{1}{2}[e_i(\eta(e_j))+e_j(\eta(e_i))]\\
&= \frac{1}{2}[e_i(f\langle\tau(\varphi),\mathrm{d}\varphi(e_j)\rangle)+e_j(f\langle\tau(\varphi),\mathrm{d}\varphi(e_i)\rangle)]\\
&= \frac{1}{2}[e_i(f)\langle\tau(\varphi),\mathrm{d}\varphi(e_j)\rangle+f\langle\nabla^{\varphi}_{e_i}\tau(\varphi),\mathrm{d}\varphi(e_j)\rangle\\
&\quad +f\langle\tau(\varphi),\nabla^{\varphi}_{e_i}\mathrm{d}\varphi(e_j)\rangle+e_j(f)\langle\tau(\varphi),\mathrm{d}\varphi(e_i)\rangle)\\
&\quad +f\langle\nabla^{\varphi}_{e_j}\tau(\varphi),\mathrm{d}\varphi(e_i)\rangle+f\langle\tau(\varphi),\nabla_{e_j}\mathrm{d}\varphi(e_i)\rangle]\\
&= (\mathrm{d}f\odot\mathrm{d}\varphi\cdot\tau(\varphi))(e_i,e_j)+f(\nabla^{\varphi}\tau(\varphi)\odot\mathrm{d}\varphi)(e_i,e_j)\\
&\quad +f(\tau(\varphi)\cdot\nabla^{\varphi}\mathrm{d}\varphi)(e_i,e_j).
\end{aligned}
\tag{13}
$$

By replacing (12) and (13) in (11), we have

$$
\begin{aligned}
\delta(F(v_{g_t})) &= \int_M \Big\langle -\frac{1}{4}f|\tau(\varphi)|^2 g - f(\tau(\varphi)\cdot\nabla\mathrm{d}\varphi)+\mathrm{d}f\odot\mathrm{d}\varphi\cdot\tau(\varphi)\\
&\quad + f\nabla\tau(\varphi)\odot\mathrm{d}\varphi-\frac{1}{2}\langle\tau(\varphi),\mathrm{d}\varphi(\mathrm{grad}\,f)\rangle g-\frac{1}{2}f\langle\mathrm{d}\varphi,\nabla^{\varphi}\tau(\varphi)\rangle g\\
&\quad + f(\tau(\varphi)\cdot\nabla\mathrm{d}\varphi),\omega\Big\rangle v_g\\
&= -\frac{1}{2}\int_M \Big\langle f\Big[\frac{1}{2}|\tau(\varphi)|^2+\langle\mathrm{d}\varphi,\nabla^{\varphi}\tau(\varphi)\rangle+\langle\tau(\varphi),\mathrm{d}\varphi(\mathrm{grad}f)\rangle\Big]g\\
&\quad -2\mathrm{d}f\odot\mathrm{d}\varphi\cdot\tau(\varphi)-2f\nabla\tau(\varphi)\odot\mathrm{d}\varphi,\omega\Big\rangle v_g.
\end{aligned}
\tag{14}
$$

Then

$$
\begin{aligned}
S_{2,f} &= \Big[\frac{1}{2}f|\tau(\varphi)|^2+f\langle\mathrm{d}\varphi,\nabla^{\varphi}\tau(\varphi)\rangle+\langle\tau(\varphi),\mathrm{d}\varphi(\mathrm{grad}f)\rangle\Big]g\\
&\quad -2\mathrm{d}f\odot\mathrm{d}\varphi\cdot\tau(\varphi)-2f\nabla\tau(\varphi)\odot\mathrm{d}\varphi\\
&= fS_2+\langle\tau(\varphi),\mathrm{d}\varphi(\mathrm{grad}\,f)\rangle g-2\mathrm{d}f\odot(\mathrm{d}\varphi\cdot\tau(\varphi)).
\end{aligned}
$$

□

Remarks.

1. If $\varphi\colon (M,g)\to(N,h)$ is a Riemannian immersion, then

$$S_{2,f}=fS_2.$$

2. If $f=1$, then

$$S_{2,f}=S_2=\Big[\frac{1}{2}|\tau(\varphi)|^2+\langle\mathrm{d}\varphi,\nabla^{\varphi}\tau(\varphi)\rangle\Big]g-2\mathrm{d}\varphi\odot\nabla\tau(\varphi).$$

Theorem 2.4. *Let $\varphi\colon (M,g) \to (N,h)$ be a smooth map, M compact and orientable, $f \in \mathcal{C}^\infty(M)$, then*

$$\operatorname{div} S_{2,f} = -\langle \tau_{2,f}(\varphi), \mathrm{d}\varphi\rangle - \frac{1}{2}|\tau(\varphi)|^2 \mathrm{d}f.$$

Proof. Write $S_{2,f} = T_1 + T_2 + T_3$, where $T_1, T_2, T_3 \in C(\odot^2 T^*M)$ are defined by

$$T_1(X,Y) = f S_2(X,Y),$$
$$T_2(X,Y) = \langle \tau(\varphi), \mathrm{d}\varphi(\operatorname{grad} f)\rangle g(X,Y),$$
$$T_3(X,Y) = -2\big(\mathrm{d}f \odot (\tau(\varphi)\cdot \mathrm{d}\varphi)\big)(X,Y).$$

Let $p \in M$ and $\{e_i\}_{i=1}^m$ a geodesic frame centered on p. We have

$$\begin{aligned}(\mathrm{div}T_1)(e_j) &= \sum_{i=1}^m e_i(T_1(e_i,e_j)) = \sum_{i=1}^m e_i[fS_2(e_i,e_j)]\\ &= \sum_{i=1}^m e_i(f)S_2(e_i,e_j) + f e_i S_2(e_i,e_j) = S_2(\mathrm{grad}f, e_j) + f(\mathrm{div}S_2)(e_j)\\ &= \frac{1}{2}|\tau(\varphi)|^2 \mathrm{d}f(e_j) + \langle \mathrm{d}\varphi, \nabla^\varphi \tau(\varphi)\rangle \mathrm{d}f(e_j) - \langle \nabla^\varphi_{e_j}\tau(\varphi), \mathrm{d}\varphi(\mathrm{grad}f)\rangle\\ &\quad - \langle \nabla_{\mathrm{grad}f}\tau(\varphi), \mathrm{d}\varphi(e_j)\rangle - f\langle \tau_2(\varphi), \mathrm{d}\varphi(e_j)\rangle,\end{aligned} \tag{15}$$

whilst

$$\begin{aligned}(\mathrm{div}T_2)(e_j) &= \sum_{i=1}^m e_i(T_2(e_i,e_j)) = \sum_{i=1}^m e_i\langle \tau(\varphi), \mathrm{d}\varphi(\operatorname{grad} f)\rangle \delta_i^j\\ &= e_j\langle \tau(\varphi), \mathrm{d}\varphi(\operatorname{grad} f)\rangle\\ &= \langle \nabla^\varphi_{e_j}\tau(\varphi), \mathrm{d}\varphi(\operatorname{grad} f)\rangle + \langle \tau(\varphi), \nabla_{e_j}\mathrm{d}\varphi(\operatorname{grad} f)\rangle\end{aligned} \tag{16}$$

and in the same way
(17)

$$\begin{aligned}(\mathrm{div}T_3)(e_j) &= \sum_{i=1}^m e_i(T_2(e_i,e_j)) = \sum_{i=1}^m e_i[-2\mathrm{d}f \odot \mathrm{d}\varphi.\tau(\varphi)(e_i,e_j)]\\ &= \sum_{i=1}^m e_i[-\mathrm{d}f(e_i)\langle \mathrm{d}\varphi(e_j), \tau(\varphi)\rangle - \mathrm{d}f(e_j)\langle \mathrm{d}\varphi(e_i), \tau(\varphi)\rangle]\\ &= \Delta(f)\langle \mathrm{d}\varphi(e_j), \tau(\varphi)\rangle - \langle \nabla^\varphi_{\operatorname{grad} f}\mathrm{d}\varphi(e_j), \tau(\varphi)\rangle - \langle \mathrm{d}\varphi(e_j), \nabla^\varphi_{\operatorname{grad} f}\tau(\varphi)\rangle\\ &\quad - \sum_{i=1}^m \mathrm{Hess}(f)(e_i,e_j)\langle \mathrm{d}\varphi(e_i), \tau(\varphi)\rangle - |\tau(\varphi)|^2 \mathrm{d}f(e_j) - \langle \mathrm{d}\varphi, \nabla^\varphi\tau(\varphi)\rangle \mathrm{d}f(e_j).\end{aligned}$$

Summing (15), (16), and (17) gives

$$\begin{aligned}
(\mathrm{div} S_{2,f})(e_j) &= -\frac{1}{2}|\tau(\varphi)|^2 \mathrm{d}f(e_j) - 2\langle \nabla_{\mathrm{grad} f}\tau(\varphi), \mathrm{d}\varphi(e_j)\rangle \\
&\quad - f\langle \tau_2(\varphi), \mathrm{d}\varphi(e_j)\rangle + \langle \tau(\varphi), \nabla_{e_j}\mathrm{d}\varphi(\mathrm{grad}\, f)\rangle \\
&\quad + \Delta(f)\langle \mathrm{d}\varphi(e_j), \tau(\varphi)\rangle - \langle \nabla^{\varphi}_{\mathrm{grad}\, f}\mathrm{d}\varphi(e_j), \tau(\varphi)\rangle \\
&\quad - \mathrm{trac}_g\, Hess(f)(\cdot, e_j)\langle \mathrm{d}\varphi(\cdot), \tau(\varphi)\rangle \\
&= -\langle f\tau_2(\varphi) - \Delta(f)\tau(\varphi) + 2\nabla^{\varphi}_{\mathrm{grad}\, f}\tau(\varphi), \mathrm{d}\varphi(e_j)\rangle \qquad (18)\\
&\quad - \frac{1}{2}|\tau(\varphi)|^2 \mathrm{d}f(e_j) + \langle \tau(\varphi), \nabla_{e_j}\mathrm{d}\varphi(\mathrm{grad}\, f)\rangle \\
&\quad - \langle \nabla^{\varphi}_{\mathrm{grad}\, f}\mathrm{d}\varphi(e_j), \tau(\varphi)\rangle - \mathrm{trac}_g\, \mathrm{Hess}(f)(\cdot, e_j)\langle \mathrm{d}\varphi(\cdot), \tau(\varphi)\rangle \\
&= -\langle \tau_{2,f}(\varphi), \mathrm{d}\varphi(e_j)\rangle - \frac{1}{2}|\tau(\varphi)|^2 \mathrm{d}f(e_j) + \langle \tau(\varphi), \nabla_{e_j}\mathrm{d}\varphi(\mathrm{grad}\, f)\rangle \\
&\quad - \langle \nabla^{\varphi}_{\mathrm{grad}\, f}\mathrm{d}\varphi(e_j), \tau(\varphi)\rangle - \mathrm{trac}_g\, \mathrm{Hess}(f)(\cdot, e_j)\langle \mathrm{d}\varphi(\cdot), \tau(\varphi)\rangle.
\end{aligned}$$

Now, we calculate the term

$$\begin{aligned}
&\langle \tau(\varphi), \nabla_{e_j}\mathrm{d}\varphi(\mathrm{grad}\, f)\rangle - \langle \nabla^{\varphi}_{\mathrm{grad}\, f}\mathrm{d}\varphi(e_j), \tau(\varphi)\rangle \\
&= \langle \nabla_{e_j}\mathrm{d}\varphi(\mathrm{grad}\, f) - \nabla^{\varphi}_{\mathrm{grad}\, f}\mathrm{d}\varphi(e_j), \tau(\varphi)\rangle \\
&= \langle \mathrm{d}\varphi(\nabla^M_{e_j}\,\mathrm{grad}\, f), \tau(\varphi)\rangle = \langle \mathrm{d}\varphi(\nabla^M_{e_j} e_k(f)e_k), \tau(\varphi)\rangle \qquad (19)\\
&= \langle \mathrm{d}\varphi\big(\mathrm{Hess}(f)(e_j, e_k)e_k\big), \tau(\varphi)\rangle = \mathrm{Hess}(f)\big(e_j, e_k\big)\langle \mathrm{d}\varphi(e_k), \tau(\varphi)\rangle \\
&= \mathrm{trace}\,\mathrm{Hess}(f)(e_j, \cdot)\langle \mathrm{d}\varphi(\cdot), \tau(\varphi)\rangle.
\end{aligned}$$

The theorem follows from (18) and (19). □

Remarks.
1) If $f = 1$, then $\mathrm{div} S_{2,f} = \mathrm{div} S_2$.
2) If f is f-bi-harmonic, then $\mathrm{div} S_{2,f} = -\frac{1}{2}|\tau(\varphi)|^2 \mathrm{d}f$.

3. Vanishing of the f-biharmonic stress-energy tensor

Clearly, from (9), harmonic implies $S_{2,f} = 0$, so it is only natural to study the converse. Not $F(g)$ is nonnegative and zero if and only if φ is harmonic. Thus our quest is for critical points ($S_{2,f} = 0$) which are minima. Before embarking on this problem, observe that ($S_{2,f} = 0$) does note en general, imply harmonicity, as illustrated by the non-geodesic curve $\gamma(t) = \mathrm{e}^t$ with $f(t) = \mathrm{e}^{-\frac{t}{2}}$. Yet, if we impose arc-length parametrization, we have the following proposition

Proposition 3.1. *Let $\gamma\colon I \subset \mathbb{R} \to (N, h)$ be a curve parameterized by arc-length, assume $S_{2,f} = 0$, then γ is geodesic.*

Proof. A direct compotation shows

$$\begin{aligned}
0 = S_{2,f}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) &= \frac{1}{2}f|\tau(\gamma)|^2 - f\Big\langle d\gamma\Big(\frac{\partial}{\partial t}\Big), \nabla_{\frac{\partial}{\partial t}}\tau(\gamma)\Big\rangle - f'\Big\langle \tau(\gamma), d\gamma\Big(\frac{\partial}{\partial t}\Big)\Big\rangle \\
&= \frac{3}{2}f|\tau(\gamma)|^2.
\end{aligned}$$

□

Proposition 3.2. *Let $\varphi\colon (M^2, g) \to (N, h)$ be a map from surface, and $f \in \mathcal{C}^\infty(M)$ positive, then*

$$S_{2,f} = 0 \quad \textit{implies} \quad \varphi \quad \textit{harmonic.}$$

Proof. Let $\{e_i\}_{i=1}^2$, an orthonormal frame around $p \in M$, the trace of $S_{2,f}$ give

$$\begin{aligned}
0 &= \Big[\frac{1}{2}f|\tau(\varphi)|^2 + f\langle \mathrm{d}\varphi, \nabla^\varphi \tau(\varphi)\rangle + \langle \tau(\varphi), \mathrm{d}\varphi(\mathrm{grad} f)\rangle\Big] \sum_{i=1}^{2} g(e_i, e_i) \\
&\quad - 2\sum_{i=1}^{2} \mathrm{d}f(e_i)\langle \mathrm{d}\varphi(e_i), \tau(\varphi)\rangle - 2f\sum_{i=1}^{2}\langle \nabla_{e_i}\tau(\varphi), \mathrm{d}\varphi(e_i)\rangle \\
&= f|\tau(\varphi)|^2 + 2\langle \mathrm{d}\varphi(\mathrm{grad}\, f), \tau(\varphi)\rangle + 2f\langle \mathrm{d}\varphi, \nabla^\varphi\tau(\varphi)\rangle \\
&\quad - 2\langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle - 2f\langle \mathrm{d}\varphi, \nabla^\varphi\tau(\varphi)\rangle \\
&= f|\tau(\varphi)|^2,
\end{aligned}$$

then $\tau(\varphi) = 0$, i.e, φ is harmonic. □

Proposition 3.3. *Let $\varphi\colon (M^m, g) \to (N, h)$ be a smooth map, with $m \neq 2$, and $f \in \mathcal{C}^\infty(M)$ positive, then $S_{2,f} = 0$ if and only if*

$$\frac{1}{2-m} f|\tau(\varphi)|^2 g - 2\mathrm{d}f \odot (\mathrm{d}\varphi.\tau(\varphi)) - 2f(\mathrm{d}\varphi \odot \nabla\tau(\varphi)) = 0 \tag{20}$$

Proof. Let $\{e_i\}_{i=1}^2$ be an orthonormal frame around $p \in M$. We have $S_{2,f} = 0$ implies $\mathrm{trace}\, S_{2,f} = 0$, so

$$\begin{aligned}
0 &= \Big[\frac{1}{2}f|\tau(\varphi)|^2 + f\langle \mathrm{d}\varphi, \nabla^\varphi \tau(\varphi)\rangle + \langle \tau(\varphi), \mathrm{d}\varphi(\mathrm{grad} f)\rangle\Big] \sum_{i=1}^{m} g(e_i, e_i) \\
&\quad - 2\sum_{i=1}^{m} \mathrm{d}f(e_i)\langle \mathrm{d}\varphi(e_i), \tau(\varphi)\rangle - 2f\sum_{i=1}^{m}\langle \nabla^\varphi_{e_i}\tau(\varphi), \mathrm{d}\varphi(e_i)\rangle \\
&= \frac{m}{2} f|\tau(\varphi)|^2 + m\langle \tau(\varphi), \mathrm{d}\varphi(\mathrm{grad} f)\rangle + mf\langle \mathrm{d}\varphi, \nabla^\varphi\tau(\varphi)\rangle \\
&\quad - 2\langle \tau(\varphi), \mathrm{d}\varphi(\mathrm{grad} f)\rangle - 2\langle \mathrm{d}\varphi, \nabla^\varphi\tau(\varphi)\rangle
\end{aligned}$$

and then

$$f\langle \mathrm{d}\varphi, \nabla^\varphi\tau(\varphi)\rangle = -\langle \tau(\varphi), \mathrm{d}\varphi(\mathrm{grad} f)\rangle - \frac{m}{2(m-2)} f|\tau(\varphi)|^2. \tag{21}$$

Substituting (21) into the definition of $S_{2,f}$, we obtain

$$\begin{aligned}
0 &= \frac{1}{2} f|\tau(\varphi)|^2 g - \frac{m}{2(m-2)} f|\tau(\varphi)|^2 g - 2\mathrm{d}f \odot (\mathrm{d}\varphi.\tau(\varphi)) - 2f\mathrm{d}\varphi \odot \nabla\tau(\varphi) \\
&= \frac{1}{2-m} f|\tau(\varphi)|^2 g - 2\mathrm{d}f \odot (\mathrm{d}\varphi.\tau(\varphi)) - 2f\mathrm{d}\varphi \odot \nabla\tau(\varphi).
\end{aligned}$$

The converse follows in the same way. □

Proposition 3.4. *A map $\varphi\colon (M^m, g) \to (N, h)$, $(m > 2)$, with $S_{2,f} = 0$ and rank $\varphi \leq m - 1$ is harmonic, where $f \in \mathcal{C}^\infty(M)$ positive.*

Proof. We take $p \in M$. Since $\operatorname{rank} \varphi(p) \leq m-1$, there exists a unit vector field $X_p \in \ker \mathrm{d}\varphi$. Then for $X = Y = X_p$, (20) becomes

$$\frac{1}{2-m} f|\tau(\varphi)|^2 g(X_p, X_p) - 2\mathrm{d}f(X_p)\langle \mathrm{d}\varphi(X_p), \tau(\varphi)\rangle - 2f\langle \mathrm{d}\varphi(X_p), \nabla^{\varphi}_{X_p}\tau(\varphi)\rangle = 0,$$

hence $\tau(\varphi) = 0$. □

Corollary 3.1. *Let* $\varphi\colon (M^m, g) \to (N, h^n)$ *be a submersion* $(m\rangle n)$. *If* $S_{2,f} = 0$, *then* φ *is harmonic.*

Theorem 3.1. *A map* $\varphi\colon (M^m, g) \to (N, h^n)$, $(m \neq 4)$, *with* $S_{2,f} = 0$, M *compact and orientable, is harmonic.*

Proof. We have $S_{2,f} = 0$, then

$$\begin{aligned} 0 = \mathrm{trace} S_{2,f} &= \frac{m}{2} f|\tau(\varphi)|^2 + (m-2)\langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle \\ &\quad + (m-2) f \mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle, \end{aligned} \tag{22}$$

and integrating over M, we get

$$\begin{aligned} 0 &= \frac{m}{2}\int_M f|\tau(\varphi)|^2 v_g + (m-2)\int_M \langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle v_g \\ &\quad + (m-2)\int_M f\mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle v_g. \end{aligned} \tag{23}$$

For the term $(m-2)\int_M f\mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle v_g$, we have

$$\begin{aligned} \int_M f\mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle v_g &= \int_M f\langle \mathrm{d}\varphi(e_i), \nabla_{e_i}\tau(\varphi)\rangle v_g \\ &= \int_M \big(f e_i\langle \mathrm{d}\varphi(e_i), \tau(\varphi)\rangle - f|\tau(\varphi)|^2\big) v_g \\ &= -\int_M f|\tau(\varphi)|^2) v_g + \int_M \nabla^{\varphi}_{e_i}\langle \mathrm{d}\varphi(e_i), f\tau(\varphi)\rangle v_g \\ &\quad - \int_M \langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle v_g. \end{aligned} \tag{24}$$

Consider a one form $\eta = \langle \mathrm{d}\varphi, f\tau(\varphi)\rangle$, then, as M is compact, we have

$$\int_M \nabla^{\varphi}_{e_i}\langle \mathrm{d}\varphi(e_i), f\tau(\varphi)\rangle v_g = \int_M \mathrm{div}(\eta) v_g = 0,$$

then, we can rewrite (23)

$$0 = \frac{4-m}{2}\int_M f|\tau(\varphi)|^2 v_g. \tag{25}$$

As f positive and $m \neq 4$, φ is harmonic. □

Proposition 3.5. *A Riemannian immersion* $\varphi\colon (M, g) \to (N, h)$ *with* $S_{2,f} = 0$, $m \neq 4$, *is minimal.*

Proof. φ is Riemannian immersion, then $\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle = 0$ and $S_{2,f} = 0$ implies

$$\begin{aligned}
0 &= \mathrm{trace} S_{2,f} \\
&= \frac{m}{2} f|\tau(\varphi)|^2 + (m-2) f \mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle \\
&= \frac{m}{2} f|\tau(\varphi)|^2 + (m-2) f \langle \nabla^{\varphi}_{e_i}\tau(\varphi), \mathrm{d}\varphi(e_i)\rangle \\
&= \frac{m}{2} f|\tau(\varphi)|^2 + (m-2) f e_i \langle \tau(\varphi), \mathrm{d}\varphi(e_i)\rangle - f(m-2)|\tau(\varphi)|^2 \\
&= \frac{m}{2} f|\tau(\varphi)|^2 - f(m-2)|\tau(\varphi)|^2 \\
&= \frac{4-m}{2} f|\tau(\varphi)|^2,
\end{aligned}$$

as f is positive and $m \neq 4$, φ is minimal. □

The next result introduces integral conditions ensuring that $S_{2,f} = 0$ reveals harmonicity. First we cite Yau's version of Stokes theorem.

Lemma 3.2 ([**8**]). *Let (M^m, g) be a complete Riemannian manifold and ω a smooth integrable $(m-1)$-form defined on M. Then there exists a sequence of domains B_i in M such that $M = \bigcup_i B_i$, $B_i \subset B_{i+1}$, and $lim_{i\to\infty} \int_{B_i} d\omega = 0$.*

Theorem 3.3. *Let (M^m, g), $m \neq 4$ be an orientable complete Riemannian manifold and $\varphi\colon (M^m, g) \to (N, h)$ a map with $S_{2,f} = 0$, where $f \in \mathcal{C}^\infty(M)$ positive function. If $\int_M f|(\mathrm{d}\varphi \cdot \tau(\varphi))^\sharp| v_g \langle \infty$ then φ is harmonic.*

Proof. For $m = 2$, this follows from Proposition (3.2), so now assume $m \neq 2$. We have

$$\begin{aligned}
\mathrm{trace} S_{2,f} = \frac{m}{2} f|\tau(\varphi)|^2 &+ (m-2)\langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle \\
&+ (m-2) f \mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle = 0,
\end{aligned}$$

then

$$f \mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle = \frac{-m}{2(m-2)} f|\tau(\varphi)|^2 - \langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle, \tag{26}$$

and from (12), we get

$$\mathrm{div}(f \mathrm{d}\varphi.\tau(\varphi))^\sharp = f|\tau(\varphi)|^2 + \langle \mathrm{d}\varphi(\mathrm{grad} f), \tau(\varphi)\rangle + f \mathrm{trace}\langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle, \tag{27}$$

replacing (26) in (27), we obtain

$$\mathrm{div}(f \mathrm{d}\varphi \cdot \tau(\varphi))^\sharp = \frac{m-4}{2(m-2)} f|\tau(\varphi)|^2,$$

and this results in

$$\frac{m-4}{2(m-2)} f|\tau(\varphi)|^2 v_g = \mathrm{div}(X) v_g = d(i_X v_g),$$

where $X = (f \mathrm{d}\varphi \cdot \tau(\varphi))^\sharp$. We now apply Lemma (3.2) to $\omega = i_X v_g$. To compute the norm of ω, choose $p \in M$ and a local normal chart $(U, x^k)_{k=1}^m$ around p

$$v_{g(p)} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m, \qquad (i_X(p) v_g)_{i\ldots\hat{k}\ldots m} = (-1)^{k+1}\xi^k.$$

So
$$|\omega|(p) = |i_X v_g|^2(p) = \sum_{i_1,\dots,i_m=1}^{m} (i_X(p)v_g)_{i_1,\dots,i_m} = (m-1)!|X|^2(p).$$
Now,
$$\int_M |X|v_g = \int_M |(f\mathrm{d}\varphi \cdot \tau(\varphi))^\sharp|v_g < \infty,$$
so ω is integrable. By Lemma (3.2), we get
$$\lim_{i\to\infty}\int_{B_i} d\omega v_g = \frac{m-4}{2(m-2)}\lim_{i\to\infty}\int_{B_i} f|\tau(\varphi)|^2 v_g = 0,$$
hence φ is harmonic. □

Corollary 3.2. *Let (M,g), $m \neq 4$, be an orientable complete Riemannian manifold and $\varphi\colon (M,g) \to (N,h)$ a map with finite energy and bienergy. If $S_{2,f} = 0$, then φ is harmonic.*

Theorem 3.4. *A non-minimal Riemannian immersion $\varphi\colon (M^4,g) \to (N,h)$ satisfies $S_{2,f} = 0$ if and only if it is pseudo-umbilical.*

Proof. First note that for a Riemannian immersion, $S_{2,f}(X,Y) = 0$ reduces to
$$\frac{1}{2}f|\tau(\varphi)|^2 g(X,Y) = 2f\langle\tau(\varphi), B(X,Y)\rangle, \tag{28}$$
$B = \nabla\mathrm{d}\varphi$ being its second fundamental form. Recall that a Riemannian immersion is pseudo-umbilical if and only if its shape operator A satisfies
$$A_{\tau(\varphi)} = \frac{1}{m}|\tau(\varphi)|^2 I,$$
equivalently,
$$\langle B(X,Y), \tau(\varphi)\rangle = \frac{1}{m}|\tau(\varphi)|^2 g(X,Y).$$
Comparing it with (28) ends the proof. □

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