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THE FAREY MAPS MODULO n

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ABSTRACT. The Farey map is the universal triangular map whose automorphism group is the classical modular group. We study the quotients of the Farey map by the principal congrience subgroups of the modular group. These include many well-known regular triangular maps. We also study the underlying graphs of these quotients.

1. Introduction

Let $\mathcal{U}^*$ denote the upper-half plane extended by adding the points $\mathbb{Q} \cup \{\infty\}$ to the upper half plane $\mathcal{U}$. On $\mathcal{U}^*$ we have the universal triangular map $\mathcal{M}_3$ which can be realised by the well-known Farey map as described below. These have as vertices the extended rationals $\mathbb{Q} \cup \{\infty\}$. The automorphism group of $\mathcal{M}_3$ is the classical modular group Γ=PSL(2, $\mathbb{Z}$). The most important subgroups of Γ are the principal congruence subgroups $\Gamma(n)$. Our aim in this paper is to discuss the maps (or clean dessin d'enfants) $\mathcal{M}_3/\Gamma(n)$ which lies on the Riemann surface $\mathcal{U}^*/\Gamma(n)$. These have as vertices rational numbers "modulo n". These were introduced in [**6**] and also discussed in [**10**].

The purpose of this paper is to show how some Riemann surfaces and Regular maps can be constructed using elementary concepts in number theory such as modular arithmetic. Compact Riemann surfaces are some of the most important objects in mathematics as they can be thought of as complex algebraic curves or complex manifolds or orbifolds. In recent years it has been shown that thy are of crucial importance to graph embeddings. The first author showed in [**11**] that associated to any graph embedding in a compact surface there is a natural complex structure that makes this surface into a Riemann surface. About the same time Grothendieck [**4**] observed that Belyi's theorem implies that the Riemann surfaces that admit these graph embeddings correspond to complex algebraic curves that are defined over the field of algebraic numbers. He christened these graph embeddings "dessins d'enfants", the idea being that even simple things such as a child's drawing may correspond to sophisticated mathematics.

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Outline of the paper. In Section 3 we describe the Farey map. In Section 4, we discuss Farey rationals mod n, which are the vertices of our maps. In Section 5 we summarize some well-known results about (2,m,n) groups. In Section 6 we describe the vertices, faces and edges of our maps. We also give the most important examples for low values of n. For example, when n=5 we get an icosahderon, and other triangular Platonic solids occur similarly. For n=7 we get the famous Klein map. For $n = 8$ we get the interesting Farey-Klein map of genus 5. Thus Farey maps mod n give us some of the most important regular maps and for this reason the general theory is well worth studying. In Section 7, we go over some previous work on Petrie Polygons just to show how the number theory nicely describes the geometry. In Sections 8 and 9 we find the stars of vertices, again showing a close connection between arithmetic and geometry. In Section 10 we introduce the poles. The centre is always a pole, otherwise they are extremal points of the map. This is demonstrated when we study the graph-theoretic distance. We show in Theorem 12 that the diameter of these maps is equal to 3, (except for small values of n) an easy result we find surprising as these maps can be arbitrarily large. Poles are always distance 3 apart.

2. The Farey map

Vertices of the Farey map (see Figure 1) are the extended rationals, i.e. $\mathbb{Q} \cup \{\infty\}$ and two rationals $\frac{a}{c}$ and $\frac{b}{d}$ are joined by an edge if and only if $ad - bc = \pm 1$. These edges are drawn as semicircles or vertical lines, perpendicular to the real axis, (i.e. hyperbolic lines). Here $\infty = \frac{1}{0}$. This map has the following properties.

(a) There is a triangle with vertices $\frac{1}{0}, \frac{1}{1}, \frac{0}{1}$, called the principal triangle.
(b) The modular group Γ= PSL(2, $\mathbb{Z}$) acts as a group of automorphisms of $\mathcal{M}_3$.
(c) The general triangle has vertices $\frac{a}{c}, \frac{a+b}{c+d}, \frac{b}{d}$.

This forms a triangular tessellation of the upper half plane. Note that the triangle in (c) is just the image of the principal triangle under the Möbius transformation corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In **[9]** it was shown that $\mathcal{M}_3$ is the universal triangular map in the sense that any triangular map on a surface is the quotient of $\mathcal{M}_3$ by a subgroup of the modular group Γ and any regular triangular map is the quotient of $\mathcal{M}_3$ by a normal subgroup of Γ. The aim of this paper is to study the maps $\mathcal{M}_3(n) = \mathcal{M}_3/\Gamma(n)$ where $\Gamma(n)$ is the principal congruence subgroup mod n of the Modular group Γ, see §5.

3. PSL(2, $\mathbb{Z}_n$) as a $(2, m, n)$ group

In this section we go over some well-known material which some may wish to ignore. A group G is called a $(2, m, n)$-group if is a homomorphic image of the $(2, m, n)$ triangle group $\Gamma(2, m, n)$. This is the group with generators (X, Y, Z) which obey the relations $X^2 = Y^m = Z^n = XYZ = 1$. If G is a finite $(2, m, n)$-group then

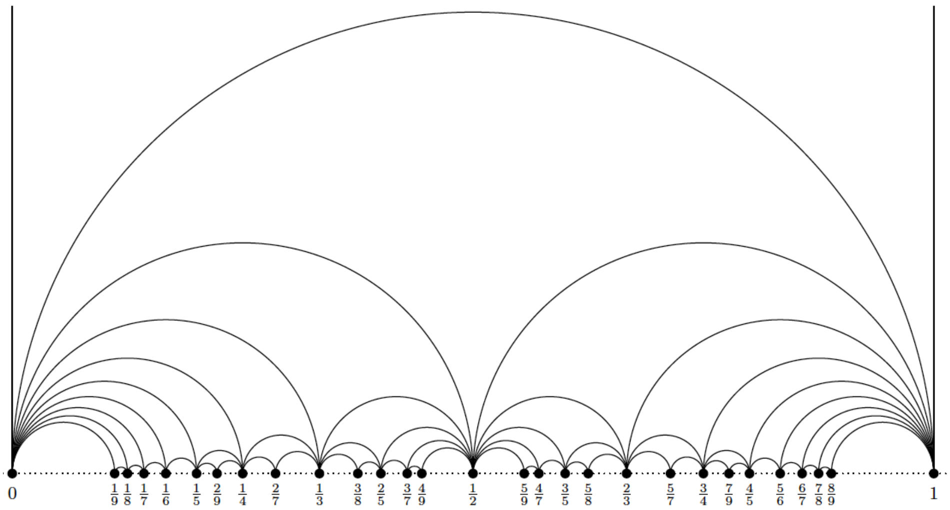

Figure 1. The Farey map (drawn by Jan Karabaš).

there is an epimorphism $\theta\colon \Gamma(2,m,n) \mapsto G$ with kernel M, say. If $\mathcal{U}$ is the upper-half complex plane then $\mathcal{U}/M$ is a compact Riemann surface that carries a map $\mathcal{M}$ of type (m,n). In [**7**], M is called a *map subgroup* for $\mathcal{M}$. If we are just interested in triangular maps then we only deal with $(2,3,n)$ triangle groups. If we are not particularly concerned with the vertex valencies then we are just looking at factor groups of the $(2,3,\infty)$ triangle group which means that we are not concerned with the order of the image of Z. Now the $(2,3,\infty)$ triangle group is isomorphic to the classical modular group Γ, which is defined by:

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, \quad ad-bc = \pm 1 \right\} / \{\pm I\}.$$

The definition of the principal congruence subgroup $\Gamma(n)$ is as follows

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}.$$

Now $\Gamma(n) \triangleleft \Gamma$ and the quotient group $\Gamma/\Gamma(n) = PSL(2,\mathbb{Z}_n)$. This group has the same generators as Γ except now the integers are taken modulo n and hence $X^2 = Y^3 = Z^n = XYZ = 1$ and thus $PSL(2,\mathbb{Z}_n)$ is a finite $(2,3,n)$ group. There is an epimorphism $\theta\colon \Gamma \to PSL(2,\mathbb{Z}_n)$. The kernel of θ is denoted by $\Gamma(n)$ and called the prinicipal congruence subgroup of Γ of level n. Now $\Gamma(n)$ is a map subgroup for a regular triangular map which we denote by $\mathcal{M}_3(n)$.

The vertices of these maps $\mathcal{M}_3(n)$ are the Farey rationals mod n. These rationals are of the form $\frac{a}{c}$ where $(a,c,n) = 1$, excluding $\frac{0}{0}$. We can think of these as fractions $\frac{a}{c}$ where now $a, c \in \mathbb{Z}_n$, (not both 0) and where we identify $\frac{a}{c}$ with $\frac{-a}{-c}$ and two vertices $\frac{a}{b}$ and $\frac{c}{d}$ are joined by an edge if and only if $ad - bc \equiv \pm 1 \pmod{n}$. Following are two examples for $n = 5$ and $n = 8$

Example 1. For $n = 5$, the vertices are:

$$\frac{1}{0}, \frac{2}{0}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}.$$

This map gives an icosahderon which is shown in figure 2.

As for $n = 8$, the vertices are:

$$\frac{1}{0}, \frac{3}{0}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \frac{7}{1}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{1}{4}, \frac{3}{4}.$$

Which gives a map on the Frickle-Klein surface which is shown in Figure 5 and discussed in Example 3. (Section 5)

4. The vital statistics of $\mathcal{M}_3(n)$

Every element of Γ is represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now Γ acts transitively on the darts of $\mathcal{M}_3$, where a *dart* is a directed edge. A dart of $\mathcal{M}_3$ is an ordered pair $(\frac{a}{c}, \frac{b}{d})$ and the matrix represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps the dart $(\frac{1}{0}, \frac{0}{1})$ to $(\frac{a}{c}, \frac{b}{d})$ and so $G = \Gamma/\Gamma(n)$ acts transitvely on the darts of $\mathcal{M}_3/\Gamma(n)$. By definition, a map is regular if its automorphism group acts transitively on its darts which makes $\mathcal{M}_3$ a regular map and for similar reasons so is $\mathcal{M}_3(n)$. Note that $\frac{1}{0}$ is joined to $\frac{k}{1}$ in $\mathcal{M}_3(n)$ for $k = 0, 1, 2, \ldots, \frac{n-1}{1}$, so that $\frac{1}{0}$ has valency n and hence by regularity, every vertex has valency n. We want to find the numbers of darts, edges, faces and vertices of $\mathcal{M}_3(n)$. Being a regular map, for $\mathcal{M}_3(n)$, we have:

- The number of darts $= |G|$
- The number of edges $= |G|/2$
- The number of faces $= |G|/3$
- Every vertex has valency n
- The number of vertices $= |G|/n$

For $n > 2, \Gamma(n)$ is a normal subgroup of Γ of index

$$\mu(n) = \frac{n^3}{2}\Pi_{p|n}\Big(1 - \frac{1}{p^2}\Big). \tag{1}$$

For $n = 2$ the index is equal to 6. For this see any text on modular functions, e.g. **[2]**. Thus $|G| = \mu(n)$ is the number of darts so that the number of edges, vertices and faces is $\mu(n)/2$, $\mu(n)/n$ and $\mu(n)/3$. If $g(n)$ is the genus of the map $\mathcal{M}_3(n)$ then the Euler characteristic is given by

$$2 - 2g(n) = \mu(n)\Big(\frac{1}{n} - \frac{1}{2} + \frac{1}{3}\Big) = \mu(n)\frac{(6-n)}{6n}$$

from which we deduce the following formula for the genus of $\mathcal{M}_3(n)$

$$g(n) = 1 + \frac{n^2}{24}(n-6)\Pi_{p|n}\Big(1 - \frac{1}{p^2}\Big)$$

which is the well-known formula for the genus of $\Gamma(n)$, i.e. the genus of the surface $\mathcal{U}/\Gamma(n)$ which carries the map $\mathcal{M}_3(n)$.

We can also find the number of vertices using group theory. An important subgroup of Γ is

$$\Gamma_1(n) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod n \Big\}$$

where $0 \leq b < n$. As the stabilizer of ∞ in Γ is just the infinite cyclic group generated by $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ it easily follows that the stabilizer of $[\infty]_\Gamma(n) \in \mathcal{M}_3(n)$ is equal to $\Gamma_1(n)/\Gamma(n)$. (See [**6**, Lemma 3.2]). By the Orbit-Stabilizer Theorem there exists a one-to one correspondence between the left cosets of $\Gamma_1(n)$ in Γ and the vertices of $\mathcal{M}_3(n)$.

Now there is a epimomorphism $\chi\colon \Gamma_1(n) \mapsto \mathbb{Z}_n$ defined by

$$\chi\Big(\pm \begin{pmatrix} an+1 & b \\ cn & dn+1 \end{pmatrix} \Big) \mapsto b \pmod{n}.$$

The kernel of χ is $\Gamma(n)$. For $n > 2$, $\Gamma_1(n)$ is a normal subgroup of Γ of index

$$\frac{n^2}{2}\Pi_{p|n}\Big(1 - \frac{1}{p^2}\Big) \tag{2}$$

and the index of $\Gamma(2)$ in Γ is equal to 6, Hence the number of vertices of $\mathcal{M}_3(n)$ is given by (2) if $n > 2$ and the number of vertices of $\mathcal{M}_3(2)$ is equal to 3.

Example 2. The maps $\mathcal{M}_3(n)$ were described in [**6**] for $n = 2, \dots, 7$. For $n = 2$ we get a triangle, for $n = 3$ we get a tetrahedron, for $n = 4$ we get an octahedron and for $n = 5$ we get an icosahedron. All these are planar maps and so the corresponding Riemann surface is the Riemann sphere. For $n = 6$ we get the torus map $\{3,6\}_{2,2}$ and the corresponding Riemann surface is the hexagonal torus and for $n = 7$ we get the Klein map which lies on Klein's Riemann surface of genus 3, known as the Klein quartic [**8**]. See Figures 2, 3, 4 in the appendix.

Example 3. $n = 8$ We find that $\mu(8) = 192$, so that the number of edges of $\mathcal{M}_3(8)$ is equal to 96, the number of faces is 64 and the number of vertices is 24 and so the genus of $\mathcal{M}_3(8)$ is equal to 5. The underlying Riemann surface is the unique Riemann surface of genus 5 with 192 automorphisms and this is known as the *Fricke-Klein surface of genus* 5. This map is particularly important as it is the regular map underlying the *Grunbaum polyhedron* which is one of the few regular maps that are known to admit a polyhedral embedding into Euclidean 3-space with convex faces [**3**]. For the uniqueness of this map and Riemann surface we refer to [**1**]. This map and Riemann surface also appears in [**5**, Figure 2]. See Figure 5 in the appendix.

5. Petrie paths in $\mathcal{M}_3(n)$

A *Petrie path* in a (regular) map is a zig-zag path on the map. This is a path in which two consecutive edges but no three consecutive edges can belong to the same face. If this is a closed path then it is called a *Petrie polygon* and the number of edges of this polygon is called the *Petrie length* of the map.

In **[10]** it was shown that the kth vertex of the Petrie path of the universal map is $\frac{f_{k-1}}{f_k}$ where f_k is the kth element of the Fibonacci sequence given by $f_0 = 1$, $f_1 = 0$, $f_{k+1} = f_k + f_{k-1}$. For $\mathcal{M}_3(n)$ we get exactly the same results except now the integers f_i are taken modulo n. As an example we give the result from **[10]**. The Petrie polygon in $\mathcal{M}_3(7)$ is $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{1}$, $\frac{1}{6}$, $\frac{6}{0} = \frac{-1}{0}$ so we have closed the path which in this case has length equal to 8. In general, we find that the length of the Petrie polygon on $\mathcal{M}_3(n)$ is equal to $\sigma(n)$ where $\sigma(n)$ is the *semiperiod* of the Fibonacci sequence mod n. This means the period of the Fibonacci sequence up to sign, so for $n = 7$ we get 1,0,1,1,2,3,5,1,-1, 0,-1,-1.. so $\sigma(7) = 8$. (This was one of the main topics of **[10]**.) As another example, let us find the Petrie path in $\mathcal{M}_3(8)$. It is easily seen that that it has length $\sigma(8) = 12$. See Figure 5 in the appendix noting that the Petrie paths are drawn in red.

6. The star of a vertex

In a graph G the star of a vertex x consists of all vertices of G that are joined to x by an edge, including x itself. Now let $\frac{a}{c}$ be a vertex of $\mathcal{M}_3(n)$, where $(a, c, n) = 1$.

Lemma 4. *There exists $b, d \in \mathbb{Z}$ such that $ad - bc \equiv 1 \pmod{n}$.*

Proof. Suppose that $(a, c) = k$. Then $(k, n) = 1$ so there exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha k + \beta n = 1$. Now there exists $u, v \in \mathbb{Z}$ such that $ua + vc = k$ so that $\alpha(ua + vc) + \beta n = 1$ and hence $(\alpha u)a + (\alpha v)c + \beta n = 1$. □

Theorem 5. *The star of $\frac{a}{c}$ consists of $\frac{a}{c}$ together with all vertices of the form $\frac{ak+b}{ck+d}$ where $k = 0, 1, \dots, n-1$ and b, d are as in* Lemma 4.

Proof. First we find the star of $\frac{1}{0}$. This consists of $\frac{1}{0}$ together with the vertices $\frac{0}{1}, \frac{1}{1}, \dots, \frac{n-1}{1}$. Here $\frac{a}{c} = \frac{1}{0}$ so $\frac{b}{d} = \frac{0}{1}$ so $a = d = 1$ and $b = c = 0$ so $\frac{b+ak}{d+ck} = \frac{k}{1}$, $k = 0, 1, \dots, n-1$ as required. More generally, let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $T(\frac{1}{0}) = \frac{a}{c}$. The stabilizer of $\frac{1}{0}$ is the cyclic group generated by U so the stabilizer of $\frac{a}{c}$ consists of elements of the form by TU^kT^{-1}. Now

$$S := TU^kT^{-1} = \begin{pmatrix} 1 - akc & a^2k \\ -c^2k & 1 + akc \end{pmatrix}$$

Therefore if $\frac{a}{c} \longleftrightarrow \frac{b}{d}$ then $S(\frac{a}{c}) \longleftrightarrow S(\frac{b}{d})$ so that $\frac{a}{c} \longleftrightarrow S(\frac{b}{d})$. Now

$$S\begin{pmatrix} b \\ \overline{d} \end{pmatrix} = \begin{pmatrix} 1-akc & a^2 \\ -c^2k & 1+akc \end{pmatrix}\begin{pmatrix} b \\ \overline{d} \end{pmatrix} = \frac{b-abck+a^2kd}{-bc^2k+d+acdk}$$

On using $ad = 1 + bc$ this becomes

$$\frac{ak+b}{ck+d}. \tag{3}$$

This is true for $k = 0, 1, \ldots, (n-1)$ so that these are the n points in the star of $\frac{a}{c}$. $\square$

Thus the numerators form an arithmetic progression of length n whose first term is b and common difference is a and the denominators form an arithmetic progression of length n whose first term is d and common difference is c.

Therefore to find the star of $\frac{a}{c}$ we find $b, d \in \mathbb{Z}$ such that $ad - bc \equiv 1 \pmod{n}$, construct the unimodular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then calculate the star as is given by (3).

Example 6. Find the star of $\frac{3}{5}$ in $\mathcal{M}_3(7)$. Our unimodular matrix is now $\begin{pmatrix} 3 & 0 \\ 5 & 5 \end{pmatrix}$ so that the numerators form an arithmetic progression whose first term is 0 and common difference 3 and whose denominators form an arithmetic progression whose first term is 5 and whose common difference is 5. Thus the star of $\frac{3}{5}$ in $\mathcal{M}_3(7)$ is $\frac{3}{5}$ together with

$$\left\{\frac{0}{5}, \frac{3}{3}, \frac{6}{1}, \frac{2}{6}, \frac{5}{4}, \frac{1}{2}, \frac{4}{0}\right\}.$$

See Figure 4 noting that $\frac{3}{5} = \frac{4}{2}$.

Example 7. Find the star of $\frac{1}{2}$ in $\mathcal{M}_3(8)$. Our unimodular matrix is now $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ so that the numerators form an arithmetic progression whose first term is 2 and whose common difference is 1 and the denominators form an arithmetic progression whose first term is 3 and whose common difference is 2. Thus the star of $\frac{1}{2}$ in $\mathcal{M}_3(8)$ is $\frac{1}{2}$ together with

$$\left\{\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{1}, \frac{6}{3}, \frac{7}{5}, \frac{0}{7}, \frac{1}{1}\right\}.$$

See Figure 5 in the appendix, noting that $\frac{3}{5} = \frac{5}{3}$, $\frac{4}{7} = \frac{4}{1}$ and $\frac{7}{5} = \frac{1}{3}$. Note that these give the cyclic orderings of the stars around our two points $\frac{3}{5}$ and $\frac{1}{2}$. This is because the star of $\frac{1}{0}$ is $\frac{0}{1}, \frac{1}{1}, \frac{2}{1} \ldots$.

Note. The vertices of the star of $\frac{a}{c}$ form a polygonal face whose centre is $\frac{a}{c}$.

7. The stars of $\mathcal{M}_3(p)$, p an odd prime

Theorem 8. *For a prime, p, the stars of $\frac{1}{0}, \frac{2}{0}, \ldots, \frac{(p-1)/2}{0}$ are disjoint and cover $\mathcal{M}_3(p)$.*

Proof. Consider the vertex $\frac{k}{0}$. Let K be the inverse of k mod p. Then the star of $\frac{k}{0}$ is $\frac{k}{0}$ together with

$$\left\{\frac{0}{K}, \frac{1}{K}, \ldots, \frac{(p-1)}{K}\right\}.$$

Consider a distinct vertex $\frac{l}{0}$, so that $l \neq \pm k$. Let L be the inverse of l mod p. Then $L \neq \pm K$ so that the star of $\frac{l}{0}$ is $\frac{l}{0}$ together with

$$\left\{\frac{0}{L}, \frac{1}{L}, \ldots, \frac{(p-1)}{L}\right\}.$$

As $L \neq \pm K$ these stars are disjoint. There are $(p-1)/2)$ stars each containing p vertices. Thus there are $p(p-1)/2$ vertices which is the total number of vertices $\mathcal{M}_3(p)$. $\square$

Example 9. $\mathcal{M}_3(5)$. We find the stars of $\frac{1}{0}$ and $\frac{2}{0}$. The star of $\frac{1}{0}$ $\frac{1}{0}$ together with is $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}\}$. The star of $\frac{2}{0}$ is $\frac{2}{0}$ together with $\{\frac{0}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3}, \frac{3}{3}\}$. Thus the stars of $\frac{1}{0}$ and $\frac{2}{0}$ give us 12 vertices which is the number of vertices of $\mathcal{M}_3(5)$,

8. Poles of Farey maps

From the diagrams in the appendix we see that the points of the form $\frac{a}{0}$ play a significant role. We call these the *poles* of $\mathcal{M}_3(n)$. As $\frac{x}{y} = \frac{-x}{-y}$ we see that for $n > 2$ the number of poles of $\mathcal{M}_3(n)$ is equal to $\phi(n)/2$.

8.1. Graphical distance

If u, v are two vertices in a graph then the distance $\delta(u, v)$ between them is defined to be the length of the shortest path joining u and v, The *diameter* of a graph or map is the maximum distance between two of its points.

Theorem 10. *The distance between two distinct poles of $\mathcal{M}_3(n)$ is equal to* 3.

Proof. By regularity we may assume that one of the poles is $\frac{1}{0}$. This is not adjacent to $\frac{a}{0}$. There is no path of length 2 between $\frac{1}{0}$ and $\frac{a}{0}$, where $a \neq \pm 1$, for otherwise there would be $x, y \in \mathbb{Z}$ such that $\frac{1}{0} \to \frac{x}{y} \to \frac{a}{0}$. Then $y = \pm 1$ and then $a = \pm 1$, a contradiction. However, we can always construct a path of length 3 of the form

$$\frac{1}{0} \to \frac{x}{1} \to \frac{xa^{-1}+1}{a^{-1}} \to \frac{a}{0}.$$

where a^{-1} is the inverse of a modulo n. This inverse exists because $(a, 0, n) = 1$ and hence $(a, n) = 1$. $\square$

Lemma 11. *Let a, c, n be integers such that $(a, c, n) = 1$. Then there exists an integer k so that $(a + ck, n) = 1$.*

Proof. Let $p_1, p_2, \ldots, p_r$ be a list without repetition of those prime divisors of n that are not divisors of a or c. Define $k = p_1 \cdot \ldots \cdot p_r$, the product of the p_i.

Now suppose that q is a prime divisor of n. If $q = p_i$ for some i then q is not a divisor of a so is not a divisor of $a + ck$.

If q is not equal to p_i for any i, then it is coprime to k and is a divisor of one, but not both of a and c. If q were a divisor of both a and c then $(a, c, n) = q \neq 1$. If q is a divisor of a, then it cannot divide $a + ck$, as q is coprime to ck. If q is a divisor of c, then q does not divide a and so does not divide $a + ck$. Hence $a + ck$ and n are coprime. □

Theorem 12. *The diameter of $\mathcal{M}_3(n)$ is equal to 3 for all $n \geq 5$.*

Proof. By transitivity we can assume one of the points to be $\frac{1}{0}$ and denote the other by $\frac{a}{c}$. Now the neighbours of $\frac{1}{0}$ are precisely those of the form $\frac{k}{1}$ for $k = 0, \ldots, n-1$. This reduces to finding an $\frac{x}{y} \in \mathcal{M}_3(n)$ and a $k, 0 \leq k \leq (n-1)$ such that such that $\frac{k}{1} \to \frac{x}{y} \to \frac{a}{c}$.

In terms of congruences this means that we have two simultaneous congruences

$$\begin{aligned} ky - x &\equiv 1 \mod (n) \\ cx - ay &\equiv 1 \mod (n) \\ \Longrightarrow\ y(ck - a) &\equiv c + 1 \mod (n) \end{aligned}$$

By Lemma 11 we know that we can find k such that $(ck - a)$ is co-prime to n and therefore has a inverse mod n. Hence the last congruence can be solved for y which, in turn, determines x. □

For $n = 2, 3, 4$ the diameters are $1, 1, 2$ respectively. For $n = 2$ this corresponds to the triangle and for $n = 3, 4$ this is the tetrahedron and octahedron respectively which can be observed in Figure 2.

Theorem 13. *Let $\frac{b}{d} \in \mathcal{M}_3(n)$ where $d \neq \pm 1$. Then $\delta(\frac{1}{0}, \frac{b}{d}) = 2$ if and only if $\gcd(d, n) | (b \pm 1)$.*

Proof. By our supposition $\frac{1}{0}$ is not adjacent to $\frac{b}{d}$ therefore assume that there exist a vertex $\frac{x}{y}$ such that $\frac{1}{0} \to \frac{x}{y} \to \frac{b}{d}$. Now for $\frac{x}{y}$ to be adjacent to $\frac{1}{0}$ we must have $y = \pm 1$. Setting $y = 1$ and looking at $\frac{x}{y} \to \frac{b}{d}$ we arrive at the equation:

$$dx - b \equiv \pm 1 \mod (n)$$

which has solutions if and only if $\gcd(d, n) | (b \pm 1)$. □

To see if any two points, $\frac{a}{c}$, $\frac{x}{y}$, are distance two apart let T be the transformation where $T(\frac{a}{c}) = \frac{1}{0}$. Apply this transformation to the other vertex, $T(\frac{x}{y}) = \frac{b}{d}$, and then apply Theorem 13.

Corollary 14. *Let $\frac{b}{d}$ in $\mathcal{M}_3(p)$, where p is a prime. Then $\delta(\frac{1}{0}, \frac{b}{d}) = 2$ if and only if $d \neq 0, \pm 1$.*

Proof. If $d \neq 0, \pm 1$, then the proof follows from Theorem 12.
If $\delta(\frac{1}{0}, \frac{b}{d}) = 2$ then $d \neq 0, \pm 1$. For if $d = 0$ and $\frac{b}{d} \neq \frac{1}{0}$ then $\frac{b}{d}$ is a distinct pole so that the distance δ is equal to 3 by Theorem 10. If $d = \pm 1$ then the distance δ is equal to 1 by definition. □

Theorem 15. *Given two points, $\frac{a}{c}, \frac{b}{d}$ in $\mathcal{M}_3(p)$, where p is prime, then $\delta\left(\frac{a}{c}, \frac{b}{d}\right) = 3$ if and only if $\Delta = 0$, where $\Delta = ad - bc$.*

Proof. ($\Rightarrow$) Apply the transformation $T \in \Gamma$ such that $T(\frac{a}{c}) = \frac{1}{0}$ and $T(\frac{b}{d}) = \frac{e}{f}$. By definition $\frac{1}{0}$ and $\frac{e}{f}$. are adjacent if and only if $f = \pm 1$. By Corollary 14, the distance would be 2 if and only if $f \neq 0, \pm 1$. From Theorem 12 the maximum distance is 3 which therefore can only be the case when $f = 0$. The determinant of these two points, $\frac{1}{0}$ and $\frac{e}{0}$, is equal to 0. As T preserves the determinants of points we see that the determinants of the original points is also equal zero.

($\Leftarrow$) If $\Delta = 0$ then we can find a $T \in \Gamma$ such that $T\left(\frac{a}{c}, \frac{b}{d}\right) = \left(\frac{x}{0}, \frac{y}{0}\right)$. Let $r = c, s = -a$. Then $ra + sc = 0$. Also $(r, s, p) = 1$ so there exist $u, q \in \mathbb{Z}$ such that $us - qr = 1 \mod (p)$ so let $T = \begin{pmatrix} u & q \\ r & s \end{pmatrix}$ Then $T(\frac{a}{c}) = \frac{ua+qc}{0}, T(\frac{b}{d}) = \frac{ub+qd}{0}$. Hence $\frac{a}{c}$ and $\frac{b}{d}$ are in the orbit of poles and therefore, from Theorem 10, have $\delta(\frac{a}{c}, \frac{b}{d}) = 3$. □

Therefore we can now completely categorize the distances in $\mathcal{M}_3(p)$ when p is prime

Theorem 16. *Let $\frac{a}{c}, \frac{b}{d}$ be distinct Farey fractions in $\mathcal{M}_3(p)$, where p is prime, and let $\Delta = ad - bc$. Then:*

$$\delta\left(\frac{a}{c}, \frac{b}{d}\right) = \begin{cases} 1 & \textit{if and only if } |\Delta| = 1, \\ 3 & \textit{if and only if } \Delta = 0, \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. (1) Follows from definition and (2) follows from Theorem 15. For part (3) we know from Theorem 12 that the maximum distance between two distinct points is 3. But in the first two parts when have shown the conditions for distance 1 and 3 are $\Delta = \pm 1$ and $\Delta = 0$. The only remaining option for points with $\Delta \neq 0, \pm 1$ is to be distance 2 apart. □

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Appendix: Pictures of $\mathcal{M}_3(n)$

For $n = 3, \ldots, 8$.

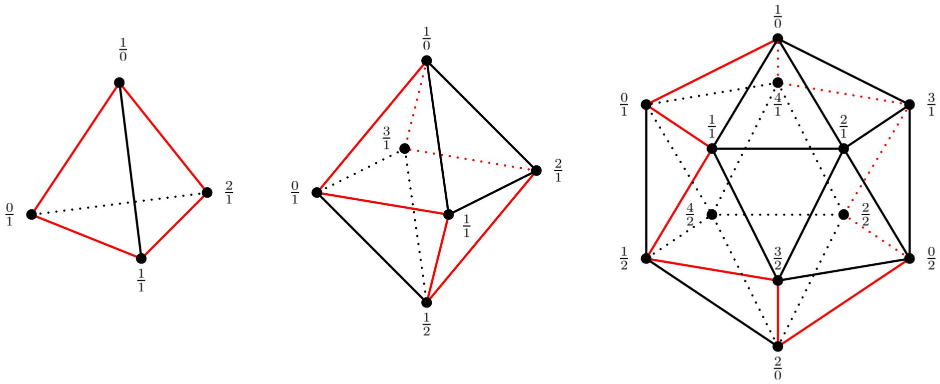

Figure 2. $\mathcal{M}_3(n)$ for $n = 3, 4, 5$.

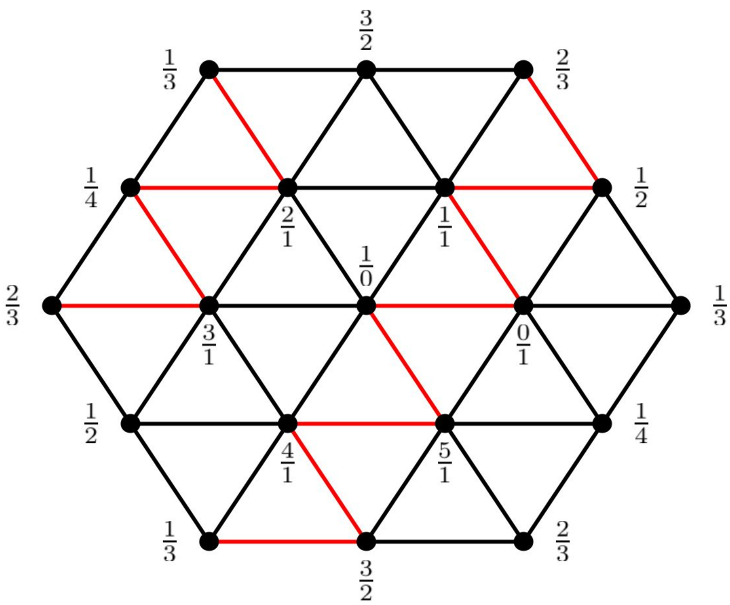

Figure 3. $\mathcal{M}_3(6)$.

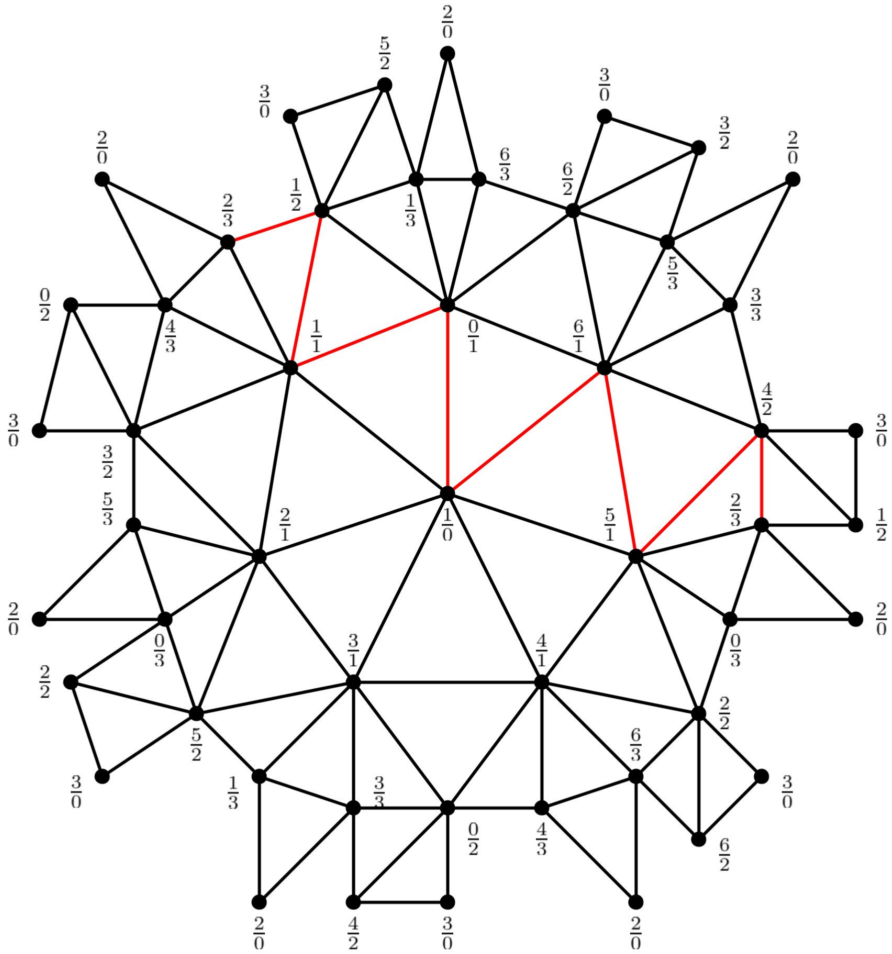

Figure 4. $\mathcal{M}_3(7)$ Klein's Riemann surface of genus 3.

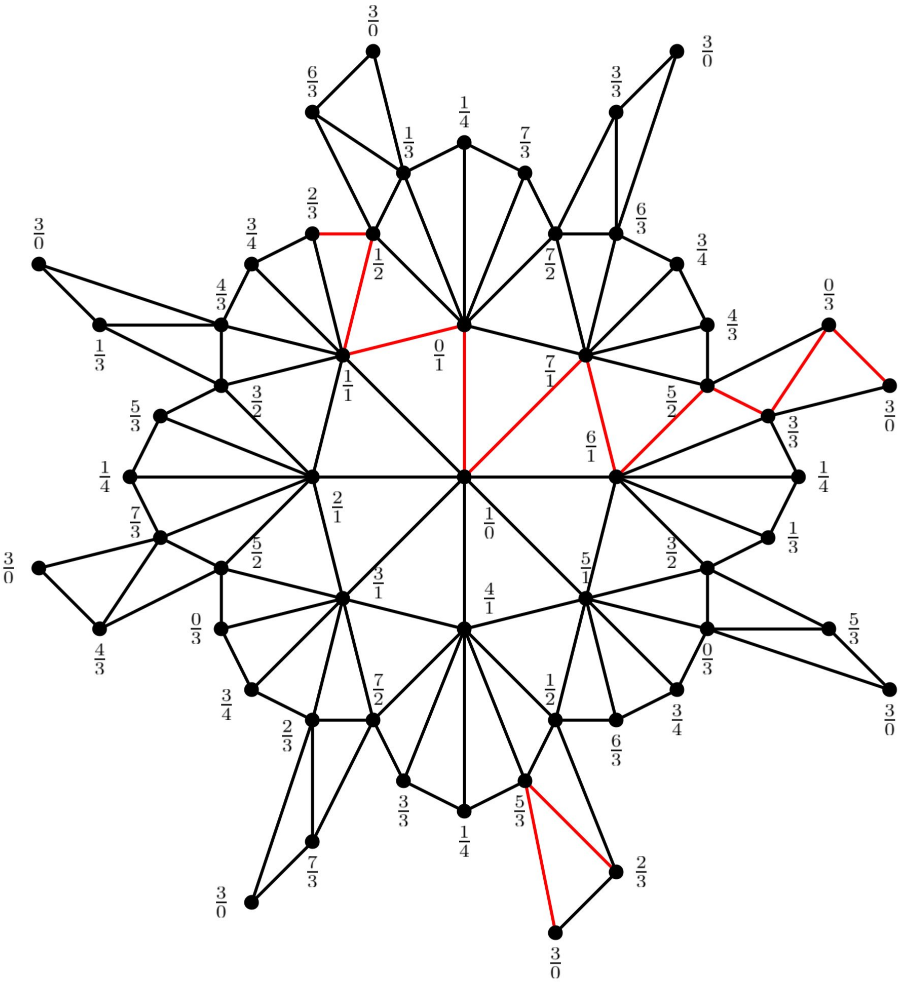

Figure 5. $\mathcal{M}_3(8)$ Fricke-Klein surface of genus 5. (Grünbaum polytope.).

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