Acta Math. Univ. Comenianae
Vol. LXXXIII, 2 (2014), pp. 267–279

CAUCHY TYPE RESULTS CONCERNING LOCATION OF ZEROS OF POLYNOMIALS

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Abstract. Let $p(z)$ be a polynomial with complex coefficients. In this paper, we obtain some new results concerning the location of zeros of polynomials $p(z)$. Our results sharpen Cauchy's result, along with some of the other known results, which are based on the classical Cauchy's work. Finally, we prove the results concerning the bounds for the number of zeros for the polynomial $p(z)$, which generalize some known results.

1. Introduction and statement of results

Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n, then according to a classical result by Cauchy [**8, 9**], the polynomial $f(z)$ has all its zeros in $|z| \leq 1+M$, where

$$M = \max|\frac{a_j}{a_n}|, \qquad j = 0, 1, 2, \ldots, n-1. \tag{1.1}$$

Also, if $f(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n with real coefficients satisfying

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0, \tag{1.2}$$

then according to the famous result due to Eneström-Kakeya [**8, 9**], the polynomial $f(z)$ has all its zeros in $|z| \leq 1$.

Joyal, Labelle and Rahman [**7**] extended the Eneström-Kakeya theorem to the polynomials whose coefficients are monotonic but not necessarily non-negative, and proved the following theorem.

Theorem A. *If $f(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0, \tag{1.3}$$

then the polynomial $f(z)$ has all its zeros in

$$|z| \leq \frac{1}{|a_n|}\left\{a_n - a_0 + |a_0|\right\}. \tag{1.4}$$

Aziz and Zargar [**2**] generalized Theorem A and proved the next theorem.

Received July 16, 2013; revised December 26, 2013.
2010 *Mathematics Subject Classification.* Primary 30C10, 30C15.
Key words and phrases. Cauchy theorem; Eneström-Kakeya theorem; zeros.

Theorem B. *If $f(z)=\sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n with real coefficients such that for some $\lambda \geq 1$,*

$$\lambda a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0, \tag{1.5}$$

then $f(z)$ has all its zeros in the disk

$$|z+\lambda-1| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}. \tag{1.6}$$

A related result from Govil and Rahman [**6**] concerns a restriction on the moduli and arguments of coefficients and proves the following theorem.

Theorem C. *If $f(z)=\sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n with complex coefficients such that*

$$|\arg a_k - \beta| \leq \alpha \leq \frac{\pi}{2}, \qquad k=0,1,2,\ldots,n \tag{1.7}$$

for some real β, and

$$|a_n| \geqslant |a_{n-1}| \geqslant \cdots \geqslant |a_1| \geqslant |a_0|, \tag{1.8}$$

then $f(z)$ has all its zeros in

$$|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|. \tag{1.9}$$

Also, Aziz and Qayoom [**1**] used a finite set of complex numbers and got a strip in complex plane included zeros of polynomials. In fact, they proved the following theorem.

Theorem D. *Let $p(z)=\sum_{i=0}^{n} a_i z^i$ be a non-constant complex polynomial of degree n. If $\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ is any set of n real or complex numbers such that*

$$\sum_{i=1}^{n}|\lambda_i| \leq 1,$$

then all the zeros of $p(z)$ lie in the annulus

$$R=\{z\in\mathbb{C}: r_1 \leq |z| \leq r_2\}, \tag{1.10}$$

where

$$r_1=\min_{1\leq k\leq n}\left|\lambda_k\frac{a_0}{a_k}\right|^{\frac{1}{k}} \qquad \text{and} \qquad r_2=\max_{1\leq k\leq n}\left|\frac{1}{\lambda_k}\frac{a_{n-k}}{a_k}\right|^{\frac{1}{k}}. \tag{1.11}$$

Recently M. Dehmer investigated two classes of bounds for the zeros of complex polynomials, namely explicit and implicit zeros bounds [**3, 4, 5**]. By using special classes of polynomials, he showed that his results might be suitable and optimal from classical Cauchy's result. In fact, he proved the following results.

Theorem E. *Let $f(z)=\sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n. If for any $p>1$, $q>1$,*

$$\frac{1}{p}+\frac{1}{q}=1,$$

then all the zeros of $f(z)$ *lie in the closed disk* $K\left(0,\ \left(1+M^q n^{\frac{q}{p}}\right)^{\frac{1}{q}}\right)$, *where*

$$M=\max\left|\frac{a_j}{a_n}\right|,\qquad j=0,1,2,\dots,n-1. \tag{1.12}$$

Theorem F. *Let* $f(z)=\sum_{i=0}^{n}a_iz^i$ *be a complex polynomial of degree* n, *then all the zeros of* $f(z)$ *lie in the closed disk* $K(0,1+\widetilde{M})$, *where*

$$\widetilde{M}=\max_{0\le j\le n}\left|\frac{a_{n-j}-a_{n-j-1}}{a_n}\right|,\qquad a_{-1}=0. \tag{1.13}$$

Theorem G. *Let* $f(z)=\sum_{i=0}^{n}a_iz^i$, $a_na_{n-1}\neq 0$ *be a complex polynomial. All zeros of* $f(z)$ *lie in the closed disk*

$$K\left(0,\frac{1+\phi_2}{2}+\frac{\sqrt{(\phi_2-1)^2+4M_1}}{2}\right), \tag{1.14}$$

where

$$M_1:=\max_{0\le j\le n-2}\left|\frac{a_j}{a_n}\right|,\qquad \phi_2:=\left|\frac{a_{n-1}}{a_n}\right|. \tag{1.15}$$

Also, concerning the number of zeros of a polynomial in the given region, we have the following results due to Shah and Liman [**10**].

Theorem H. *If* $p(z)=\sum_{i=0}^{n}a_iz^i$ *is a complex polynomial satisfying*

$$\sum_{i=1}^{n}|a_i|<|a_0|, \tag{1.16}$$

then $p(z)$ *does not vanish in* $|z|<1$.

Theorem I. *If* $p(z)=\sum_{i=0}^{n}a_iz^i$ *is a complex polynomial satisfying*

$$\sum_{i=0}^{n-1}|a_i|<|a_n|, \tag{1.17}$$

then $p(z)$ *has all its zeros in* $|z|<1$.

In this paper, first we prove the following theorem without any restrictions on the coefficients of a polynomial which include not only Cauchy's theorem and Eneström-Kakeya theorem simultaneously but also some other well-known results.

Theorem 1. *Let* $f(z)=\sum_{i=0}^{n}a_iz^i$ *be a complex polynomial of degree* n. *If for any* $p>1$, $q>1$,

$$\frac{1}{p}+\frac{1}{q}=1,$$

then all the zeros of $f(z)$ *lie in the closed disk* $K\left(0,\ \left(1+A_p^q\right)^{\frac{1}{q}}\right)$, *where* $A_p=\min_{-1\le i\le n}\{A_{p,i}\}$,

$$A_{p,i}=\left\{\sum_{j=0}^{n}\left|\frac{a_ia_{n-j}-a_na_{n-j-1}}{a_n^2}\right|^p\right\}^{\frac{1}{p}},\qquad a_{-1}=0,\ -1\le i\le n. \tag{1.18}$$

If A_p is obtained for $i = -1$, Theorem 1 reduces to a result by Tôya ([**12**], see also [**8**, Theorem 27.4, pp. 124]). Also, for $p = q = 2$, Theorem 1 gives the bound investigated by Carmichael and Mason (for reference see [**9**, pp. 247]). Furthermore, by taking

$$M = \max \left| \frac{a_j}{a_n} \right|, \qquad j = 0, 1, 2, \cdots, n-1, \tag{1.19}$$

we get

$$A_{p,-1} = \left\{ \sum_{j=0}^{n} \left| \frac{a_{n-j-1}}{a_n} \right|^p \right\}^{\frac{1}{p}} \le M n^{\frac{1}{p}}, \qquad a_{-1} = 0. \tag{1.20}$$

Therefore the bound obtained in Theorem 1 is better than that one in Theorem E due to Dehmer. Finally, if A_p is obtained for $i = n$, then we get the following result.

Corollary 1. *Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n. If for any $p > 1$, $q > 1$,*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then all the zeros of $f(z)$ lie in the disk $K\left(0, \left(1 + A_{p,n}^q\right)^{\frac{1}{q}}\right)$, where

$$A_{p,n} = \left\{ \sum_{j=0}^{n} \left| \frac{a_{n-j} - a_{n-j-1}}{a_n} \right|^p \right\}^{\frac{1}{p}}, \qquad a_{-1} = 0. \tag{1.21}$$

Remark 1. For $p = q = 2$, Corollary 1 reduces to a result of Williams ([**13**], see also [**8**, pp. 126]). If $p \to \infty$ in Corollary 1, then it reduces to Theorem F.
Also, by letting $p \to \infty$ in Theorem 1, we have $q = 1$ and

$$\lim_{p \longrightarrow \infty} A_{p,i} = M_i,$$

where

$$M_i = \max_{0 \le j \le n} \left| \frac{a_i a_{n-j} - a_n a_{n-j-1}}{a_n^2} \right|, \qquad a_{-1} = 0,$$

so we get the following result.

Corollary 2. *Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n. Then all the zeros of $f(z)$ lie in the $K\,(0, 1 + M)$, where $M = \min_{-1 \le i \le n}\{M_i\}$.*

Remark 2. It is clear that Corollary 2 is an improvement of Theorem F and Cauchy's theorem, when M is obtained for $i \ne n, -1$. For example, if we consider the polynomial $f(z) = z^3 + 0.1z^2 + 0.3z + 0.7$, then by Cauchy's theorem and Theorem F of Dehmer, it has all the zeros in the closed disk $K(0, 1.7)$, but by Corollary 2, it has all the zeros in the closed disk $K(0, 1.49)$.

Since

$$\lim_{q \to \infty} \left(1 + A_{p,i}^q\right)^{\frac{1}{q}} = \left\{ \begin{array}{cl} 1 & \text{if } A_{1,i} \le 1 \\ A_{1,i} & \text{if } A_{1,i} > 1 \end{array} \right., \tag{1.22}$$

hence, we get the following interesting result.

Corollary 3. *Let* $f(z)=\sum_{i=0}^{n} a_i z^i$ *be a complex polynomial of degree* n*. Then all the zeros of* $f(z)$ *lie in the* $K(0,R)$ *where*

$$R=\min_{-1\leq i\leq n} R_i,$$

$$R_i=\begin{cases} 1 & \text{if } A_{1,i}\leq 1 \\ A_{1,i} & \text{if } A_{1,i}>1 \end{cases} \tag{1.23}$$

and

$$A_{1,i}=\left\{\sum_{j=0}^{n}\left|\frac{a_i a_{n-j}-a_n a_{n-j-1}}{a_n^2}\right|\right\}, \qquad a_{-1}=0. \tag{1.24}$$

Remark 3. Under assumption in Theorem C and by using the inequality in [**6**]

$$|a_k-a_{k-1}|\leq(|a_k|-|a_{k-1}|)\cos\alpha+(|a_k|+|a_{k-1}|)\sin\alpha,$$

we conclude that

$$\begin{aligned} R\leq R_n&=\frac{1}{|a_n|}\left\{\sum_{j=0}^{n-1}|a_{n-j}-a_{n-j-1}|+|a_0|\right\} \\ &\leq\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|. \end{aligned} \tag{1.25}$$

Consequently, Corollary 3 is an improvement of Theorem C. In some cases, our result is significantly better than that one of Theorem C. We can illustrate this by the following examples.

Example 1.

i) For the polynomial $f_1(z)=\mathrm{i}\,z^3+z^2+\mathrm{i}\,z+1$, Theorem C (with $\beta=\alpha=\frac{\pi}{4}$) results in fact that $f_1(z)$ has all its zeros in $|z|\leq 5\sqrt{2}\approx 7.07$. While our result shows that all the zeros of $f_1(z)$ lie in $|z|\leq 3$.

ii) For the polynomial $f_2(z)=iz^3+z^2-\mathrm{i}\,z+1$, Theorem C (with $\beta=0$, $\alpha=\frac{\pi}{2}$) results in fact that $f_2(z)$ has all its zeros in $|z|\leq 9$. While our result shows that all the zeros of $f_2(z)$ lie in $|z|\leq 3$.

iii) For the polynomial $f_3(z)=-\mathrm{i}\,z^3-z^2+\mathrm{i}\,z+1$, Theorem C is not applicable. While our result shows that all the zeros of $f_3(z)$ lie in $|z|\leq 1$.

Furthermore, in general, the comparison between zeros bound of Theorem G and Theorem 1 is not possible. But in some cases, our result is better than that one of Theorem G. We can illustrate this by the following examples.

Example 2.

i) For the polynomial $f_4(z)=100z^9+100z^8+100z^7+100z^6+100z^5+100z^4+100z^3+100z^2+1$, Theorem G results in fact that $f_4(z)$ has all zeros in $K(0,2)$. While Theorem 1 for $p=1.00002$, $q=50001$, shows that all the zeros of $f_4(z)$ lie in $|z|\leq 1$.

ii) For the polynomial $f_5(z) = 20z^3 + 20z^2 + 19z + 19$, Theorem G results in fact that $f_5(z)$ has all zeros in $K(0, 2)$. While for $p = q = 2$, Theorem 1 shows that all the zeros of $f_5(z)$ lie in $K(0, 1.3)$.

iii) For the polynomial $f_6(z) = z^3 + 0.1z^2 + 0.3z + 0.7$, Theorem G results in fact that $f_6(z)$ has all zeros in $K(0, 1.5)$. While for $p = q = 2$, Theorem 1 shows that all the zeros of $f_6(z)$ lie in $K(0, 1.2)$.

The following theorem gives the lower bound for the zeros of polynomials.

Theorem 2. *Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n. If for any $p > 1$, $q > 1$,*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then all the zeros of $f(z)$ lie outside the disk $K\left(0, \frac{1}{(1+B_p^q)^{\frac{1}{q}}}\right)$, where

$$B_p = \min_{-1 \le i \le n} \{B_{p,i}\},$$

$$B_{p,i} = \left\{ \sum_{j=0}^{n} \left| \frac{a_i a_j - a_0 a_{j+1}}{a_0^2} \right|^p \right\}^{\frac{1}{p}}, \qquad a_{-1} = a_{n+1} = 0, \ -1 \le i \le n. \tag{1.26}$$

Many interesting results can be deduced from Theorem 2 in exactly the same way as we have done from Theorem 1. Since

$$\lim_{q \to \infty} \left(1 + B_{p,i}^q\right)^{\frac{1}{q}} = \begin{cases} 1 & \text{if } B_{1,i} \le 1 \\ B_{1,i} & \text{if } B_{1,i} > 1 \end{cases}, \tag{1.27}$$

hence, we get the following interesting result.

Corollary 4. *Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n, then all the zeros of $f(z)$ lie outside the disk $K\left(0, \frac{1}{r}\right)$, where*

$$r = \min_{-1 \le i \le n} r_i,$$

$$r_i = \begin{cases} 1 & \text{if } B_{1,i} \le 1 \\ B_{1,i} & \text{if } B_{1,i} > 1 \end{cases} \tag{1.28}$$

and

$$B_{1,i} = \sum_{j=0}^{n} \left| \frac{a_i a_j - a_0 a_{j+1}}{a_0^2} \right|, \qquad a_{-1} = a_{n+1} = 0, \ -1 \le i \le n. \tag{1.29}$$

Remark 4. If

$$|a_0| \ge |a_1| \ge \cdots \ge |a_n|, \tag{1.30}$$

similar to (1.25), we have

$$r \le r_0 = \frac{1}{|a_0|} \left\{ \sum_{j=0}^{n-1} |a_j - a_{j+1}| + |a_n| \right\} \le \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|} \sum_{k=1}^{n} |a_k|. \tag{1.31}$$

Therefore Corollary 4 is an improvement of a result of Govil [**6**].

Next, we prove the result which generalizes Eneström-Kakeya theorem.

Theorem 3. *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$, *be a non-constant complex polynomial. If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *such that for some* $\lambda \geq 1$ *and* $t \geq 1$,

$$\begin{aligned} &\lambda\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0, \\ &t\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \end{aligned} \tag{1.32}$$

then all the zeros of $f(z)$ *lie in*

$$\left| z + \frac{(\lambda-1)\alpha_n + (t-1)\beta_n \, \mathrm{i}}{a_n} \right| \leq \frac{\lambda\alpha_n + t\beta_n + |\alpha_0| + |\beta_0| - \alpha_0 - \beta_0}{|a_n|}. \tag{1.33}$$

If we take $t = 1$ in Theorem 3, then we get the following result.

Corollary 5. *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ *be a non-constant complex polynomial. If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *such that for some* $\lambda \geq 1$,

$$\begin{aligned} &\lambda\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0, \\ &\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \end{aligned} \tag{1.34}$$

then all the zeros of $f(z)$ *lie in*

$$\left| z + \frac{(\lambda-1)\alpha_n}{a_n} \right| \leq \frac{\lambda\alpha_n + \beta_n + |\alpha_0| + |\beta_0| - \alpha_0 - \beta_0}{|a_n|}. \tag{1.35}$$

For $\beta_0 > 0$, Corollary 5 reduces to the result of Shah and Liman [**11**, Theorem 2]. If we take $\lambda = 1$ in Theorem 3, then we get the following result.

Corollary 6. *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ *be a non-constant complex polynomial. If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *such that for some* $t \geq 1$,

$$\begin{aligned} &\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0, \\ &t\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \end{aligned} \tag{1.36}$$

then all the zeros of $f(z)$ *lie in*

$$\left| z - \frac{(t-1)\beta_n \, \mathrm{i}}{a_n} \right| \leq \frac{\alpha_n + t\beta_n + |\alpha_0| + |\beta_0| - \alpha_0 - \beta_0}{|a_n|}. \tag{1.37}$$

Also if we take $\lambda = t$ in Theorem 3, then we get the following result.

Corollary 7. *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ *be a non-constant complex polynomial. If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, *such that for some* $\lambda \geq 1$,

$$\begin{aligned} &\lambda\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0, \\ &\lambda\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \end{aligned} \tag{1.38}$$

then all the zeros of $f(z)$ lie in

$$\left|z+(\lambda-1)\frac{\overline{a_n}}{a_n}\right| \le \frac{\lambda a_n+|\alpha_0|+|\beta_0|-\alpha_0-\beta_0}{|a_n|}. \tag{1.39}$$

Remark 5. If $\beta_j=0$ for $j=0,1,2,\dots,n$, then Corollary 7 reduces to Theorem B due to Aziz and Zargar [**2**]. If $\lambda=1$ and $\beta_j=0$ for $j=0,1,2,\dots,n$, then Corollary 7 reduces to Theorem A due to Joyal [**7**]. And if $\lambda=1,\alpha_0>0$ and $\beta_j=0$ for $j=0,1,2,\dots,n$, then Corollary 7 reduces to Enesrtöm–Kakeya theorem.

Finally, we prove the following generalization of Theorems H and I .

Theorem 4. *Let $p(z)=a_0+\sum_{i=\mu}^{n}a_iz^i$ be a complex polynomial of degree n. If*

$$R^{n-k}\sum_{i=0,i\neq j\in A}^{n}|a_i|<|a_k|, \tag{1.40}$$

for some k with $a_k\neq 0$ and $R\ge 1$, where $A=\{1,2,\dots,\mu-1,k\}$, then $p(z)$ has exact k zeros in $|z|<R$.

Remark 6. If $k=0$ and $R=1$, then Theorem 4 reduces to Theorem H. Also, for $k=n$ and $R=1$, Theorem 4 reduces to Theorem I.

If $\mu=k=R=1$, then we have the following result.

Corollary 8. *If $p(z)=\sum_{i=0}^{n}a_iz^i$ is a polynomial of degree n and*

$$\sum_{i=0,i\neq 1}^{n}|a_i|<|a_1|, \tag{1.41}$$

then $p(z)$ has exact one zeros in $|z|<1$.

Remark 7. If we define

$$\lambda_1=\frac{a_0}{a_1},\qquad \lambda_2=\frac{a_2}{a_1},\qquad \lambda_3=\frac{a_3}{a_1},\qquad \dots\quad ,\lambda_n=\frac{a_n}{a_1}, \tag{1.42}$$

then we have

$$\sum_{i=1}^{n}|\lambda_i|=\frac{1}{|a_1|}\sum_{i=0,i\neq 1}^{n}|a_i|<1 \qquad (by\ (1.41)). \tag{1.43}$$

So, by applying Theorem D, all the zeros of $p(z)$ lie in annulus $\{z\in\mathbb{C}: r_3\le|z|\le r_4\}$, where

$$r_3=\min_{1\le k\le n}\left|\lambda_k\frac{a_0}{a_k}\right|^{\frac{1}{k}},\qquad r_4=\max_{1\le k\le n}\left|\frac{1}{\lambda_k}\frac{a_{n-k}}{a_n}\right|^{\frac{1}{k}}. \tag{1.44}$$

By substituting λ_i in r_3, we have

$$\begin{aligned} r_3 &= \min\Big\{\Big|\frac{a_0}{a_1}\cdot\frac{a_0}{a_1}\Big|, \Big|\frac{a_2}{a_1}\cdot\frac{a_0}{a_2}\Big|^{\frac{1}{2}}, \dots, \Big|\frac{a_n}{a_1}\cdot\frac{a_0}{a_n}\Big|^{\frac{1}{n}}\Big\} \\ &= \min\Big\{\Big|\frac{a_0}{a_1}\Big|^2, \Big|\frac{a_0}{a_1}\Big|^{\frac{1}{2}}, \dots, \Big|\frac{a_0}{a_1}\Big|^{\frac{1}{n}}\Big\} \\ &= \Big|\frac{a_0}{a_1}\Big|^2. \end{aligned} \tag{1.45}$$

Therefore, we conclude the following result.

Corollary 9. *If $p(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n and*

$$\sum_{i=0, i\neq 1}^{n} |a_i| < |a_1|, \tag{1.46}$$

then, $p(z)$ does not vanish in $|z| < |\frac{a_0}{a_1}|^2$.

If $\mu = k \geq 1$ in Theorem 4, then we have the following result.

Corollary 10. *Let*

$$p(z) = a_0 + \sum_{i=\mu}^{n} a_i z^i, \tag{1.47}$$

be a complex polynomial of degree n. If

$$R^{n-\mu} \sum_{i=0,\ i\neq j\in B}^{n} |a_i| < |a_\mu| \tag{1.48}$$

where $B = \{1, 2, \dots, \mu - 1\}$, then $p(z)$ has exact μ zeros in $|z| < R$.

Remark 8. If we define

$$\begin{aligned} &\lambda_1 = \frac{a_0}{R^{n-\mu}a_\mu}, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_\mu = 0, \quad \lambda_{\mu+1} = \frac{a_{\mu+1}}{R^{n-\mu}a_\mu}, \\ &\quad \dots, \quad \lambda_n = \frac{a_n}{R^{n-\mu}a_\mu}, \end{aligned} \tag{1.49}$$

then by Theorem D, all the zeros of $p(z)$ lie in annulus $\{z \in \mathbb{C} : r_5 \leq |z| \leq r_6\}$, where

$$\begin{aligned} r_5 &= \min_{1\leq k\leq n} \Big|\lambda_k \frac{a_0}{a_k}\Big|^{\frac{1}{k}} \\ &= \min\Big\{\Big|\frac{a_0}{R^{n-\mu}a_\mu}\cdot\frac{a_0}{a_1}\Big|, \Big|\frac{a_{\mu+1}}{R^{n-\mu}a_\mu}\cdot\frac{a_0}{a_{\mu+1}}\Big|^{\frac{1}{\mu+1}}, \dots, \Big|\frac{a_n}{R^{n-\mu}a_\mu}\cdot\frac{a_0}{a_n}\Big|^{\frac{1}{n}}\Big\} \\ &= \min\Big\{\Big|\frac{a_0}{R^{n-\mu}a_\mu}.\frac{a_0}{a_1}\Big|, \Big|\frac{a_0}{R^{n-\mu}a_\mu}\Big|^{\frac{1}{\mu+1}}, \dots, \Big|\frac{a_o}{R^{n-\mu}a_\mu}\Big|^{\frac{1}{n}}\Big\} \\ &= \min\{\Big|\frac{a_0^2}{R^{n-\mu}a_1}\Big|, \Big|\frac{a_0}{R^{n-\mu}a_\mu}\Big|^{\frac{1}{\mu+1}}\}. \end{aligned} \tag{1.50}$$

Hence, we get the following result.

Corollary 11. *If the condition of Corollary 9 holds, then* $p(z)$ *does not vanish in* $\{z \in \mathbb{C} : |z| < r_5\}$, *where*

$$r_5 = \min\{|\frac{a_0^2}{R^{n-\mu}a_1}|, |\frac{a_0}{R^{n-\mu}a_\mu}|^{\frac{1}{\mu+1}}\}.$$

2. Proofs of the Theorems

Proof of Theorem 1. For the zeros with $|z| \leq 1$, we have nothing to prove. Assuming $|z| > 1$ and defining $q(z) = (a_i - a_n z)f(z)$, we obtain

$$\begin{aligned} q(z) = &- a_n^2 z^{n+1} + (a_i a_n - a_n a_{n-1})z^n + (a_i a_{n-1} - a_n a_{n-2})z^{n-1} \\ &+ \cdots + (a_i a_1 - a_n a_0)z + a_i a_0. \end{aligned} \tag{2.1}$$

Or

$$\begin{aligned} |q(z)| &\geq |a_n^2 z^{n+1}| - \{|a_i a_n - a_n a_{n-1}||z|^n + |a_i a_{n-1} - a_n a_{n-2}||z|^{n-1} \\ &\quad + \cdots + |a_i a_1 - a_n a_0||z| + |a_i a_0|\} \\ &= |a_n^2||z|^{n+1}\left[1 - \sum_{j=0}^{n}\left|\frac{a_i a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|\frac{1}{|z|^{j+1}}\right]. \end{aligned} \tag{2.2}$$

By using Holder's inequality for $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have for $|z| > 1$,

$$\begin{aligned} |q(z)| &\geq |a_n^2||z|^{n+1}\left(1 - \left(\sum_{j=0}^{n}\left|\frac{a_i a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{n}\frac{1}{|z|^{q(j+1)}}\right)^{\frac{1}{q}}\right) \\ &= |a_n^2||z|^{n+1}\left(1 - A_{p,i}\left(\sum_{j=0}^{n}\frac{1}{|z|^{q(j+1)}}\right)^{\frac{1}{q}}\right) \\ &> |a_n^2||z|^{n+1}\left(1 - A_{p,i}\left(\sum_{j=0}^{\infty}\frac{1}{|z|^{q(j+1)}}\right)^{\frac{1}{q}}\right) \\ &= |a_n^2||z|^{n+1}\left(1 - A_{p,i}\frac{1}{(|z|^q - 1)^{\frac{1}{q}}}\right) > 0 \end{aligned} \tag{2.3}$$

if $|z| > \left(1 + A_{p,i}^q\right)^{\frac{1}{q}}$.

Therefore, $|q(z)| > 0$ if $|z| > \left(1 + A_{p,i}^q\right)^{\frac{1}{q}}$. This shows that all the zeros of $q(z)$ and hence, those of $f(z)$ lie in the closed disk $K\left(0, \left(1 + A_p^q\right)^{\frac{1}{q}}\right)$, where $A_p = \min_{-1 \leq i \leq n}\{A_{p,i}\}$. □

Proof of Theorem 2. Consider the polynomial $g(z) = z^n f(1/z)$. For the proof of this theorem, it is sufficient that $g(z)$ has all its zeros in the closed disk $K\left(0, \left(1 + B_p^q\right)^{\frac{1}{q}}\right)$, where $B_p = \min_{-1 \leq i \leq n}\{B_{p,i}\}$.

For the zeros with $|z| \leq 1$, we have nothing to prove. We assume that $|z| > 1$ and define

$$\begin{aligned} q(z) &= (a_i - a_0 z)g(z) \\ &= -a_0^2 z^{n+1} + (a_i a_0 - a_0 a_1)z^n + (a_i a_1 - a_0 a_2)z^{n-1} \\ &\quad + \cdots + (a_i a_{n-1} - a_0 a_n)z + a_i a_n. \end{aligned} \tag{2.4}$$

Then we have

$$\begin{aligned} |q(z)| &\geq |a_0^2 z^{n+1}| - \{|a_i a_0 - a_0 a_1||z|^n + |a_i a_1 - a_0 a_2||z|^{n-1} \\ &\quad + \cdots + |a_i a_{n-1} - a_0 a_n||z| + |a_i a_n|\} \\ &= |a_0^2||z|^{n+1}\left(1 - \sum_{j=0}^{n} \left|\frac{a_i a_j - a_0 a_{j+1}}{a_0^2}\right| \frac{1}{|z|^{j+1}}\right). \end{aligned} \tag{2.5}$$

By using Holder's inequality for $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, for $|z| > 1$, we have

$$\begin{aligned} |q(z)| &\geq |a_0^2||z|^{n+1}\left(1 - \left(\sum_{j=0}^{n}\left|\frac{a_i a_j - a_0 a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{n} \frac{1}{|z|^{q(j+1)}}\right)^{\frac{1}{q}}\right) \\ &= |a_0^2||z|^{n+1}\left(1 - B_{p,i}\left(\sum_{j=0}^{n} \frac{1}{|z|^{q(j+1)}}\right)^{\frac{1}{q}}\right) \\ &> |a_0^2||z|^{n+1}\left(1 - B_{p,i}\left(\sum_{j=0}^{\infty} \frac{1}{|z|^{q(j+1)}}\right)^{\frac{1}{q}}\right) \\ &= |a_0^2||z|^{n+1}\left(1 - B_{p,i}\frac{1}{(|z|^q - 1)^{\frac{1}{q}}}\right) > 0 \end{aligned} \tag{2.6}$$

if $|z| > \left(1 + B_{p,i}^q\right)^{\frac{1}{q}}$.

Therefore, $|q(z)| > 0$ if $|z| > \left(1 + B_{p,i}^q\right)^{\frac{1}{q}}$. This completes the proving of Theorem 2. □

Proof of Theorem 3. Consider the following polynomial

$$\begin{aligned} q(z) &= (1 - z)f(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\ &\quad + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\quad + \mathrm{i}\big((\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \cdots + (\beta_1 - \beta_0)z + \beta_0\big) \\ &= -a_n z^{n+1} - (\lambda - 1)\alpha_n z^n + (\lambda\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\ &\quad + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\quad + \mathrm{i}\big(-(t-1)\beta_n z^n + (t\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} \\ &\quad + \cdots + (\beta_1 - \beta_0)z + \beta_0\big). \end{aligned} \tag{2.7}$$

Hence, we have

$$\begin{aligned}|q(z)| \geq\ & |a_n z^{n+1} + (\lambda-1)\alpha_n z^n - \mathrm{i}(t-1)\beta_n z^n| \\ & - \big(|\lambda\alpha_n - \alpha_{n-1}||z|^n + |\alpha_{n-1}-\alpha_{n-2}||z|^{n-1} \\ & + \cdots + |\alpha_1-\alpha_0||z| + |\alpha_0| + |t\beta_n - \beta_{n-1}||z|^n + |\beta_{n-1}-\beta_{n-2}||z|^{n-1} \\ & + \cdots + |\beta_1-\beta_0||z| + |\beta_0|\big).\end{aligned} \tag{2.8}$$

Now if $|z| > 1$, then by using hypothesis, we get

$$\begin{aligned}|q(z)| \geq |a_n z^n| \times \Bigg(& \left| z + \frac{(\lambda-1)\alpha_n + (t-1)\beta_n \mathrm{i}}{|a_n|} \right| \\ & - \frac{\lambda\alpha_n - \alpha_0 + |\alpha_0| + t\beta_n - \beta_0 + |\beta_0|}{|a_n|}\Bigg) > 0\end{aligned} \tag{2.9}$$

if

$$\left| z + \frac{(\lambda-1)\alpha_n + (t-1)\beta_n \mathrm{i}}{a_n} \right| > \frac{\lambda\alpha_n + t\beta_n + |\alpha_0| + |\beta_0| - \alpha_0 - \beta_0}{|a_n|}. \tag{2.10}$$

Hence all the zeros of $q(z)$ whose modulus is greater than one lie in the disk

$$\left| z + \frac{(\lambda-1)\alpha_n + (t-1)\beta_n \mathrm{i}}{a_n} \right| \leq \frac{\lambda\alpha_n + t\beta_n + |\alpha_0| + |\beta_0| - \alpha_0 - \beta_0}{|a_n|}. \tag{2.11}$$

But those zeros of $q(z)$ whose modulus is less than or equal to one already satisfy the inequality (2.11). Since all the zeros of $f(z)$ are also zeros of $q(z)$, the proving of Theorem 3 completes. □

Proof of Theorem 4. If we set $A = \{1, 2, \ldots, \mu-1, k\}$ and

$$g(z) = \frac{1}{a_k} \sum_{i=0, i\neq j\in A}^{n} a_i z^i, \tag{2.12}$$

then we have

$$\begin{aligned}|g(z)| &= \frac{1}{|a_k|} \Big| \sum_{i=0,\, i\neq j\in A}^{n} a_i z^i \Big| \\ &\leq \frac{1}{|a_k|} \sum_{i=0,\, i\neq j\in A}^{n} |a_i||z^i| \\ &= \frac{1}{|a_k|} \sum_{i=0, i\neq j\in A}^{n} |a_i| R^i \qquad \text{for } |z| = R \\ &< R^n \frac{1}{|a_k|} \sum_{i=0, i\neq j\in A}^{n} |a_i| \\ &< R^k \qquad\qquad \text{(by (1.40)).}\end{aligned} \tag{2.13}$$

Now, we have $|g(z)| < |z^k| = R^k$ for $|z| = R$. By Rouche's theorem, $g(z) + z^k$ has exactly k zeros in $|z| < R$. Hence, equation $p(z) = 0$ has exactly k solutions in $|z| < R$. And theorem proof is obtained. □

Acknowledgment. The authors are grateful to the anonymous referee for his careful reading and useful suggestions.

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