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ON SUMS OF BINOMIAL COEFFICIENTS, WAVELETS, COMPLEX ANALYSIS, AND OPERATOR THEORY

O. HUTNÍK AND M. HUTNÍKOVÁ

Abstract. This note aims to highlight an interesting connection among the theory of wavelets, special functions, complex analysis, and operator theory in order to evaluate certain sums involving binomial coefficients. Our approach is based on a relationship between spectral functions corresponding to Toeplitz operators acting on weighted Bergman spaces and Toeplitz operators acting on true-poly-analytic Bergman spaces over the upper half-plane. Choosing various generating symbols of Toeplitz operators we may prove many interesting identities for sums of binomial coefficients. Some concrete examples are provided.

1. Introduction

The shortest path between two truths in the real domain passes through the complex domain.

Jacques Hadamard (1865–1963)

Usually, three approaches are used to prove identities involving binomial coefficients as well as identities for sums of binomial coefficients:

- *an algebraic proof* – it transforms one side of the equation with the aid of substitutions and of arithmetic operations into the expression on the other side,
- *a combinatorial proof* – it shows that the expressions on both sides count the same things,
- *a probabilistic proof* – it involves the computation of the probability of a certain event in two different ways and equats them.

However, a more challenging problem is to find a simple closed formula for an expression (such as the sums of combination coefficients), for instance, considering

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$n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, find the value of the sum

$$\sum_{k=0}^{2n} (-1)^k \sum_{r=0}^{k} \binom{k}{r} \binom{n}{k-r} \binom{n}{r} \tag{1}$$

in dependence on the value of n. Several techniques are well-known for evaluating the sum varying from very simple ones (transformations, substitutions, or exchanging the order of summation to rewrite the sums in terms of known expressions summarized in many tables in the available literature) to some more sophisticated ones (recurrences, divided differences, generating functions, etc.). Also, mathematical software may be useful for this purpose.

The quotation from the beginning of this section expresses a deep insight of Hadamard that many problems of real analysis (computation of integrals, summing of series, solving differential equations) are easier (and perhaps elegantly) solved by passing to the complex domain and using its methods. The aim of this note is to present an interesting tool coming from a seemingly unrelated (and distant!) areas of mathematics glued together – wavelets, special functions, poly-analytic complex function theory, and Toeplitz operators[1]. On the one hand, it contributes to the Hadamard's viewpoint, and on the other hand, this machinery demonstrates a useful method for summing certain binomial coefficients using the "continuous mathematics", as it is, for example, the already mentioned probabilistic approach, see [**12, 14, 15**], or the residue approach, see [**7**], or the Rice's integral [**11**].

2. Complex analysis, wavelets, special functions, operator theory – a mixture of all of that

Polyanalytic is the new analytic.

Luís Daniel Abreu

Before describing our main analytic tool, we recall some basic facts from the mathematical areas mentioned in the title. All of the complex analysis theory needed for this note can be found in the Vasilevski's monograph [**13**].

Weighted Bergman spaces and Toeplitz operators acting on them

In this note, the basic spaces, we restrict our attention to, are the weighted Bergman spaces on the upper half plane $\Pi = \{w = u + \mathrm{i}v \in \mathbb{C}; v > 0\}$ in the complex plane $\mathbb{C}$. Given a weight parameter $\lambda \in (-1, +\infty)$, we introduce the standard weighted measure on Π,

$$\mathrm{d}\mu_\lambda(w) = (\lambda + 1)(2v)^\lambda \, \frac{\mathrm{d}u\mathrm{d}v}{\pi}.$$

[1]Certain appearance of binomial sums in the theory of Toeplitz operators is mentioned in Wolf's paper [**16**]. However, Wolf does not study any connection between the two topics. Indeed, he studies the compactness of the difference of the Toeplitz operator T_a and the discrete Wiener-Hopf operator W_a for the case of symbol a being of a special form. Then Wolf's computations result in verifying a certain combinatorial identity.

Then the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ of analytic functions is the closed subspace of $L_2(\Pi, \mathrm{d}\mu_\lambda)$ consisting of measurable functions f on Π, for which

$$\left(\int_\Pi |f(w)|^2 \,\mathrm{d}\mu_\lambda(w)\right)^{1/2} < +\infty.$$

Toeplitz operators form one of the most significant classes of concrete operators because of their importance in both pure and applied mathematics, and in many other sciences. Despite their simple definition, Toeplitz operators exhibit a very rich spectral theory and appear in relation with many problems in physics, probability, finance, hydrology, information theory, signal processing, etc. In general, if X is a function space and P is a projection from X to a closed subspace Y of X, then the Toeplitz operator $T_g\colon X \to Y$ with a symbol g is given as $T_g f = P(gf)$, i.e., it is the compression of the multiplication operator into a suitable subspace. In the literature, two of the most studied cases of Y are the Hardy and the Bergman spaces. In our particular case, given a function (generating symbol) $g \in L_\infty(\Pi)$, the Toeplitz operator $T_g^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ is defined as

$$T_g^{(\lambda)} f = B_{\Pi,\lambda}(gf),$$

where $B_{\Pi,\lambda}$ is the orthogonal projection of $L_2(\Pi, \mathrm{d}\mu_\lambda)$ onto $\mathcal{A}^2_\lambda(\Pi)$.

Behavior of Toeplitz operators $T_g^{(\lambda)}$ and their algebras clearly depends on the behaviour of its symbol g. From now on, we consider very special, the so-called vertical symbols: a measurable function $g\colon \Pi \to \mathbb{C}$ is called *vertical* if there exists a measurable function $a\colon \mathbb{R}_+ \to \mathbb{C}$ such that $g(w) = a(\Im(w))$ for almost all $w \in \Pi$. Very important result of Vasilevski reads as follows, see the summarizing Vasilevski's book [**13**].

Theorem 2.1. *The Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ with a bounded vertical symbol a on Π is unitarily equivalent to the multiplication operator $M_{\gamma_{a,\lambda}}$ acting on $L_2(\mathbb{R}_+)$, where the "spectral" function $\gamma^V_{a,\lambda}\colon \mathbb{R}_+ \to \mathbb{C}$ is given by*

$$\gamma^V_{a,\lambda}(x) = \frac{2x}{\Gamma(\lambda+1)} \int_0^{+\infty} a(v)\,(2vx)^\lambda\, \mathrm{e}^{-2vx}\,\mathrm{d}v.$$

True-poly-analytic Bergman spaces and Calderón-Toeplitz operators

Now we move a step further to uncover an interesting connection among wavelets, special functions, complex analysis, and Toeplitz operators. As it is well-known, the standard Cauchy-Riemann operator

$$\partial_{\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right), \qquad z = x + \mathrm{i}y,$$

stands behind the scene of analytic functions. However, its iterations yield the notion of poly-analytic function. More precisely, a function f is called *poly-analytic* (or analytic of order n) if $\partial^n_{\overline{z}} f = 0$. Poly-analytic functions of order $n > 1$ which are not poly-analytic of order $n-1$ are called *true-poly-analytic* of order n. Interestingly, poly-analytic functions inherit some of the properties of analytic functions, often in a nontrivial form. However, many of the properties break down once we leave the analytic setting. The origin of investigations of poly-analytic functions goes back to 60ties (mainly to works of Mark B. Balk [**2**]), see also the recent survey [**1**].

There exists a close connection between the classical Bergman space of analytic functions, the time-scale (or wavelet) analysis, and certain special functions. Recently, a quite unexpected connection in the poly-analytic setting has been obtained: the true-poly-analytic Bergman space $\mathcal{A}^{(n+1)}(\Pi)$ may be viewed as the space of wavelet transforms with respect to Laguerre functions of order n. Recall that the (simple) Laguerre functions are given as $\ell_n(x) := \mathrm{e}^{-x/2}L_n(x)$ with

$$L_n(x) := \frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x}x^n\right) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{x^i}{i!}, \qquad x \in \mathbb{R}_+,$$

being the Laguerre polynomial of order $n \in \mathbb{Z}_+$. For further reading and more information about the time-scale approach to poly-analytic Bergman spaces we recommend the recent papers [**1, 9**].

It was observed in several cases that Toeplitz operators can be transformed into pseudo-differential operators by means of certain unitary maps constructed as an exact analog of the Bargmann transform mapping the Segal-Bargmann-Fock space $F_2(\mathbb{C}^n)$ of Gaussian square-integrable entire functions on complex n-space onto $L_2(\mathbb{R}^n)$. A key result that gives an easy and direct access to the properties of Toeplitz operators acting on true-poly-analytic Bergman spaces (or alternatively, of Calderón-Toeplitz operators acting on wavelet subspaces, see [**8**, Theorem 3.2]), is the following theorem.

Theorem 2.2. *Let a be a bounded vertical symbol on Π. Then the Toeplitz operator $T_a^{(n)}$ acting on $\mathcal{A}^{(n)}(\Pi)$ is unitarily equivalent to the multiplication operator $M_{\gamma_{a,n}}$ acting on $L_2(\mathbb{R}_+)$, where the "spectral" function $\gamma_{a,n}\colon \mathbb{R}_+ \to \mathbb{C}$ is given by*

$$\gamma_{a,n}(x) = 2x\int_0^{+\infty} a(v)\ell_n^2(2vx)\,\mathrm{d}v = 2x\int_0^{+\infty} a(v)\,\mathrm{e}^{-2vx}L_n^2(2vx)\,\mathrm{d}v.$$

Sketch of proof (for the benefit of the reader). The key ingredient here is the unitary operator $R_n\colon \mathcal{A}^{(n)}(\Pi) \to L_2(\mathbb{R})$ given by

$$(R_nF)(\xi) = \chi_+(\xi)\sqrt{2\xi}\int_\Pi F(u,v)\ell_n(2v\xi)\,\mathrm{e}^{-2\pi\mathrm{i}\xi u}\,\frac{\mathrm{d}u\mathrm{d}v}{v},$$

which diagonalizes each Toeplitz operator $T_a^{(n)}$ with bounded vertical symbol a in the sense that $R_nT_a^{(n)}R_n^* = \gamma_{a,n}I$. This implies the assertion. □

In fact, the function $\gamma_{a,n}$ is obtained by integrating a dilation of a bounded vertical symbol a of $T_a^{(n)}$ against a Laguerre function of order n, and it sheds a new light upon the investigation of main properties of the corresponding Toeplitz operator $T_a^{(n)}$, such as boundedness, spectrum, invariant subspaces, norm value, etc., see [**9**]. Recently, an explicit description of the C*-algebras generated by the set of spectral functions was obtained in [**10**] via membership of all gammas $\gamma_{a,n}$

in the algebra $\mathrm{VSO}(\mathbb{R}_+)$ of very slowly oscillating functions on $\mathbb{R}_+$ being the set of all bounded functions uniformly continuous with respect to logarithmic metric $\rho(x,y) = |\ln x - \ln y|$. An independent and beautiful reasoning of this result starts from the simple estimate

$$|\gamma_{a,n}(x) - \gamma_{a,n}(y)| \le \|a\|_\infty \left(\int_{\mathbb{R}_+} \left| 2vxL_n^2(2vx)\,\mathrm{e}^{-vx} - 2vyL_n^2(2vy)\,\mathrm{e}^{-vy} \right| \frac{\mathrm{d}v}{v} \right)$$

with $a \in L_\infty(\mathbb{R}_+)$ and $x, y > 0$. Replacing $zL_n^2(z)$ with an arbitrary polynomial P of the form

$$P(x) = \sum_{j=0}^{m} \alpha_j x^j$$

with the special polynomial norm

$$\mathcal{N}(P) := \sum_{j=0}^{m} j!\,|\alpha_j| = \sum_{j=0}^{\infty} |P^{(j)}(0)|,$$

we can estimate the function $J_P \colon \mathbb{R}_+^2 \to \mathbb{C}$ given by

$$J_P(x,y) := \int_{\mathbb{R}_+} \left| P(vx)\,\mathrm{e}^{-vx} - P(vy)\,\mathrm{e}^{-vy} \right| \frac{\mathrm{d}v}{v}$$

as follows: $J_P(x,y) \le 2\mathcal{N}(P)\rho(x,y)$. Thus, for each $x, y > 0$, we get

$$|\gamma_{a,n}(x) - \gamma_{a,n}(y)| \le \|a\|_\infty J_P(x,y) \le C\rho(x,y)$$

with $P(z) = zL_n^2(z)$ and $C := 2\,\|a\|_\infty \mathcal{N}(P)$. This proves that $\gamma_{a,n} \in \mathrm{VSO}(\mathbb{R}_+)$ for each $n \in \mathbb{Z}_+$. This enables to provide an explicit description of the C*-algebra generated by vertical Toeplitz operators on true-poly-analytic Bergman spaces over the upper half-plane.

3. How to get sums via gammas? A toy example

In mathematics the most exiting and welcome results usually combine facts, notions and ideas of very remote fields.

In the beginning, there is an easy observation about the connection of true-poly-analytic gammas and Vasilevski's gammas. In the case $\lambda = n = 0$, we have trivially $\gamma_{a,0}^V = \gamma_{a,0}$, but in general, the situation is much more interesting. We can see that the main difference between the true-poly-analytic gamma and Vasilevski's gamma function integral is the square of Laguerre polynomials appearing in $\gamma_{a,n}$. To uncover the connection between the gammas, we use *Feldheim's formula* [**3**] which provides the product of two generalized Laguerre polynomials in terms of other generalized Laguerre polynomials. More precisely, for $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}\alpha > -1$,

$\operatorname{Re}\beta > -1$, and $\operatorname{Re}(\alpha+\beta) > -1$, it holds

$$\begin{aligned} L_m^{(\alpha)}(x)\cdot L_n^{(\beta)}(x) &= \sum_{k=0}^{m+n} A_k(m,n,\alpha,\beta) L_k^{(\alpha+\beta)}(x) \\ &= (-1)^{m+n}\sum_{k=0}^{m+n} A_k(m,n,\beta-m+n,\alpha+m-n)\frac{x^k}{k!}, \end{aligned}$$

where

$$A_k(m,n,\alpha,\beta) := (-1)^{m+n+k}\sum_{r=0}^{k}\binom{k}{r}\binom{m+\alpha}{n-k+r}\binom{n+\beta}{m-r}.$$

Using this result with $\alpha=\beta=0$ and $m=n$, we have

$$L_n^2(x) = \sum_{k=0}^{2n}\frac{A_k(n)}{k!}\,x^k, \qquad \text{where } A_k(n) := (-1)^k\sum_{r=0}^{k}\binom{k}{r}\binom{n}{k-r}\binom{n}{r}.$$

Thus, we get the basic relation.

Theorem 3.1. *For each $a \in L_\infty(\mathbb{R}_+)$, each $n \in \mathbb{Z}_+$, and $x > 0$, it holds*

$$\gamma_{a,n}(x) = \sum_{k=0}^{2n} A_k(n)\frac{2x}{k!}\int_0^{+\infty} a(v)(2xv)^k\,\mathrm{e}^{-2vx}\,\mathrm{d}v = \sum_{k=0}^{2n} A_k(n)\,\gamma_{a,k}^V(x). \tag{2}$$

There are many good reasons for being interested in this formula itself (mentioning some consequences for Toeplitz operators and the whole machinery for the study of operator algebras), but we leave those reasons aside in this note (we recommend our overview paper [**9**], where an interested reader may find many interesting results and connections). Our goal is simply to see how this formula may be a useful tool when expressing some sums. Of course, it is not an accident that $A_k(n)$ appears in our motivational task to evaluate the sum (1). Now we may give an elegant solution.

In fact, for the generating symbol $a \equiv 1$ on $\mathbb{R}_+$, we easily get $\gamma_{a,n} \equiv 1$ for each $n \in \mathbb{Z}_+$. Indeed, it is enough to realize that the system $\{\ell_n(x)\}_{n\in\mathbb{Z}_+}$ forms an orthonormal basis in the space $L_2(\mathbb{R}_+)$. Also, using the definition of Euler Gamma function Γ, we have

$$\gamma_{a,k}^V(x) = \frac{2x}{k!}\int_0^{+\infty}(2vx)^k\,\mathrm{e}^{-2vx}\,\mathrm{d}v = \frac{\Gamma(k+1)}{k!} = 1$$

for each $k \in \mathbb{Z}_+$ and $x > 0$, and therefore, by (2), we conclude that

$$\sum_{k=0}^{2n} A_k(n) = \sum_{k=0}^{2n}(-1)^k\sum_{r=0}^{k}\binom{k}{r}\binom{n}{k-r}\binom{n}{r} \equiv 1 \qquad \text{for each } n \in \mathbb{Z}_+.$$

In addition to the inner beauty of this reasoning, it allows us to say that *each $\gamma_{a,n}$ is an affine combination of $\gamma_{a,k}^V$* with $k = 0, 1, \ldots, 2n$.

We may observe that Feldheim's formula was the main ingredient to express true-poly-analytic gammas via Vasilevski's ones. Using other formulas for square of

Laguerre polynomials, we may recover some known formulas for sums of binomial coefficients.

Remark. Starting with the (definition) formula for Laguerre polynomial

$$L_n(x) = \frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x}x^n\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!}, \qquad x \in \mathbb{R}_+,$$

we immediately get

$$\ell_n^2(x) = \mathrm{e}^{-x}L_n^2(x) = \mathrm{e}^{-x}\sum_{r=0}^{n}\sum_{s=0}^{n}\frac{(-1)^{r+s}}{r!s!}\binom{n}{r}\binom{n}{s}x^{r+s}.$$

Then the formula for $\gamma_{a,n}$ may be written as follows

$$\gamma_{a,n}(x) = 2x\int_0^{+\infty} a(v)\ell_k^2(2vx)\,\mathrm{d}v = \sum_{r=0}^{n}\sum_{s=0}^{n} B_{r,s}(n)\,\gamma_{a,r+s}^V(x),$$

where

$$B_{r,s}(n) := (-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{r+s}{r},$$

and for each $n \in \mathbb{Z}_+$, it holds

$$\sum_{r=0}^{n}\sum_{s=0}^{n} B_{r,s}(n) = 1.$$

Remark. Furthermore, from *Howell's formula* [**6**], see also [**5**, formula 8.976 (3^6)],

$$\left(L_n^{(\alpha)}(x)\right)^2 = \frac{\Gamma(1+\alpha+n)}{2^{2n}n!}\sum_{r=0}^{n}\binom{2n-2r}{n-r}\frac{(2r)!}{r!}\frac{1}{\Gamma(1+\alpha+r)}L_{2r}^{(2\alpha)}(2x),$$

we have

$$L_n^2(x) = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_k(n)L_{2k}(2x), \qquad \text{where} \;\; C_k(n) := \binom{2n-2k}{n-k}\binom{2k}{k}.$$

Similarly as above, we then recover the known formula

$$\frac{1}{2^{2n}}\sum_{k=0}^{n} C_k(n) = 1 \qquad \text{for each } n \in \mathbb{Z}_+,$$

see, e.g., [**4**, formula (6.10)].

4. A more sophisticated example

...it seems worth enough for a mathematician to think (and sometime to write) not only about new mathematical facts but also about old original ideas. Particularly, relations between new facts and old ideas deserve special attention...

Vladimir V. Kisil

Now we demonstrate the usefulness of the described procedure considering the oscillating symbol $a(v) = \mathrm{e}^{2v\mathrm{i}}$. Then the explicit form of the corresponding function $\gamma_{a,n}$ is

$$\begin{aligned}\gamma_{a,n}(x) &= 2x \int_0^{+\infty} \mathrm{e}^{-2v(x-\mathrm{i})} L_n^2(2vx)\,\mathrm{d}v = \frac{x}{x-\mathrm{i}} \int_0^{+\infty} \mathrm{e}^{-t} L_n^2\left(\frac{tx}{x-\mathrm{i}}\right) \mathrm{d}t\\ &= \frac{(-1)^n}{(x-\mathrm{i})^{2n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^{2k+1},\end{aligned}$$

where [**5**, formula 7.414.2] has been used. On the other hand, for each $k \in \mathbb{Z}_+$, we have

$$\gamma_{a,k}^V(x) = \frac{2x}{k!} \int_0^{+\infty} \mathrm{e}^{-2v(x-\mathrm{i})} (2vx)^k\,\mathrm{d}v = \left(\frac{x}{x-\mathrm{i}}\right)^{k+1},$$

where [**5**, formula 3.381.4] has been used. Then from the equality (2), we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^{2k} = (-1)^n (x-\mathrm{i})^{2n} \sum_{k=0}^{2n} A_k(n) \left(\frac{x}{x-\mathrm{i}}\right)^k, \qquad x > 0.$$

Especially, for $x = 1$, we have

$$(-1)^n \sum_{k=0}^{2n} A_k(n)(1-\mathrm{i})^{2n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2.$$

This identity seems to be interesting itself because it provides a sum of combinations of powers of complex number $w = 1 - \mathrm{i}$ in real terms only.[2] Moreover, the sum on the right-hand side is well known – it equals the coefficient of r^n in $(1-r)^n(1+r)^n = (1-r^2)^n$. More precisely,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^{[\frac{n}{2}]} \binom{n}{[\frac{n}{2}]} \frac{1+(-1)^n}{2} = (-1)^n 2^{2n} \binom{\frac{n-1}{2}}{n},$$

see [**4**, formula (1.35), (1.45)], and we have

$$\sum_{k=0}^{2n} A_k(n)(1-\mathrm{i})^{2n-k} = 2^{2n} \binom{\frac{n-1}{2}}{n}.$$

Similarly as in Remarks above, we may obtain the corresponding identities with coefficients $B_{r,s}(n)$ (a triple sum identity, in fact) and $C_k(n)$ (a double sum identity, in fact). Moreover, choosing various symbols, we may provide many other identities for sums of binomial coefficients. These are left to the interested reader.

[2] In some sense, it may resemble the famous *Binet formula* – the expression of Fibonacci (natural) numbers via powers of irrational numbers (golden ratio, in fact).

Final comments

Of course, the identities proved here might be proved by a variety of other means. For example, using hypergeometric functions, using orthogonality ideas and the generating function for Laguerre polynomials, the Chu-Vandermonde summation, the Rice integral formulas, etc. However, we hope the interested reader will enjoy the presented interesting (and perhaps unexpected) connections among sums of binomial coefficients, time-frequency (wavelet) analysis, and spectral functions of Toeplitz operators associated with functions on the upper half-plane in the complex plane.

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O. Hutník, Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University in Košice, *current address*: Jesenná 5, SK 040 01 Košice, Slovakia,
e-mail: `ondrej.hutnik@upjs.sk`

M. Hutníková, Department of Physics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, *current address*: Park Komenského 2, SK 042 00 Košice, Slovakia,
e-mail: `maria.hutnikova@tuke.sk`