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FRACTIONAL HYBRID INITIAL VALUE PROBLEM FEATURING q-DERIVATIVES

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Abstract. We have perused about the existence of a solution toward hybrid initial value problem (HIVP) featuring fractional q-derivative

$$\begin{cases} \mathfrak{D}_q^\delta\Big[\frac{\nu(t)}{h\big(t,\nu(t),\ \max\limits_{0\le\tau\le t}|\nu(\tau)|\big)}\Big] = \rho\,(t,\ \nu\,(t))\,, & t\in(0,1),\ 0<\delta\le 1,\\ \nu(0)=0, \end{cases}$$

in which $\mathfrak{D}_q^\delta$ denotes the Riemann-Liouville fractional q-derivative in the order of δ. In Banach algebra, by making use of a fixed point theorem based Dhage along with mixed Lipschitz and Caratheodory condition, a way of solving the above fractional Hybrid initial value problem (FHIVP) featuring q-derivatives verified, in this study.

1. Introduction

Fractional calculus has a long history and it dates back to the birth of the classical calculus. However, during these years, some research works have been carried out but in the last few decades, this new calculus together with dynamic equation has gained more popularity. There are several classes of fractional derivative, but the most prevalent definitions are Riemann-Liouville and Caputo fractional derivatives. The former has an abstraction mathematically but the latter is mostly used by engineers.

Research on differential equations featuring fractional derivative has appreciably improved within current years, which indicates the significance of the calculus featuring fractional derivative and fractional integral in engineering, sciences, and technologies [**16, 11, 18, 5, 1, 3**].

In the last few years, solvability of initial differential equations featuring fractional derivative with regard to particular functions were investigated in these kinds of problems in which there exist positive solutions or a way of solving propounded problems using Leray-Schoder theory and fixed point theorem [**7, 12, 13, 14, 11, 20**].

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As a matter of truth, there are similar requirements toward boundary conditions [**19, 20, 23, 21, 25, 15, 26**].

In recent years, hybrid differential equation (quadrature perturbations of non linear differential equation) has attracted much attention.

In [**9**], Dhage and his associate reviewed the following hybrid differential equation featuring linear perturbations of first order

$$\begin{cases} \mathfrak{D}_t\left[\frac{\nu(t)}{h(t,\nu(t))}\right] = \rho\left(t,\ \nu\left(t\right)\right), \qquad \rho \in C\left(\mathcal{J} \times \mathbb{R},\ \mathbb{R}\right),\ h \in C\left(\mathcal{J} \times \mathbb{R},\ \mathbb{R} - \{0\}\right), \\ \nu(t_0) = x_0 \in \mathbb{R}. \end{cases}$$

$\mathcal{J} = [0,\ 1]$.

Zhao and his associates [**26**] utilizing a fixed point theorem in Banach algebras researched a way of solving featuring the fractional version of the above IVP, i.e.,

$$\begin{cases} \mathfrak{D}_t^{\delta}\left[\frac{\nu(t)}{h(t,\ \nu(t))}\right] = \rho\left(t,\ \nu\left(t\right)\right), \qquad 0 < \delta \le 1. \\ \nu(0) = 0. \end{cases}$$

There exists a way of solving the next fractional HIVP, featuring supermom

$$\begin{cases} \mathfrak{D}_t^{\delta}\left[\frac{\nu(t)}{h\left(t,\ \nu(t),\ \max\limits_{0\le\tau\le t}|\nu(\tau)|\right)}\right] = \rho\left(t,\ \nu\left(t\right)\right), \qquad 0 < \delta \le 1, \\ \nu(0) = 0, \end{cases}$$

where $\rho \in C\left(\mathcal{J} \times \mathbb{R},\ \mathbb{R}\right)$ and $h \in C\left(\mathcal{J} \times \mathbb{R}^2,\ \mathbb{R} - \{0\}\right)$ investigated by Caballero and his associates [**6**], in which the way of measuring non compactness within the Banach space was the basic tool applied by the scholars.

Now, we deal with the next HIVP featuring fractional q-derivatives

$$ {}_t\mathfrak{D}_q^{\delta}\left[\frac{\nu\left(t\right)}{h\left(t, \nu\left(t\right),\ \max\limits_{0\le\tau\le t}\left|\nu\left(\tau\right)\right|\right)}\right] = \rho\left(t,\ \nu\left(t\right)\right), \qquad 0 < \delta \le 1, \tag{1}$$

$$\nu(0) = 0, \tag{2}$$

$t \in (0, 1)$, $h \in C\left([0,\ 1] \times \mathbb{R}^2, \mathbb{R} - \{0\}\right)$, ${}_t\mathfrak{D}_q^{\delta}$ is q-derivative of Rimann-Liouville the order of δ, ρ is the measurable map for each $\nu \in \mathbb{R}$ and continuous to every $t \in \mathbb{R}$.

In Banach algebra, by employing a fixed point theorem based on Dhage, under the following hypothesis, featuring Caratheodory condition and mixed Lipschitz, there exists a way of solving the hybrid equations featuring fractional derivative.

Hypotheses:

(H1) There exists $L > 0$, which is a constant wherein

$$\left|h\left(t,\ r_1,\ r_2\right) - h\left(t,\ r_3,\ r_4\right)\right| \le L\left(\max\left|r_1 - r_3\right|,\ \left|r_2 - r_4\right|\right)$$

for any $t \in \mathcal{J} = [0, 1]$ and $r_1, r_2, r_3, r_4 \in \mathbb{R}$.

(H2) There is a function $g \in L^1\left([0, 1], \mathbb{R}\right)$, so that toward every $x \in \mathbb{R}$

$$\left|\rho\left(t, x\right)\right| \le g\left(t\right),$$

$t \in [0, 1]$.

2. q-CALCULUS

We here present preludes q-calculus, lemmas and definitions that employed in the rest of this paper. The display here may be found in [**10, 2**], for example, presume $a \in \mathbb{R}$ and $q \in (0,1)$

$$[a]_q = \frac{1-q^a}{1-q}.$$

Whereas $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, the q-analogue of the power function $(a-b)^n$ featuring $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is expressed

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k).$$

Consequently, if $a \neq 0$ and $\delta \in \mathbb{R}$, therefore,

$$(a-b)^{(\delta)} = a^\delta \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\delta+k}}.$$

Obviously, in case $b = 0$, $a^{(\delta)} = a^\delta$. For $\tau \in \mathbb{R} - \{0, -1, -2, \dots\}$, the q-gamma function is expressed

$$\Gamma_q(\tau) = \frac{(1-q)^{(\tau-1)}}{(1-q)^{\tau-1}},$$

and gladdens $\Gamma_q(\tau+1) = [\tau]_q \Gamma_q(\tau)$.
The q-derivative of

- a function u is expressed

$$(\mathfrak{D}_q u)(\tau) = \frac{u(q\tau) - u(\tau)}{(q-1)\tau}, \qquad (\mathfrak{D}_q u)(0) = \lim_{\tau \to 0} (\mathfrak{D}_q u)(\tau),$$

- higher order is expressed with

$$(\mathfrak{D}_q^n u)(\tau) = \mathfrak{D}_q(\mathfrak{D}_q^{n-1} u)(\tau), \qquad n \in \mathbb{N},$$

wherein $(\mathfrak{D}_q^0 u)(\tau) = u(\tau)$.
The q-integral of a function u expressed within the distance $[0, b]$ is certain as

$$(I_q u)(\tau) = \int_0^\tau u(s) \mathrm{d}_q s = u(1-q) \sum_{k=0}^{\infty} u(\tau q^k) q^k, \quad \tau \in [0, b].$$

Assuming that u is expressed within $[0, b]$, thus

$$\int_a^b u(s) \mathrm{d}_q s = \int_0^b u(s) \mathrm{d}_q s - \int_0^a u(s) \mathrm{d}_q s,$$

wherein $a \in [0, b]$.
The same is true for derivatives, an operator I_q^n may be represented as

$$(I_q^0 u)(\tau) = u(\tau), \quad (I_q^n u)(\tau) = I_q(I_q^{k-1} u)(\tau), \qquad k \in \mathbb{N}.$$

With regard to the basic theorem of arithmetic utilizing I_q and $\mathfrak{D}_q$, i.e.,

$$(\mathfrak{D}_q I_q u)(\tau) = u(\tau),$$

and if u is continuous at $\tau = 0$, then

$$(I_q \mathfrak{D}_q u)(\tau) = u(\tau) - u(0).$$

The equalities below utilized subsequently:

$$\int_0^\tau u(s)(\mathfrak{D}_q g)(s)\mathrm{d}_q s = \Big[u(s)g(s)\Big]_{s=0}^{s=\tau} - \int_0^\tau (\mathfrak{D}_q u)(s)g(qs)\mathrm{d}_q s,$$
$$[a(\varsigma - s)]^{(\delta)} = a^\delta (\varsigma - s)^{(\delta)},$$
$${}_t\mathfrak{D}_q (\varsigma - s)^{(\delta)} = [\delta]_q (\varsigma - s)^{(\delta-1)},$$
$${}_s\mathfrak{D}_q (\varsigma - s)^{(\delta)} = -[\delta]_q (\varsigma - qs)^{(\delta-1)},$$
$$({}_\tau\mathfrak{D}_q \int_0^x u(\tau, s)\mathrm{d}_q s)(\tau) = \int_0^\tau {}_\tau\mathfrak{D}_q u(\tau, s)\mathrm{d}_q s + u(q\tau, \tau).$$

${}_t\mathfrak{D}_q$ indicates the derivative in relation to the variable ς.

Definition 2.1. Permit u be to a function expressed on $[0, 1]$. The q-integral of Riemann-Liouville kind of the fractional order $\delta > 0$ is

$$(I_q^\delta u)(\tau) = \frac{1}{\Gamma_q(\delta)} \int_0^\tau (\tau - qs)^{(\delta-1)} u(s)\mathrm{d}_q s, \qquad \tau \in [0, 1],$$

meanwhile $(I_q^0 u)(\tau) = u(\tau)$.

Definition 2.2. The q-integral of Riemann-Liouville kind featuring the fractional derivative of order $\delta > 0$, is expressed by $(\mathfrak{D}_q^\delta u)(\tau) = (\mathfrak{D}_q^{[\delta]} I_q^{[\delta]-\delta} u)(\tau)$, $[\delta]$ is equivalent to or the least integer greater than δ and $(\mathfrak{D}_q^0 u)(\tau) = u(\tau)$.

Lemma 2.3 ([**10**])**.** *Presume $\delta \geq 0$ and $a \leq b \leq \varsigma$, therefore, $(\varsigma - b)^{(\delta)} \leq (\varsigma - a)^{(\delta)}$.*

Lemma 2.4 ([**10**])**.** *Suppose u a function expressed on $[0, 1]$, δ and $\sigma \geq 0$. Next, we get the expressions below:*
(a) $(I_q^\sigma I_q^\delta u)(\tau) = (I_q^{\delta+\sigma} u)(\tau)$,
(b) $(\mathfrak{D}_q^\delta I_q^\delta u)(\tau) = u(\tau)$.

Lemma 2.5 ([**10**])**.** *Presume δ and $n > 0$ an integer. Therefore, the parity below holds*

$$(I_q^\delta \mathfrak{D}_q^n u)(\tau) = (\mathfrak{D}_q^n I_q^\delta u)(\tau) - \sum_{k=0}^{n-1} \frac{\tau^{\delta-n+k}}{\Gamma_q(\delta + k - n + 1)} (\mathfrak{D}_q^k u)(0).$$

Lemma 2.6 ([**2**])**.** *Presume $\delta \in \mathbb{R}^+$ and $f\colon (0, a] \to \mathbb{C}$ a function. If $g \in L_q^1[0, a]$, then $I_q^\delta g \in L_q^1[0, a]$ and $I_q^\delta f \leq \frac{a^\delta \|g\|_{L^1}}{\Gamma_q(\delta+1)}$.*

3. Main results

In this part, we demonstrate the existing outcomes for the q-FHIVP (1)–(2) within the bounded and closed interval $\mathcal{J} = [0, 1]$ following combined Lipschitz and

Caratheodory circumstances within the nonlinearities included in them. We locate the q-FHIVP (1)–(2) in the space $C(\mathcal{J},\mathbb{R})$ of continuous functions featuring real-value expressed within $\mathcal{J}$, which describes a supremum norm $\|.\|$ within $C(\mathcal{J},\mathbb{R})$ given below

$$\|\nu\| = \max_{t\in\mathcal{J}} |\nu(t)| \tag{3}$$

and a multiplication within $C(\mathcal{J},\mathbb{R})$ given as

$$(\nu\mu)(t) = \nu(t)\,\mu(t) \tag{4}$$

for $\nu,\mu \in C(\mathcal{J},\mathbb{R})$. Plainly $C(\mathcal{J},\mathbb{R})$ regarding the norm and multiplication in (3)–(4) is a Banach algebra. By $L^1(\mathcal{J},\mathbb{R})$, we represents the region of Lebesgue integrable real-valued functions on $\mathcal{J}$, equipped featuring the norm $\|\cdot\|_{L^1}$ defined by

$$\|\nu\|_{L^1} = \int_0^1 |\nu(s)|\mathrm{d}s.$$

The subsequent Banach algebra featuring fixed point theorem based on Dhage [**8**] is the basic theorem of our important result.

Theorem 3.1 ([**8**]). *Assume ψ, is a bounded subset, closed, convex and non-empty of the Banach algebra χ and assume $A\colon \psi \to \chi$ and $B\colon \chi \to \chi$ be operators in which:*

i) *A is completely continuous,*

ii) *B is Lipschitz featuring a Lipschitz constant γ,*

iii) *$\varsigma = A\varsigma\, B\,\mu \Longrightarrow \varsigma \in \psi$ for all $\mu \in \psi$, and*

iv) *$\mathfrak{M}\,\gamma < 1$, where $\mathfrak{M} = \|A(\psi)\| = \sup\{\|A(\varsigma)\| : \varsigma \in \psi\}$. Then, the operator equation $B\,\varsigma\, A\,\varsigma = \varsigma$ has a solution in ψ.*

Lemma 3.2. *Let ν belong to $C\,[0,\,1]$, $C\left([0,\,1]\times\mathbb{R}^2,\,\mathbb{R}-\{0\}\right)$ and $0<\delta<1$. Then, the unique solution of the below IVP*

$$\mathfrak{D}_q^{\delta}\Big[\frac{\nu\,(t)}{h\big(t,\nu\,(t)\,,\ \max\limits_{0\le\tau\le t}|\nu\,(\tau)|\,\big)}\Big] = \rho(t), \tag{5}$$

$$\nu(0) = 0 \tag{6}$$

for $0\le t\le 1$, $\nu(t) = \dfrac{h\big(t,\nu(t),\ \max\limits_{0\le\tau\le t}|\nu(\tau)|\big)}{\Gamma_q(\delta)}\displaystyle\int_0^t \frac{\rho(s)}{(t-qs)^{(1-\delta)}}\mathrm{d}_q s.$

Proof. Utilizing both sides of (5), the operator I^{δ} and applying Definition 2.2, we get an equivalent equation

$$I_q^{\delta}\,\mathfrak{D}_q\Big[I_q^{1-\delta}\Big(\frac{\nu\,(t)}{h\big(t,\nu\,(t)\,,\ \max\limits_{0\le\tau\le t}|\nu\,(\tau)|\,\big)}\Big)\Big] = I_q^{\delta}\,\rho(t).$$

Here, from Lemma 2.5, we have

$$\mathfrak{D}_q\Big[I_q^{\delta}\left(g(t)\right)\Big] - \frac{t^{\delta-1}g(0)}{\Gamma_q\,(\delta)} = I_q^{\delta}\rho(t),$$

where we let $h(t) = I_q^{1-\delta}\Big(\frac{\nu(t)}{h\big(t,\nu(t),\,\max_{0\le\tau\le t}|\nu(\tau)|\big)}\Big)$.

Afterwards, with Lemma 2.4, we get

$$\nu(t) = \frac{h\big(t, \nu\,(t)\,,\ \max_{0\le\tau\le t}|\nu\,(\tau)|\,\big)}{\Gamma_q\,(\delta)}\int_0^t \frac{\rho(s)}{(t-qs)^{(1-\delta)}}\mathrm{d}_q s. \tag{7}$$

Thus, it lets (7) be satisfied.

Dividing by $h\big(t, \nu\,(t),\ \max_{0\le\tau\le t}|\nu\,(\tau)|\,\big)$, exerting on both sides and getting Lemma 2.4 into account, we obtain (5). Also, setting $t = 0$ in (7), we get the expression

$$\nu(0) = h\,(0,\,\nu(0),\,|\nu(0)|) \times 0 = 0.$$

This demonstrates $\nu(t)$ is a solution to problem (5)–(6). $\square$

Theorem 3.3. *Presume that hypotheses* (H1) *and* (H2) *retain. Moreover, q-FHIVP* (1)–(2) *has a solution expressed on* $\mathcal{J}$, *when* $\frac{L\|g\|_{L^1}}{\Gamma_q(\delta+1)} < 1$.

Proof. Let $\chi = C\,(\mathcal{J},\mathbb{R})$ and

$$\psi = \left\{\nu \in \chi,\, \|\nu\| \le \frac{kL\,T_{\delta,\,q}}{1 - L\,T_{\delta,\,q}}\right\},$$

where $k = \max|h\,(t,\,0,\,0)|$ and $T_{\delta,\,q} = \frac{\|g\|_{L^1}}{\Gamma_q\,(1+\delta)}$. Obviously, ψ is a convex, closed, and bounded subset of the Banach space χ.

With Lemma 3.1, BVP (1)–(2) is equal to the next equation

$$\begin{aligned}\nu(t) &= h\Big(t,\,\nu(t),\ \max_{0\le\tau\le t}|\nu\,(\tau)|\Big)\frac{1}{\Gamma_q(\delta)}\int_0^t (t-q\,s)^{(\delta-1)}\,\rho\,(s,\,\nu\,(s))\,\mathrm{d}_q s\\ &= h\Big(t,\,\nu(t),\ \max_{0\le\tau\le t}|\nu\,(\tau)|\Big)I_q^{\delta}\rho\,(s,\,\nu\,(s))\,.\end{aligned} \tag{8}$$

This defines $A\colon \chi \to \chi$ and $B\colon \psi \to \chi$, with $A\,\nu\,(t) = h(t,\,\nu\,(t)\,,\,\max_{0\le\tau\le t}|\nu\,(\tau)|)$ and $B\,\nu\,(t) = I_q^{\delta}\rho\,(s,\,\nu\,(s))$.
Afterwards, equation (8) is converted in the following operator

$$A\,\nu(t)B\,\nu(t) = \nu(t),$$

whereas $t \in \mathcal{J}$.

We ought to display in Theorem 3.1 which of the two operators A and B convince whole requirements.

Imprimis, we display operator A, featuring Lipschitz constant L is a Lipschitz operator on χ. Next, by hypothesis (H1),

$$\begin{aligned}\big|A\,\nu\,(t) - A\,\mu\,(t)\,\big| &= \Big|h\Big(t,\,\nu\,(t)\,,\ \max_{0\le\tau\le t}|\nu\,(\tau)|\Big) - h\Big(t,\,\mu\,(t)\,,\ \max_{0\le\tau\le t}|\mu\,(\tau)|\Big)\Big|\\ &\le L\max\Big(|\nu(t)-\mu(t)|,\big|\max_{0\le\tau\le t}|\nu(\tau)| - \max_{0\le\tau\le t}|\mu(\tau)|\big|\Big)\\ &\le L\max\big(\|\nu-\mu\|\,,\,\|\nu-\mu\|^{*}\big)\end{aligned}$$

$$\leq L \max\Big(\|\nu-\mu\|, |\nu-\mu\Big\|\Big)$$
$$\leq L\,\|\nu-\mu\|\,,$$

where $\|\nu-\mu\|^{*} = ||\,(\nu-\mu)\,\big|_{[0,\,t]}||$, $t\in j$.

Catching supremum over t, for ν, $\mu\in\chi$, we get

$$||A\,\nu\,(t) - A\,\mu\,(t)|| \leq L\,\|\nu-\mu\|\,.$$

Next, we display B is completely continuous within χ on ψ. For this, we display B is continuous and compact operator within χ on ψ.

Now, we display that B is continuous on ψ. Suppose that $\{\nu_n\}$ a sequence into ψ converging to a point $\nu\in\psi$. Thus, with Lebesgue dominating convergence theorem

$$\begin{aligned}\lim_{n\to\infty} B\,\nu_n\,(t) &= \lim_{n\to\infty} I_q^{\delta}\rho\,(s,\,\nu_n\,(t))\\ &= \lim_{n\to\infty}\frac{1}{\Gamma_q(\delta)}\int_0^t (t-q\,s)^{(\delta-1)}\,\rho\,(s,\,\nu_n\,(s))\,\mathrm{d}_q s\\ &= \frac{1}{\Gamma_q(\delta)}\int_0^t (t-q\,s)^{(\delta-1)}\lim_{n\to\infty}\,\rho\,(s,\,\nu_n\,(s))\,\mathrm{d}_q s\\ &= \frac{1}{\Gamma_q(\delta)}\int_0^t (t-q\,s)^{(\delta-1)}\,\rho\,(s,\,\nu\,(s))\,\mathrm{d}_q s = I_q^{\delta}\rho\,(s,\,\nu\,(t)) = B\,\nu(t).\end{aligned}$$

Then, we display B on ψ is a compact operator. That is sufficient to display $B(s)$ in χ is an equicontinuous set and uniformly bounded.

Otherwise, assume $\nu\in\psi$ be arbitrary. By (H2), we have

$$|B\,\nu(t)| = |I_q^{\delta}\rho\,(s,\,\nu\,(t))\,| \leq I_q^{\delta} g\,(t)\,,\; t\in\mathcal{J}.$$

Getting supremum over t, for all $\nu\in\psi$ from Lemma 3.2, $\|B\,\nu\,(t)\|\leq T_{a,q}$. This represents B on ψ is uniformly bounded.

Because $\rho(t,\nu(t))$ within $[0,\,1]\times\mathbb{R}$ is continuous, and within compact set $[0,\,1]\times[-r,\,r]$ is bounded. Assume $\mathfrak{N} = \sup\{\rho\,(t,\nu(t))\,,\; t\in[0,\,1]\,,\; \nu\in[-r,\,r]\}$.

Here, we suggest that B is equcontinuous on ψ, given $\epsilon>0$. Presume $\delta < \left(\frac{\epsilon\,\Gamma_q(\delta)}{2\mathfrak{N}}\right)^{\frac{1}{\delta}}$. Next, for any $\nu\in\psi$, s_1, $s_2\in[0,\,1]$, $s_1<s_2$, $0<s_2-s_1<\sigma$, have

$$\begin{aligned}&|B\,\nu\,(s_2) - B\,\nu\,(s_1)|\\ &= \frac{1}{\Gamma_q(\delta)}\Big|\int_0^{s_2}(s_2-q\,s)^{(\delta-1)}\,\rho\,(s,\,\nu\,(s))\,\mathrm{d}_q s - \int_0^{s_1}(s_1-q\,s)^{(\delta-1)}\,\rho\,(s,\,\nu\,(s))\,\mathrm{d}_q s\Big|\\ &= \frac{\mathfrak{N}}{\Gamma_q(\delta)}\Big|\int_0^{s_2}\frac{{}_s\mathfrak{D}_q\,(s_2-s\;)^{(\delta)}}{-\,[\delta]}\mathrm{d}_q s - \int_0^{s_1}\frac{{}_s\mathfrak{D}_q\,(s_1-s\;)^{(\delta)}}{-\,[\delta]}\mathrm{d}_q s\Big|\\ &\leq \frac{\mathfrak{N}}{\Gamma_q(\delta+1)}\Big\{\Big|\int_0^{s_1}{}_s\mathfrak{D}_q\,(s_2-s)^{(\delta)}\,\mathrm{d}_q s - \int_0^{s_1}{}_s\mathfrak{D}_q\,(s_1-s)^{(\delta)}\,\mathrm{d}_q s\Big|\\ &\quad + \Big|\int_{s_1}^{s_2}{}_s\mathfrak{D}_q\,(s_2-s\;)^{(\delta)}\,\mathrm{d}_q s\Big|\Big\}\end{aligned}$$

$$= \frac{\mathfrak{N}}{\Gamma_q(\delta+1)} \Big\{ \big[\, (s_2 - s_1)^{(\delta)} - (s_2)^{(\delta)} + (s_1)^{(\delta)} \,\big] + (s_2 - s_1)^{(\delta)} \Big\}$$
$$= \frac{2\mathfrak{N}}{\Gamma_q(\delta+1)} \, (s_2 - s_1)^{(\delta)} < \frac{2\mathfrak{N}}{\Gamma_q(\delta+1)} \sigma^{\delta}.$$

Therefore, we receive $|B\,\nu\,(s_2) - B\,\nu\,(s_1)| < \epsilon$, for all $s_1,\, s_2 \in [0,\,1]$, toward all $\nu \in \psi$. This demonstrates $B(\psi)$ in χ is an equi-continuous set. Here, the set $B(\psi)$ is the equi-continuous set within χ and uniformly bounded, so that is compact via Arzela-Ascoli theorem.

Eventually, B on ψ is a complete continuous operator. Afterwards, we conclude that in Theorem 3.1, the hypothesis (iii) is achieved.

Assume $\nu \in \chi$ and $\mu \in \psi$ arbitrary way that $\nu = A\,\nu$. Then, by hypothesis (H1) and Lemma 2.6, we have

$$\begin{aligned} |\nu\,(t)| &= |A\,\nu(t) - B\,\mu(t)| = \big|h(t, \nu(t), \max_{0\le\tau\le t} |\nu\,(\tau)|)\big| \, \big|I_q^{\delta} g(t, \mu(t))\big| \\ &\le \big|h(t, \nu(t), \max_{0\le\tau\le t} |\nu\,(\tau)|) - h(t,0,0) + h(t,0,0)\big| \, \big|I_q^{\delta} g(t)\big| \\ &\le \Big(L \Big[\max \big\{ \nu(t),\, \max_{0\le\tau\le t} |\nu\,(\tau)|\, \big\} \Big] + k \Big) \frac{\|g\|_{L^1}}{\Gamma_q\,(\delta+1)} \le L\,(\|\nu\| + k)\, \frac{\|g\|_{L^1}}{\Gamma_q\,(\delta+1)} \end{aligned}$$

where $k = \max_{0\le\tau\le t} |h(t,\,0,\,0)|$.

Thus, $|\nu\,(t)| \le L\,(\|\nu\| + k)\, \frac{\|g\|_{L^1}}{\Gamma_q(\delta+1)}$. Taking supremum over t

$$\|\nu\| \le \frac{Lk \frac{\|g\|_{L^1}}{\Gamma_q(\delta+1)}}{1 - L \frac{\|g\|_{L^1}}{\Gamma_q(\delta+1)}} = \frac{LkT_{\delta,q}}{1 - LT_{\delta,q}}. \tag{9}$$

This demonstrates Theorem 3.1, hypothesis (iii) is satisfied. At last, we have

$$\mathfrak{M} = \|B(\psi)\| = \sup\{\|B\,\nu\|\,,\, \nu \in \psi\} \le \frac{\|g\|_{L^1}}{\Gamma_q\,(\delta+1)}. \tag{10}$$

So, $\mathfrak{M}\gamma \le 1$. Accordingly, in Theorem 3.1, all conditions are satisfied, and thus the operator equation $A\nu\, B\mu = \nu$ has a way of solving, which is ψ. Hence, the BVP (1)–(2) defined on $\mathcal{J}$ has a solution. □

4. An example

Regarding the following q-FHIVP:

$$\begin{cases} \mathfrak{D}_q^{\delta} \left[\dfrac{\nu(t)}{\frac{1}{2} + \frac{1}{4} \tan^{-1}\Big(\max\limits_{0\le\tau\le t} |\nu(\tau)| \Big)} \right] = \cos t, \\ \nu(0) = 0 \end{cases} \tag{11}$$

where $\delta = q = \frac{1}{2}$, $\rho\,(t, \nu\,(t)) = \cos t$,

$$h\big(t, \nu(t), \max_{0\le\tau\le t} |\nu(\tau)|\,\big) = \frac{1}{2} + \frac{1}{4} \tan^{-1} \Big(\max_{0\le\tau\le t} |\nu(\tau)|\, \Big)$$

and $g(t) = 1$.

We get

$$
\begin{aligned}
&|h(t, r_1, r_3) - h(t, r_2, r_4)| \\
&= \Big|\Big(\frac{1}{2} + \frac{1}{4}\tan^{-1}|r_3|\Big) - \Big(\frac{1}{2} + \frac{1}{4}\tan^{-1}|r_4|\Big)\Big| \le \frac{1}{4}\left|\tan^{-1}(|r_3| - |r_4|)\right| \\
&\le \frac{1}{4}\left|\tan^{-1}(|r_3 - r_4|)\right| \le \frac{1}{4}\left|\tan^{-1}(\max(|r_1 - r_2|, |r_3 - r_4|))\right| \\
&\le \frac{1}{4}\max(|r_1 - r_2|, |r_3 - r_4|),
\end{aligned}
$$

because $\tan^{-1} t < t$ as $t > 0$. Also, $|\rho(t, \nu(t))| = |\cos t| \le 1 = g(t)$.

Thus hypotheses (H1) and (H2) hold also for $q = \delta = \frac{1}{2}$, and we can obtain

$$
\Gamma_q(\delta + 1) = \frac{\left(1 - \frac{1}{2}\right)^{\left(\frac{1}{2}\right)}}{\left(1 - \frac{1}{2}\right)^{\frac{1}{2}}} \approx 0.93201317,
$$

$$
\mathfrak{M} \le \frac{\|g\|_{L^1}}{\Gamma_q(\delta + 1)} \approx \frac{1}{4}\Big(\frac{1}{0.93201317}\Big) = 1.07294529, \qquad \mathfrak{M}\gamma \le \frac{1}{4}(1.53647204).
$$

Hence, the q-FHIVP (11) has a solution.

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