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ON THE NONCOMMUTATIVE MAPPING TORUS AND RELATED STRUCTURES

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Abstract. In this paper, the noncommutative mapping torus and some of its related structures are investigated. Some generalizations are also introduced. One of these generalizations is noncommutative mapping torus telescope which is defined for a sequence of C*-morphisms and the other is double mapping which is applied for a couple of C*-morphisms between two C*-algebras. Finally, we generalize the double mapping for a couple of parallel sequences of C*-morphisms in order to introduce the double mapping telescope.

1. Introduction

Noncommutative structures are defined alongside the usual structures in the topology. Starting with a locally compact Hausdorff space X, one can obtain a C*-algebra $C_0(X)$, i.e, the space of continuous complex functions on X, vanishing at infinity. Philosophically, the C*-algebra theory may be regarded as *noncommutative topology*: each topological concept can be formulated in terms of C*-algebras and their properties.

For a continuous map $f\colon X \to X$ on a topological space X, the *mapping torus* is defined by the quotient

$$T_f := \frac{X \sqcup \mathbb{I}}{\sim}$$

by identifying $(x, 0) \sim (f(x), 1)$ for all $x \in X$. Same qoutient structures are defined for a continuous map $f\colon X \to Y$ between topological spaces to obtain *mapping cylinder* and *mapping cone*. The noncommutative for these notions are defined for the C*-morphisms between C*-algebras which are very interesting and rich. In [**2**], we investigated the noncommutative mapping cylinder and noncommutative mapping cone in cofibration point of view and extended our study for a sequence of C*-morphisms which are called *noncommutative mapping telescope*. We also considered some aspects of K-groups and noncommutative CW-complex [**3, 5**] for these generalized structures.

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In this paper, we focus on *noncommutative mapping torus* and generalize the notion to several structures. The K-theory of noncommutative mapping torus was completely studied in [**1**] and [**4**] in different ways. Our main purpose is to distribute the notion of noncommutative mapping cone to general cases.

We start by preliminaries and continue with noncommutative mapping torus. In the last section, we introduce the generalized structures.

2. Preliminaries

Throughout this paper, $\mathbb{I}$ denotes the interval $[0,1]$, $\mathbb{S}^1$ the unit circle, $\mathbb{I}A$ ($\mathbb{S}^1 A$) denotes the space of all continuous function from $\mathbb{I}$ ($\mathbb{S}^1$) to a C*-algebra A, and ε_t is the evaluation map $f \mapsto f(t)$ from $\mathbb{I}A$ to A.

We recall that a *pullback* for the C*-algebra C via C*-morphisms $\alpha_1\colon A_1 \to C$, and $\alpha_2\colon A_2 \to C$ is the C*-subalgebra $PB(C,\alpha_1,\alpha_2)$ of $A_1 \oplus A_2$ defined by

$$PB(C,\alpha_1,\alpha_2) := \{a_1 \oplus a_2 \in A_1 \oplus A_2 | \alpha_1(a_1) = \alpha_2(a_2)\}.$$

When α_1, α_2, and C are understood, we usually denote the pullback by PB. Since α_1 and α_2 are continuous maps, PB is closed in $A_1 \oplus A_2$ and so it is a C*-algebra. From this definition it follows that the pullback commutes the following diagram

$$\begin{array}{ccc} PB & \xrightarrow{\pi_2} & A_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_2} \\ A_1 & \xrightarrow{\alpha_1} & C, \end{array}$$

where π_1 and π_2 are projections onto the first and second coordinates, respectively. For a C*-algebra A, the *cone* over A, and the *suspension* of A, respectively, are defined by

$$CA := \{f \in \mathbb{I}A | f(0) = 0\},$$
$$SA := \{f \in \mathbb{I}A | f(0) = f(1) = 0\}.$$

By point-wise operations and supremum norm, CA and SA are C*-algebras. It is easy to show that CA is contractible, and SA is contractible only if A is contractible.

Let $\alpha\colon A \to B$ be a morphism between C*-algebras. The *mapping Cone* C_α and *mapping cylinder* Z_α, respectively, are defined by

$$C_\alpha := \{a \oplus f \in A \oplus CB | f(1) = \alpha(a)\},$$
$$Z_\alpha := \{a \oplus f \in A \oplus \mathbb{I}B | f(1) = \alpha(a)\}.$$

One can show that A is a deformation retraction of Z_α [**2**]. C_α and Z_α are pull-back of B through the morphisms which are presented in the following diagrams.

$$\begin{array}{ccc} C_\alpha & \xrightarrow{\pi_2} & CB \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_1} \\ A & \xrightarrow{\alpha} & B \end{array} \qquad\qquad \begin{array}{ccc} Z_\alpha & \xrightarrow{\pi_2} & \mathbb{I}B \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_1} \\ A & \xrightarrow{\alpha} & B \end{array}$$

These C*-algebras are related in following exact sequences:

$$0 \longrightarrow SA \xrightarrow{\ i\ } CA \xrightarrow{\ \varepsilon_1\ } A \longrightarrow 0,$$

$$0 \longrightarrow SB \xrightarrow{\ j\ } C_\alpha \xrightarrow{\ \pi_A\ } A \longrightarrow 0,$$

$$0 \longrightarrow C_\alpha \xrightarrow{\ i\ } Z_\alpha \xrightarrow{\ \pi_B\ } B \longrightarrow 0,$$

where i is the inclusion map, $j(f) = 0 \oplus f$, $\pi_A(a \oplus f) = a$, and $\pi_B(a \oplus f) = f(0)$. Every sequence induces a six term exact sequence of K-groups. We recall that if $0 \to J \to A \to A/J \to 0$ is an exact sequence of C*-algebras, then we have a cyclic six term exact sequences as follows [**6**]:

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J), \end{array} \tag{1}$$

where K_0 and K_1 are the K-groups of C*-algebras. Another useful result about K-groups is the following lemma.

Lemma 2.1. *Let* $\alpha\colon A \to B$ *be a surjective morphism with kernel* J. *Then* $K_i(J) \simeq K_i(C_\alpha),\ \ i = 0, 1.$

Proof. Let $\beta\colon J \longrightarrow C_\alpha$ be the morphism $i \mapsto j \oplus 0$ and $\pi_2\colon C_\alpha \longrightarrow CB$ be the obvious projection $a \oplus f \mapsto f$. Then the sequence

$$0 \longrightarrow J \xrightarrow{\ \beta\ } C_\alpha \xrightarrow{\ \pi_2\ } CB \longrightarrow 0$$

is exact and we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & 0 & \longrightarrow & J & \longrightarrow & A & \xrightarrow{\ \alpha\ } & B \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & SB & \longrightarrow & C_\alpha & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\pi_2} & & & & \\ & & & & CB & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

with exact rows and columns. Since $K_0(CB) = 0$, it follows that $\beta*\colon K_0(J) \longrightarrow K_0(C_\alpha)$ is surjective.

Now consider the mapping cone C_β for $\beta\colon J \longrightarrow C_\alpha$. The sequence

$$0 \longrightarrow CJ \longrightarrow C_\beta \longrightarrow SCB \longrightarrow 0$$

is exact. Since $K_0(CJ) = 0 = K_0(SCB)$, we conclude that $K_0(C_\beta) = 0$. On the other hand, we have the exact sequence

$$0 \longrightarrow C_\beta \xrightarrow{i} Z_\beta \xrightarrow{\pi_{C_\alpha}} C_\alpha \longrightarrow 0,$$

where i is the inclusion map and $\pi_{C_\alpha}(a \oplus g) = g(0)$. It follows that the sequence

$$K_0(C_\beta) \xrightarrow{\pi_J *} K_0(J) \xrightarrow{\beta *} K_0(C_\alpha)$$

is exact. Since $K_0(C_\beta) = 0$, we conclude that $\beta *\colon K_0(J) \longrightarrow K_0(C_\alpha)$ is injective. It completes our argument and $\beta *$ is an isomorphism. The same argument is applied for $K_1(J) \simeq K_1(C_\alpha)$. □

3. Mapping torus

Definition 3.1. Let α be an automorphism of a C*-algebra A. We define the *mapping torus* for α, by

$$T_\alpha := \{f \in \mathbb{I}A | f(1) = \alpha(f(0))\}.$$

Note that for $f \in SA$, we have $f(1) = 0 = \alpha(0) = \alpha(f(0))$, namely $f \in T_\alpha$, which means $SA \subseteq T_\alpha$. In fact, one can show that SA is an ideal of T_α.

Theorem 3.2. *Let $\varepsilon_0 \colon T_\alpha \to A$ be evaluation in 0. Then $K_1(A) \simeq K_0(C_{\varepsilon_0})$.*

Proof. First we show that ε_0 is surjective. For $a \in A$, define the function $f \in \mathbb{I}A$ by

$$f(t) = (1 - t)a + t\alpha(a).$$

Then $f(0) = a$ and $f(1) = \alpha(a)$. So we have $\alpha(f(0)) = \alpha(a) = f(1)$. Hence $f \in T_\alpha$ and $\varepsilon_0(f) = f(0) = a$.
Then we compute the kernel of ε_0.

$$\begin{aligned} \mathrm{Ker}(\varepsilon_0) &= \{f \in T_\alpha | f(0) = 0\} \\ &= \{f \in \mathbb{I}A | f(0) = 0, \alpha(f(0)) = 0\} \\ &= \{f \in \mathbb{I}A | f(0) = 0, f(1) = 0\} \\ &= SA. \end{aligned}$$

Now Lemma 2.1 implies $K_0(SA) \simeq K_0(C_{\varepsilon_0})$. But we know that $K_0(SA) = K_1(A)$, so the proof is completed. □

Corollary 3.3. *There is a six term exact sequence*

$$\begin{array}{ccccc} K_0(SA) & \longrightarrow & K_0(T_\alpha) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(T_\alpha) & \longleftarrow & K_1(A). \end{array}$$

Proof. The sequence

$$0 \longrightarrow SA \xrightarrow{i} T_\alpha \xrightarrow{\varepsilon_0} A \longrightarrow 0$$

is exact, where i is the inclusion. The corresponding six term exact sequence follows immediately from (1). □

Proposition 3.4. *If A is contractible, then $K_0(T_\alpha)$ and $K_1(T_\alpha)$ are trivial.*

Proof. SA is contractible because A is contractible. So we have $K_0(SA) = \{0\}$ and $K_1(SA) = \{0\}$. The six term exact sequence in previous corrolary will be

$$\begin{array}{ccccc} \{0\} & \longrightarrow & K_0(T_\alpha) & \longrightarrow & \{0\} \\ \uparrow & & & & \downarrow \\ \{0\} & \longleftarrow & K_1(T_\alpha) & \longleftarrow & \{0\}. \end{array}$$

By exactness, we have $K_0(T_\alpha) = \{0\} = K_1(T_\alpha)$. □

In [**2**], we generalized the notions Z_α and C_α for a sequence of C*-morphisms

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$$

to introduce *NC mapping cylindrical telescope*

$$\begin{aligned} T_n := \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus \mathbb{I}A_2 \oplus \cdots \oplus \mathbb{I}A_{n+1} | \\ f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a),\ k = 1, 2, \ldots, n\} \end{aligned}$$

and *NC mapping conical telescope*

$$\begin{aligned} T_nC := \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} | \\ f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n\}. \end{aligned}$$

A similar process can be used for T_α as follows.

Definition 3.5. Let A be a C*-algebra and

$$A \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A \xrightarrow{\alpha_n} A$$

be a sequence of C*-morphisms. The *NC mapping torus telescope* is defined by

$$T_T := \{f_1 \oplus \cdots \oplus f_n | f_k \in \mathbb{I}A, f_k(1) = \alpha_k \circ \cdots \circ \alpha_1(f(0)), k = 1, \ldots n\}.$$

For $n = 1$, the NC mapping torus telescope is just T_{α_1}. Note that each f_k is a member of $T_{\alpha_k \circ \cdots \circ \alpha_1}$ (this is the reason for term "telescope"). Hence like T_α, one can apply similar theorems to T_T.

4. Generalization

In this section, we define a construction for which the constructions Z_α, C_α, and T_α are special cases.

Definition 4.1. Let $\alpha, \beta\colon A \to B$ are C*-morphisms. We define *double mapping* for α and β

$$Z_{\alpha\beta} := \{a \oplus f \in A \oplus \mathbb{I}B | f(1) = \alpha(a)\ , f(0) = \beta(a)\}.$$

Note that $Z_{\alpha\beta} \subseteq Z_\alpha$. In fact, one can define the double mapping by

$$Z_{\alpha\beta} := \{a \oplus f \in Z_\alpha | f(0) = \beta(a)\}.$$

It follows from Definition 4.1 that

$$Z_{\alpha\beta} = PB(B, \varepsilon_1, \alpha) \cap PB(C, \varepsilon_0, \beta).$$

Since PB is closed, $Z_{\alpha\beta}$ is a C*-algebra. We also notice that $Z_{\alpha\beta}$ satisfies in both following commutative diagrams:

$$\begin{array}{ccc} Z_{\alpha\beta} & \xrightarrow{\pi_2} & \mathbb{I}B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \varepsilon_1} \\ A & \xrightarrow{\alpha} & B \end{array} \qquad\qquad \begin{array}{ccc} Z_{\alpha\beta} & \xrightarrow{\pi_2} & \mathbb{I}B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\ A & \xrightarrow{\beta} & B \end{array}$$

Simple calculations show that $Z_{\alpha\beta}$ can be regarded as previous structures in some cases. For example, if β is the identity (and hence $A = B$), then the following calculations show that $Z_{\alpha\beta} \simeq T_\alpha$:

$$\begin{aligned} Z_{\alpha\beta} &= \{a \oplus f \in A \oplus \mathbb{I}A | f(1) = \alpha(a)\ , f(0) = \beta(a)\} \\ &= \{a \oplus f \in A \oplus \mathbb{I}A | f(1) = \alpha(a)\ , f(0) = a\} \\ &= \{a \oplus f \in A \oplus \mathbb{I}A | f(1) = \alpha(f(0))\ , f(0) = a\} \\ &= \{a \oplus f \in A \oplus T_\alpha | f(0) = a\} \\ &= \{f(0) \oplus f | f \in T_\alpha\} \\ &\simeq T_\alpha. \end{aligned}$$

By similar argument one can prove the next proposition.

Proposition 4.2. *Let $Z_{\alpha\beta}$ be in Definition 4.1.*

1. *If $\beta = 0$, then $Z_{\alpha\beta} = C_\alpha$.*
2. *If both α and β are identity, then $Z_{\alpha\beta} \simeq \mathbb{S}^1 A$.*
3. *If $\alpha = \beta = 0$, then $Z_{\alpha\beta} = A \oplus SB$.*

If $\alpha, \beta\colon A \to B$ are simplicial morphisms between noncommutative CW-complexes, then all cases in Proposition 4.2 are noncommutative CW-complexes [**2, 5**]. The overall mode is complicated and is not currently known.

Theorem 4.3. *There is an exact sequence*

$$0 \longrightarrow SB \longrightarrow Z_{\alpha\beta} \longrightarrow A \longrightarrow 0.$$

Proof. Define $i\colon SB \to Z_{\alpha\beta}$ by $i(f) = 0 \oplus f$ and $\pi\colon Z_{\alpha\beta} \to A$ by $\pi(a \oplus f) = a$. It is easy to show that $\mathrm{Ker}(\pi) = \mathrm{Im}(i)$. □

Corollary 4.4. *If A and B are contractible C*-algebras, then $K_0(Z_{\alpha\beta}) = \{0\} = K_1(Z_{\alpha\beta})$.*

Proof. The proof is just the same as in Proposition 3.4, by exactness of the six term exact sequence in previous theorem. □

Definition 4.5. For two sequences of C*-algebras

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1},$$

$$A_1 \xrightarrow{\beta_1} A_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n-1}} A_n \xrightarrow{\beta_n} A_{n+1},$$

the *double mapping telescope* is defined by

$$\begin{aligned} Z_{\alpha_i\beta_i}(n) := \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} | a \in A_1, f_{k+1} \in \mathbb{I}A_{k+1},\\ f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a),\\ f_{k+1}(0) = \beta_k \circ \beta_{k-1} \circ \cdots \circ \beta_1(a),\\ k = 1, 2, \ldots, n\}. \end{aligned}$$

Note that $Z_{\alpha_i\beta_i}(n)$ is a closed subalgebra of the C*-algebra

$$A_1 \oplus \mathbb{I}A_2 \oplus \cdots \oplus \mathbb{I}A_{n+1}$$

because all α_ks are continuous. For $n = 1$, we have $Z_{\alpha_i\beta_i}(1) = Z_{\alpha\beta}$. If both sequences are exact then we have

$$Z_{\alpha_i\beta_i}(n) = Z_{\alpha_1\beta_1} \oplus SA_3 \oplus \cdots \oplus SA_{n+1},$$

if one of them (for example β_is) be exact then

$$Z_{\alpha_i\beta_i}(n) \subseteq Z_{\alpha_1\beta_1} \oplus CA_3 \oplus \cdots \oplus CA_{n+1}.$$

Theorem 4.6. *There is an exact sequence*

$$0 \longrightarrow \bigoplus_{i=2}^{n+1} SA_i \longrightarrow Z_{\alpha_i\beta_i}(n) \longrightarrow A_1 \longrightarrow 0. \tag{2}$$

Proof. Let $i\colon SA_2 \oplus \cdots \oplus SA_{n+1} \longrightarrow Z_{\alpha_i\beta_i}(n)$ be the inclusion

$$f_2 \oplus \cdots \oplus f_{n+1} \longmapsto 0 \oplus f_2 \oplus \cdots \oplus f_{n+1}$$

and $\pi\colon Z_{\alpha_i\beta_i}(n) \longrightarrow A_1$ be the projection on the first component. It is easy to show that $\mathrm{Ker}(\pi) = \mathrm{Im}(i)$. □

One can use the exact sequence 2 to obtain the corresponding six term exact sequence. It follows that if A_ks are contractible, then $K_0(Z_{\alpha_i\beta_i}(n))$ and $K_1(Z_{\alpha_i\beta_i}(n))$ are trivial (cf. Corollary 4.4).

5. More remarks

Noncommutative structures such as C_α and Z_α are cofiber [**2, 6**]. An interesting question can be raised about $Z_{\alpha\beta}$: $Z_{\alpha\beta}$ be regarded as a cofiber of a cofibration? In some cases, the answer is clear. For example, if $\beta = 0$, then $Z_{\alpha\beta} = C_\alpha$ and the answer is true. But general case needs more investigation. Every original research about $Z_{\alpha\beta}$ can be translated to particular cases such as C_α, Z_α, and T_α.

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