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CHARACTERIZING TOPOLOGICAL QUASI-BOOLEAN ALGEBRAS FROM WEAKLY TOPOLOGICAL QUASI-BOOLEAN ALGEBRAS

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Abstract. In this paper, we characterize the variety TQBA (Topological Quasi-Boolean algebra) from the variety WTQBA (Weakly Topological Quasi-Boolean Algebra).

1. Introduction

The main idea of rough set theory was to develop a Mathematical formalism to deal with vagueness which arised due to the lack of information. Though rough set proposed by Pawlak [**21**] has wide range of applications, its algebraic approach has been quite interesting since its initiation by Iwinski in 1987 [**12**]. Later many authors studied the algebraic structures of rough sets determined by an equivalence relation and arbitrary binary relations [**2, 5, 7, 8, 11, 13, 14, 15, 19, 22**]. One can find a survey on the algebras of rough sets in [**20**]. The notion of a topological quasi-Boolean algebra arose in the context of rough set theory while studying rough equality within the framework of the modal system S5 by Mohua Banerjee and Chakaraborty in [**2**].

A similar study of rough equality within the framework of modal system S4 was carried out in [**18**] and an algebraic structure of rough sets system determined by quasi order was obtained as semantic counter part of the Lindenbaum-Tarski like construction. It does not satisfy an axiom of topological quasi-Boolean algebra. From abstraction of the properties of that algebraic structure, we obtain the notion of weakly topological quasi-Boolean algebra. Similar algebraic structure was named Tarski interior lattice in [**6**].

Let us denote the class of all topological quasi-Boolean algebras by TQBA and the class of all weakly topological quasi-Boolean algebras by WTQBA. In this paper, we prove that WTQBA is a variety. We also characterize TQBA from WTQBA.

For notations, definitions and results not given here, we refer readers to [**4, 9, 10**].

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2. Weakly Topological Quasi-Boolean Algebras

Let us begin with the formal definition of a topological quasi-Boolean algebra.

Definition 2.1 ([**2**])**.** An algebra $\mathcal{A} = (A, \vee, \wedge, \neg, L, 0, 1)$ is said to be a topological quasi-Boolean algebra if $\mathcal{A}$ satisfies the following statements

1. $(A, \vee, \wedge, \neg, 0, 1)$ is a quasi-Boolean algebra ([**23**] or De Morgan algebra).
2. The unary operator L satisfies the following axioms: for every $a, b \in A$,
   - (T1) $La \leq a$,
   - (T2) $LLa = La$,
   - (T3) $L(a \wedge b) = La \wedge Lb$,
   - (T4) $L1 = 1$.

Definition 2.2 ([**18**])**.** An algebra $(A, \vee, \wedge, \neg, L, 0, 1)$ is said to be a weakly topological quasi-Boolean algebra if it satisfies the following statements

1. $(A, \vee, \wedge, \neg, 0, 1)$ is a quasi-Boolean algebra.
2. The unary operator L satisfies the following axioms: for every $a, b \in A$,
   - (WT1) $La \leq a$,
   - (WT2) $LLa = La$,
   - (WT3) $a \leq b \Rightarrow La \leq Lb$,
   - (WT4) $L1 = 1$.

An algebra $(L, \vee, \wedge, ^*, 0, 1)$ is called a distributive p-algebra if $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $x \leq a^*$ iff $a \wedge x = 0$ for all x. An algebra $(L, \vee, \wedge, ^+, 0, 1)$ is called a distributive dual p-algebra if $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a^+ \leq x$ iff $a \vee x = 1$ for all x. Let $(D_1, \vee_1, \wedge_1, ^*, 0, 1)$ be a distributive p-algebra and $(D_2, \vee_2, \wedge_2, ^+, 0', 1')$ be a distributive dual p-algebra such that $D_1 \cong D_2^\partial$, where D_2^∂ is the dual of a lattice D_2 which is obtained by swapping $\vee$ and $\wedge$ in D_2. An element a is said to be dense in D_1 if $a^* = 0$. Let F be any filter in D_1 containing all the dense elements of D_1. Then we get a weakly topological quasi-Boolean algebra, $(\langle [D_1, D_2], F\rangle, \sqcup, \sqcap, \sim, L, \mathbf{0}, \mathbf{1})$ [**17**], where $\langle [D_1, D_2], F\rangle = \{(a, b) \in D_1 \times D_2 : a \wedge_1 \psi(b) = 0 \text{ and } a \vee_1 \psi(b) \in F\}$ and for any (a, b) and $(c, d) \in \langle [D_1, D_2], F\rangle$,

$$(a, b) \sqcup (c, d) = (a \vee_1 c, b \vee_2 d),$$

$$(a, b) \sqcap (c, d) = (a \wedge_1 c, b \wedge_2 d),$$

$$\sim (a, b) = (\psi(b), \varphi(a)),$$

$$L(a, b) = (a, \varphi(a^*)),$$

$$\mathbf{0} = (0, 0'), \qquad \mathbf{1} = (1, 1'),$$

where ψ and φ are the dual isomorphisms between D_1 and D_2. If D_1, D_2 and F are as above, then $\langle [D_1, D_2], F\rangle$ is a topological quasi-Boolean algebra if and only if D_1 is a Stone algebra.

It was shown in [**17**] that the rough sets system determined by quasi order R, $\mathcal{R}^* = \{(X^\blacktriangledown, X^\blacktriangle) : X \subseteq U\}$ can be represented as $\{(X,Y) \in \mathcal{D}_R \times \mathcal{D}^R : X \cap \psi(Y) = \phi$ and $X \cup \psi(Y) \in F\}$, where F is the principal filter generated by $B = \{x \in U : R(x) = \{x\}\}$ in $\mathcal{D}_R$. The dual isomorphisms φ, ψ between $\mathcal{D}_R = \{X^\blacktriangledown : X \subseteq U\}$ and $\mathcal{D}^R = \{X^\blacktriangle : X \subseteq U\}$ are given by $\varphi(X^\blacktriangledown) = X^{c\blacktriangle}$ and $\psi(X^\blacktriangle) = X^{c\blacktriangledown}$. A similar representation of rough sets system determined by a quasi order was given by Järvinen and his co-authors in [**16**]. By the above construction, it was proved that $(\mathcal{R}^*, \sqcup, \sqcap, \sim, L, (\phi,\phi), (U,U))$ is a weakly topological quasi-Boolean algebra [**18**] and the operators are defined as follows:

$$\begin{aligned}
(A^\blacktriangledown, A^\blacktriangle) \sqcup (B^\blacktriangledown, B^\blacktriangle) &= (A^\blacktriangledown \cup B^\blacktriangledown, A^\blacktriangle \cup B^\blacktriangle),\\
(A^\blacktriangledown, A^\blacktriangle) \sqcap (B^\blacktriangledown, B^\blacktriangle) &= (A^\blacktriangledown \cap B^\blacktriangledown, A^\blacktriangle \cap B^\blacktriangle),\\
\sim (A^\blacktriangledown, A^\blacktriangle) &= ((A^c)^\blacktriangledown, (A^c)^\blacktriangle),\\
L(A^\blacktriangledown, A^\blacktriangle) &= (A^\blacktriangledown, \varphi((A^\blacktriangledown)^*)) = (A^\blacktriangledown, A^{\blacktriangledown\blacktriangle}).
\end{aligned}$$

Every topological quasi-Boolean algebra is a weakly topological quasi-Boolean algebra because (T3) implies (WT3). But the converse is not true.

An example of a weakly topological quasi-Boolean algebra but not a topological quasi-Boolean algebra is shown in Figure 1.

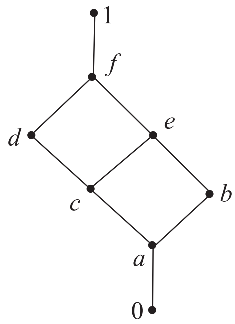


Figure 1.

The operators $\neg$ and L of the algebraic structure in Figure 1 are as follows:

$$\neg 0 = 1, \quad \neg a = f, \quad \neg c = e, \quad \neg b = d, \quad \neg e = c, \quad \neg d = b, \quad \neg f = a, \quad \neg 1 = 0,$$

$$L0 = La = Lc = 0, \qquad Le = Lb = b, \qquad Ld = d, \qquad Lf = f, \qquad L1 = 1.$$

In [**3**], it is shown that TQBA is a variety. Similarly, we can check that WTQBA is closed under homomorphisms, subalgebras and direct products. Hence it leads to the following theorem.

Theorem 2.3. *WTQBA is a variety of algebras.*

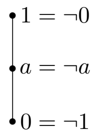
In fact, WTQBA is a subvariety of TQBA.

In an attempt to characterize TQBA from WTQBA, we try to find the minimal subdirectly irreducible elements of the class $\Omega = WTQBA \setminus TQBA$. An element of the class Ω must be a weakly topological quasi-Boolean algebra $\mathcal{A} =$

$(A, \vee, \wedge, \neg, L, 0, 1)$ and the unary operator L does not satisfy the axiom (T3). That is, it satisfies the axioms (WT1)–(WT4) but not the axiom (T3).

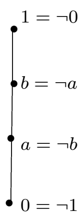
3. Minimal Subdirectly Irreducible Elements of Ω

First, we consider the smallest non-trivial quasi-Boolean algebras:
1. Three-element quasi-Boolean algebra

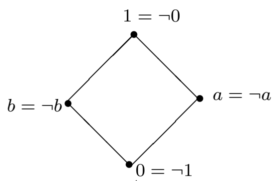


$\mathcal{A}^3$

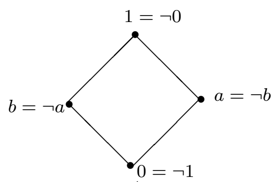
2. Four-element quasi-Boolean algebras



$\mathcal{A}_1^4$

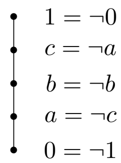


$\mathcal{A}_2^4$



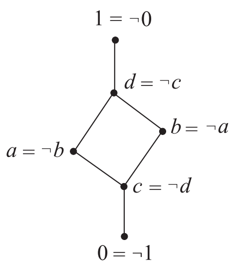
$\mathcal{A}_3^4$

3. Five-element quasi-Boolean algebra

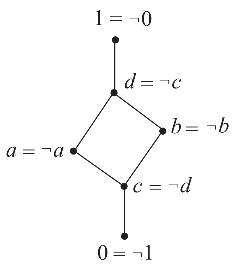


$\mathcal{A}^5$

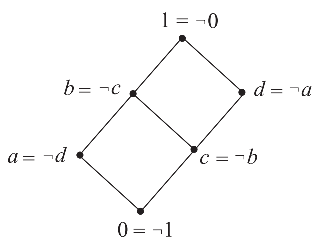
4. Six-element quasi-Boolean algebras



$\mathcal{A}_1^6$



$\mathcal{A}_2^6$



$\mathcal{A}_3^6$

Now we can identify the elements of Ω from the above quasi-Boolean algebras by imposing the unary operator L satisfying the axioms (WT1)–(WT4) but not the axiom (T3). If any two elements are comparable, i.e., $a \leq b$, then $La \leq Lb$

by the axiom (WT3). So, $L(a \wedge b) = La \wedge Lb$. Therefore, we have the following observation.

Observation 3.1. Any unary operation L satisfying the axioms (WT1)–(WT4) on chain distributes over $\wedge$, i.e., it satisfies the axiom (T3).

So, we do not consider chains. Hence $\mathcal{A}^3, \mathcal{A}^5, \mathcal{A}^4_1 \notin \Omega$. In the case of $\mathcal{A}^4_2$ and $\mathcal{A}^4_3$, all possible unary operators L on $\mathcal{A}^4_2$ and $\mathcal{A}^4_3$ satisfying the axioms (WT1)–(WT4) also satisfy the axiom (T3). So the algebras $\mathcal{A}^4_2$ and $\mathcal{A}^4_3$ are ruled out. Next, we consider the case of six element quasi-Boolean algebras. Since $\mathcal{A}^6_1$ and $\mathcal{A}^6_2$ have the same underlying lattice, we use the notation $\mathcal{A}^6_{1,2}$ to denote the underlying lattice of these algebras.

Proposition 3.2. *The only possible unary operator L on $\mathcal{A}^6_{1,2}$ satisfying the axioms* (WT1)–(WT4) *but not the axiom* (T3) *is defined by $L0 = 0$, $Lc = 0$, $La = a$, $Lb = b$, $Ld = d$, $L1 = 1$.*

Proof. From the axioms (WT1)–(WT4), $L0 = 0$ and $L1 = 1$. Because the unary operator L does not satisfy the axiom (T3), there exist two elements $x, y \in \mathcal{A}^6_{1,2}$ such that $L(x \wedge y) \neq Lx \wedge Ly$. Since every comparable pair of elements in $\mathcal{A}^6_{1,2}$ satisfies the axiom (T3), the only possible pair of elements in $\mathcal{A}^6_{1,2}$ which does not satisfy the axiom (T3) must be the non-comparable elements, i.e., a and b. Then $L(a \wedge b) \neq La \wedge Lb$. Now, let us consider the possible cases.

Case (i): $La \neq a$, $Lb = b$ (the symmetric case $Lb \neq b, La = a$ is similar).

- (a) $La = c, Lc = c$. Then $L(a \wedge b) = Lc = c = c \wedge b = La \wedge Lb$. Thus (T3) holds.
- (b) $La = c, Lc = 0$. Then $LLa \neq La$ and (WT2) fails.
- (c) If $La = 0$, then $Lc = 0$ by (WT3). Then $L(a \wedge b) = Lc = 0 = La \wedge Lb$.

Case (ii): $La \neq a$, $Lb \neq b$.

- (a) $La = c, Lb = c$. Thus, in view of (WT2), $Lc = c$. Therefore, by easy calculation, (T3) holds.
- (b) $La = 0, Lb = c$ (the symmetric case is similar). Since $Lb = c$, in view of (WT2), $Lc = c$. But in this case, $La < Lc$ which contradicts (WT3).
- (c) $La = 0, Lb = 0$. Then $Lc = 0$ and by easy calculation, one verifies that (T3) holds.

Thus, it remains only the case $La = a$ and $Lb = b$. If $Lc = c$, then (T3) holds. It follows that it must be $Lc = 0$. Moreover, if $Ld \lneq d$, then $Ld = a$ because $La = a$ and (WT3). But in this case, (WT3) does not hold, because $b \leq d$ but $Lb \nleq Ld$, which is a contradiction (the other case with $Ld = b$ is similar). Therefore, $Ld = d$.

Hence the only possible unary operator L on $\mathcal{A}^6_{1,2}$ satisfying the axioms (WT1)–(WT4) but not the axiom ($T3$) is defined by $L0 = 0$, $Lc = 0$, $Lb = b$, $La = a$, $Ld = d$, $L1 = 1$. $\square$

In the case of an algebra $\mathcal{A}^6_3$, one can prove analogously that L_2 and L_3 defined by $L_2 0 = 0$, $L_2 a = a$, $L_2 c = 0$, $L_2 b = b$, $L_2 d = d$, $L_2 1 = 1$ and $L_3 0 = 0$, $L_3 a = 0$, $L_3 c = 0$, $L_3 b = b$, $L_3 d = d$, $L_3 1 = 1$ are the only possible unary operators on $\mathcal{A}^6_3$ satisfying the axioms (WT1)–(WT4) but not the axiom (T3).

Theorem 3.3 ([4]). *An algebra* **A** *is subdirectly irreducible iff* **A** *is trivial or there exists the least congruence among its nontrivial congruences.*

The congruence lattices of the algebras $(\mathcal{A}_1^6, L_1)$, $(\mathcal{A}_2^6, L_1)$, where the unary operator L_1 is defined as in the Proposition 3.2, $(\mathcal{A}_3^6, L_2)$ and $(\mathcal{A}_3^6, L_3)$ are shown in Figure 2.

Con $(\mathcal{A}_1^6, L_1)$

Con $(\mathcal{A}_2^6, L_1)$, **Con** $(\mathcal{A}_3^6, L_2)$, **Con** $(\mathcal{A}_3^6, L_3)$

Figure 2.

It is evident from the above Hasse diagrams and Theorem 3.3 that the following theorem holds

Theorem 3.4. *$(\mathcal{A}_1^6, L_1)$, $(\mathcal{A}_2^6, L_1)$, $(\mathcal{A}_3^6, L_2)$ and $(\mathcal{A}_3^6, L_3)$ are the minimal subdirectly irreducible algebras in Ω.*

An algebra **A** is congruence-distributive if **Con A** is a distributive lattice. Therefore, the algebras $(\mathcal{A}_1^6, L_1)$, $(\mathcal{A}_2^6, L_1)$, $(\mathcal{A}_3^6, L_2)$ and $(\mathcal{A}_3^6, L_3)$ are congruence-distributive.

Since the unary operation $\neg$ has no influence over the operation L, the elements of Ω are characterized by the sublattice which is closed under the unary operation L. We shall denote the following lattice structure together with the defined unary operation L by NT_5.

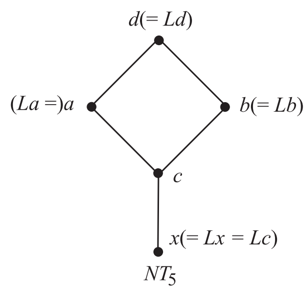


Figure 3.

All the minimal subdirectly irreducible algebras of Ω embed a copy of NT_5.

Theorem 3.5. *Let $\mathcal{A}=(A,\vee,\wedge,\neg,L,0,1)$ be a weakly topological quasi-Boolean algebra. Then $A\in\Omega$ if and only if it embeds a copy of NT_5.*

Proof. Suppose $\mathcal{A}$ embeds a copy of NT_5, then obviously $A \in \Omega$ (since $L(a \wedge b) \neq La \wedge Lb$). Conversely, assume that $\mathcal{A} \in \Omega$. Then there exist $a, b \in A$ such that $L(a \wedge b) \neq La \wedge Lb$. As discussed earlier, $a \nparallel b$. Let us assume that $La = p$ and $Lb = q$ for some $p, q \in A$. Since $LLa = La$ and $LLb = Lb$, $Lp = p$ and $Lq = q$. Let $a \wedge b = c$ and $p \wedge q = z$. Now, $z = p \wedge q = La \wedge Lb \leq a \wedge b = c$. Thus $z \leq c$. Since $Lc \lneq La \wedge Lb = p \wedge q = z$, there exists $x \in A$ such that $x \lneq z$ and $Lc = x$. We have that $LLc = Lc$, $Lx = x$. $x < z \leq c$ imply $x = Lx \leq Lz \leq Lc = x$. Therefore, $Lz = x$. Let $d = p \vee q$. We have that $p = Lp \leq Ld$ and $q = Lq \leq Ld$ imply $d = p \vee q \leq Ld$. Also, we have $Ld \leq d$. Therefore, $Ld = d$. Thus the elements of the set $N = \{x, z, p, q, d\}$ with their unary operation L defined as $Lx = x, Lz = x, Lp = p, Lq = q, Ld = d$ must be a homomorphic image of NT_5. It is enough to show that $p \nparallel q$ and all the five elements of N are distinct. If $p \parallel q$, then $p \leq q$ or $q \leq p$. If $p \leq q$, then $p \wedge q = p$. This implies $p = z$ which implies $p = Lp = Lz = x$. That is, $x = z$, which is a contradiction to $x < z$. Similar contradiction also occurs when $q \leq p$. Therefore $p \nparallel q$. If any other two elements of N are equal, then that also leads to a contradiction $x < z$. Hence all elements of N are distinct in A and they form NT_5. □

Corollary 3.6. *A weakly topological quasi-Boolean algebra* $\mathcal{A} = (A, \vee, \wedge, \neg, L, 0, 1)$ *is a topological quasi-Boolean algebra if and only if it embeds no copy of* NT_5.

4. Conclusion

In this paper, we have presented the minimal subdirectly irreducible algebras of the class $\Omega = WTQBA \setminus TQBA$. We have characterized the class of topological quasi-Boolean algebras from the class of weakly topological quasi-Boolean algebras by the structure NT_5. We have a stronger algebra than the topological quasi-Boolean algebra in the context of rough sets system determined by an equivalence relation called rough algebra [**1**] and another algebra named Q-rough algebra [**18**] stronger than weakly topological quasi-Boolean algebra in the context of rough sets system determined by a quasi order. In future, we wish to characterize the class of rough algebras from the class of Q-rough algebras. This will help us to characterize the rough sets system determined by an equivalence relation from the rough sets system determined by quasi order.

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