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OSCILLATORY BEHAVIOROF NONLINEAR ADVANCED DIFFERENTIAL EQUATIONS WITH A NON-MONOTONE ARGUMENT

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Abstract. Consider the first-order nonlinear advanced differential equation

$$x'(t) - p(t)f\left(x\left(\tau(t)\right)\right) = 0, \quad t \geq t_0,$$

where $p(t)$ is nonnegative function on $\mathbb{R}$ and $\tau(t)$ is non-monotone or nondecreasing function such that $\tau(t) \geq t$ for $t \geq t_0$. Under these assumptions we researched oscillatory behaviour of solutions of nonlinear advanced differential equations and we obtain new oscillation criteria, involving limsup and liminf. An example illustruting the result is also given.

1. Introduction

Consider the nonlinear advanced differential equation

$$x'(t) - p(t)f(x(\tau(t))) = 0, \qquad t \geq t_0, \tag{1}$$

where $p(t)$ is nonnegative function on $\mathbb{R}$ and $\tau(t)$ is non-monotone or nondecrasing function such that

$$\tau(t) \geq t \quad \text{for } t \geq t_0 \tag{2}$$

and

$$f \in C(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad xf(x) > 0 \qquad \text{for } x \neq 0. \tag{3}$$

By a solution of (1) we mean continuosly differentiable function defined on $[t_0, \infty)$ and such that (1) satisfied for $t \geq t_0$. Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called non-oscillatory.

Recently, there has been a considerable interest in the study of the oscillatory behaviour of the following special form of (1)

$$x'(t) - p(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{4}$$

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and if $\tau(t) = t + T$, where $T > 0$, then Equation (4) turns into the following differential equation

$$x'(t) - p(t)x(t+T) = 0, \qquad t \geq t_0. \tag{5}$$

In 1982, Ladas and Stavroulakis [**7**] established that if

$$\liminf_{t\to\infty} \int_t^{t+T} p(s)\mathrm{d}s > \frac{1}{\mathrm{e}},$$

then all solutions of (5) are oscillatory. When $p(t) \equiv p \in (0, \infty)$, they also proved that the condition

$$pT > \frac{1}{\mathrm{e}}$$

is necessary and sufficient condition such that

(a) The advanced differential inequality

$$x'(t) - p(t)x(t+T) \geq 0, \qquad t \geq t_0,$$

has no eventually positive solution.

(b) The advanced differential inequality

$$x'(t) - p(t)x(t+T) \leq 0, \qquad t \geq t_0,$$

has no eventually negative solution.

(c) All solutions of (5) are oscillatory.

We can give also Li and Zhu [**9**], Koplatadze and Chanturija [**4**] and Kusano [**6**] and Kulenovic and Grammatikopoulus [**5**] as references for oscillation behaviour of Equation (5).

In 1983, Fukagai and Kusano [**3**] proved that if $\tau(t)$ is nondecreasing and

$$\liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\mathrm{d}s > \frac{1}{\mathrm{e}},$$

then all solutions of (4) are oscillatory, while if

$$\int t^{\tau(t)} p(s)\mathrm{d}s \leq \frac{1}{\mathrm{e}} \qquad \text{for all sufficiently large } t,$$

then Equation (4) has a non-oscillatory solution.

Moreover, in [**3**], the authors proved that the following result; consider the following nonlinear differential equation

$$x'(t) + p(t)f(x(\tau(t))) = 0, \qquad t \geq t_0. \tag{6}$$

Suppose that $p(t) \leq 0$ and $\tau(t) \geq t$ is nondecreasing for $t \geq t_0$. Suppose moreover that

$$M = \limsup_{|x|\to\infty} \frac{|x|}{|f(x)|} < \infty. \tag{7}$$

If

$$\liminf_{t\to\infty} \int_t^{\tau(t)} [-p(s)]\,\mathrm{d}s > \frac{M}{\mathrm{e}}, \tag{8}$$

then all solutions of Equation (6) are oscillatory.

Thus, in this paper, our aim is to obtain new oscillation criteria, involving limsup and liminf, for all solutions of Equation (1) under the assumption that $\tau(t)$ should not necessarily be monotone and there is no restrictions for $f(t)$ related to monotony. As far as we can see, since $\tau(t)$ is non-monotone function, Equation (1) has not yet been studied in the literature with the stated conditions.

2. Main results

In this section, we present a new sufficient conditions for the oscillation of all solutions of Equation (1), under the assumption that $\tau(t)$ is non-monotone or nondecrasing function. Set

$$h(t) := \inf_{s\geq t} \tau(s), \qquad t \geq 0. \tag{9}$$

Obviously, $h(t)$ is nondecreasing and $\tau(t) \geq h(t)$ for all $t \geq 0$.

Suppose that the f in Equation (1) satisfies the following condition

$$\limsup_{|x|\to\infty} \frac{x}{f(x)} = M, \quad 0 \leq M < \infty. \tag{10}$$

Theorem 2.1. *Assume that* (2)*,* (3) *and* (10) *hold. If* $\tau(t)$ *is non-monotone or nondecrasing function and*

$$\liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\mathrm{d}s > \frac{M}{\mathrm{e}}, \tag{11}$$

then all solutions of (1) *oscillate.*

Proof. Assume for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1). Since $-x(t)$ is also a solution of (1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then, there exists a $t_1 \geq t_0$ such that $x(t)$, $x(\tau(t)) > 0$ for all $t \geq t_1$. Thus from (1), we have

$$x'(t) = p(t)f(x(\tau(t))) \geq 0 \quad \text{for all } t \geq t_1.$$

It means that $x(t)$ is nondecreasing. Condition (11) implies that

$$\int_a^\infty p(t)\mathrm{d}t = \infty. \tag{12}$$

In view of (12) and by the [**8**, Theorem 3.1.6] we obtain $\lim_{t\to\infty} x(t) = \infty$. Suppose that $M > 0$. Then, by virtue of (10) we can choose $t_2 \geq t_1$ so large that

$$f(x(t)) \geq \frac{1}{2M} x(t) \qquad \text{for } t \geq t_2. \tag{13}$$

On the other hand, we know from [**10**, Lemma 2.2] (see also [**1**, Lemma 2.2]) that

$$\liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\mathrm{d}s = \liminf_{t\to\infty} \int_t^{h(t)} p(s)\mathrm{d}s. \tag{14}$$

Since $h(t) \leq \tau(t)$, $x(t)$ and $h(t)$ are nondecreasing, by (1) and (13), we have

$$x'(t) - \frac{1}{2M} p(t) x(h(t)) \geq 0, \qquad t \geq t_3. \tag{15}$$

Also, from (11) and (14), it follows that there exists a constant $c > 0$ such that

$$\int_t^{h(t)} p(s) \mathrm{d}s \geq c > \frac{M}{\mathrm{e}}, \qquad t \geq t_3 \geq t_2. \tag{16}$$

So, from (16), there exists a real number $t^* \in (t, h(t))$, for all $t \geq t_3$ such that

$$\int_t^{t^*} p(s) \mathrm{d}s > \frac{M}{2\,\mathrm{e}} \qquad \text{and} \qquad \int_{t^*}^{h(t)} p(s) \mathrm{d}s > \frac{M}{2\,\mathrm{e}}. \tag{17}$$

Integrating (15) from t to t^* and using $x(t)$ and $h(t)$ are nondecreasing, then we have

$$x(t^*) - x(t) - \frac{1}{2M} \int_t^{t^*} p(s) x(h(s)) \mathrm{d}s \geq 0$$

or

$$x(t^*) - x(t) - \frac{1}{2M} x(h(t)) \int_t^{t^*} p(s) \mathrm{d}s \geq 0.$$

Thus, by (17), we have

$$x(t^*) - \frac{1}{2M} x(h(t)) \frac{M}{2\,\mathrm{e}} > 0. \tag{18}$$

Integrating (15) from t^* to $h(t)$ and using $x(t)$ and $h(t)$ are nondecreasing, we obtain

$$x(h(t)) - x(t^*) - \frac{1}{2M} \int_{t^*}^{h(t)} p(s) x(h(s)) \mathrm{d}s \geq 0$$

or

$$x(h(t)) - x(t^*) - \frac{1}{2M} x(h(t^*)) \int_{t^*}^{h(t)} p(s) \mathrm{d}s \geq 0.$$

Thus, by (17), we have

$$x(h(t)) - \frac{1}{2M} x(h(t^*)) \frac{M}{2\,\mathrm{e}} > 0. \tag{19}$$

Combining the inequalities (18) and (19), we obtain

$$x(t^*) > \frac{1}{4\,\mathrm{e}} x(h(t)) > \left(\frac{1}{4\,\mathrm{e}}\right)^2 x(h(t^*))$$

and hence we have

$$\frac{x(h(t^*))}{x(t^*)} < (4\,\mathrm{e})^2, \qquad t \geq t_4.$$

Let

$$w = \frac{x(h(t^*))}{x(t^*)} \geq 1,$$

and because of $1 \leq w < (4\,\mathrm{e})^2$, w is finite.

Now dividing (1) with $x(t)$ and then integrating from t to $h(t)$, we get

$$\int_t^{h(t)} \frac{x'(s)}{x(s)}\mathrm{d}s - \int_t^{h(t)} p(s)\frac{f(x(\tau(s)))}{x(s)}\mathrm{d}s = 0$$

and

$$\ln\frac{x(h(t))}{x(t)} - \int_t^{h(t)} p(s)\frac{f(x(\tau(s)))}{x(\tau(s))}\frac{x(\tau(s))}{x(s)}\mathrm{d}s = 0.$$

Since $x(t)$ is nondecreasing, we get

$$\ln\frac{x(h(t))}{x(t)} - \int_t^{h(t)} p(s)\frac{f(x(\tau(s)))}{x(\tau(s))}\frac{x(h(s))}{x(s)}\mathrm{d}s \geq 0$$

and

$$\ln\frac{x(h(t))}{x(t)} - \frac{f(x(\tau(\xi)))}{x(\tau(\xi))}\frac{x(h(\xi))}{x(\xi)}\int_t^{h(t)} p(s)\mathrm{d}s \geq 0$$

or

$$\ln\frac{x(h(t))}{x(t)} \geq \frac{f(x(\tau(\xi)))}{x(\tau(\xi))}\frac{x(h(\xi))}{x(\xi)}\int_t^{h(t)} p(s)\mathrm{d}s. \tag{20}$$

where ξ is defined with $t < \xi < h(t)$, while $t \to \infty, \xi \to \infty$ and because of this $h(t) \to \infty$. Then taking lower limit on both side of (20), we obtain $\ln w > \frac{w}{\mathrm{e}}$. But this is impossible since $\ln x \leq \frac{x}{\mathrm{e}}$ for all $x > 0$.

Now, we consider the case where $M = 0$. In this case, it is clear that since $\frac{x}{f(x)} > 0$ and

$$\lim_{x\to\infty}\frac{x}{f(x)} = 0, \tag{21}$$

by (21), we get

$$\frac{x}{f(x)} < \varepsilon \qquad \text{and} \qquad \frac{f(x)}{x} > \frac{1}{\varepsilon}, \tag{22}$$

where $\varepsilon > 0$ is an arbitrary real number. Thus, since $h(t) \leq \tau(t)$ and $x(t)$, $h(t)$ are nondecreasing, by (1) and (22), we have

$$x'(t) - \frac{1}{\varepsilon}p(t)x(h(t)) > 0. \tag{23}$$

Integrating (23) from t to $h(t)$, we obtain

$$x(h(t)) - x(t) - \frac{1}{\varepsilon}\int_t^{h(t)} p(s)x(h(s))\mathrm{d}s > 0,$$

and

$$x(h(t)) - \frac{1}{\varepsilon}x(h(t))\int_t^{h(t)} p(s)\mathrm{d}s > 0. \tag{24}$$

By (16) and (24), we can write

$$1 > \frac{c}{\varepsilon} \qquad \text{or} \qquad \varepsilon > c,$$

this contradicts to $\lim_{x\to\infty}\frac{x}{f(x)} = 0$. The proof of the theorem is completed. □

Theorem 2.2. *Assume that* (2), (3), (10) *and* (12) *hold with* $0 < M < \infty$. *If* $\tau(t)$ *is non-monotone and*

$$\limsup_{t\to\infty} \int_t^{h(t)} p(s)\mathrm{d}s > M, \tag{25}$$

where $h(t)$ *is defined by* (9), *then all solutions of Equation* (1) *oscillate.*

Proof. Assume for the sake of conradiction, that there exists a nonoscillatory solution $x(t)$ of (1) and it means that $x(t)$ is nondecreasing. In view of (12), we know from Theorem 2.2 that $\lim_{t\to\infty} x(t) = \infty$ for $t \geq t_1$. On the other hand, by (10), we have a constant $\theta > 1$ such that $f(x(t)) \geq \frac{1}{\theta M} x(t)$ for $t \geq t_1 \geq t_2$. So, by Equation (1) and using the functions $x(t)$ and $h(t)$ are nondecreasing, we have

$$x'(t) - \frac{1}{\theta M} p(t) x(h(t)) \geq 0, \qquad t \geq t_1 \geq t_2. \tag{26}$$

Now, integrating (26) from t to $h(t)$, then we have

$$x(h(t)) - x(t) - \frac{1}{\theta M} \int_t^{h(t)} p(s) x(h(s))\mathrm{d}s \geq 0,$$

or

$$x(h(t)) - x(t) - \frac{1}{\theta M} x(h(t)) \int_t^{h(t)} p(s)\mathrm{d}s \geq 0.$$

This implies

$$x(h(t)) \left[1 - \frac{1}{\theta M} \int_t^{h(t)} p(s)\mathrm{d}s \right] \geq 0$$

and hence

$$\int_t^{h(t)} p(s)\mathrm{d}s \leq \theta M,$$

for sufficiently large t. Therefore,

$$\limsup_{t\to\infty} \int_t^{h(t)} p(s)\mathrm{d}s \leq \theta M. \tag{27}$$

On the other hand, from (2.17), we can write

$$\limsup_{t\to\infty} \int_t^{h(t)} p(s)\mathrm{d}s = K > M.$$

So, we get $M < \frac{K+M}{2} < K$. Therefore, if we choose $\theta = \frac{K+M}{2M} > 1$, then from (27), we get

$$\limsup_{t\to\infty} \int_t^{h(t)} p(s)\mathrm{d}s = K \leq \theta M = \frac{K+M}{2}.$$

This is a contradiction to $K > \frac{K+M}{2}$. So, the proof is completed. □

Remark 2.3. We remark that if $\tau(t)$ is nondecreasing, then we have $h(t) = \tau(t)$ for all $t \geq t_0$ and the condition (25) reduces to

$$\limsup_{t\to\infty} \int_t^{\tau(t)} p(s)\mathrm{d}s > M, \tag{28}$$

Now, we have the following example.

Example 2.4. Consider the following nonlinear advanced differential equation

$$x'(t) - \frac{3}{\mathrm{e}}x(\tau(t))\ln(3 + |x(\tau(t))|) = 0, \qquad t \geq 1, \tag{29}$$

where

$$\tau(t) = \begin{cases} 4t - 6k - 2, & \text{if } t \in [2k+1, 2k+2] \\ -2t + 6k + 10, & \text{if } t \in [2k+2, 2k+3], \end{cases} \qquad k \in \mathbb{N}_0. \tag{30}$$

By (9), we see that

$$h(t) := \inf_{s \geq t} \tau(s) = \begin{cases} 4t - 6k - 2, & \text{if } t \in [2k+1, 2k+1.5] \\ 2k + 4, & \text{if } t \in [2k+1.5, 2k+3], \end{cases} \qquad k \in \mathbb{N}_0.$$

If we put $p(t) = \frac{3}{\mathrm{e}}$ and $f(x) = x\ln(3 + |x|)$, then we have

$$M = \limsup_{|x| \to \infty} \frac{x}{x\ln(3 + |x|)} = 0.$$

Now, for $t = 2k + 3$, $k \in \mathbb{N}_0$, we have

$$\liminf_{t \to \infty} \int_t^{\tau(t)} p(s)\mathrm{d}s = \liminf_{t \to \infty} \int_t^{h(t)} p(s)\mathrm{d}s = \frac{1}{\mathrm{e}} > \frac{M}{\mathrm{e}},$$

that is, all conditions of Theorem 2.2 are satisfied and therefore all solutions of (29) oscillate.

We remark that no result in the literature gives an answer for the equation (29) to be oscillatory under the (30).

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