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ON THE SOLVABILITY OF THE SEQUENCE SPACES EQUATIONS OF THE FORM $(\ell_a^p)_\Delta + F_x = F_b$ $(p > 1)$ WHERE $F = c_0$, c OR ℓ_∞

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Abstract. Given any sequence $z = (z_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_z for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/z = (y_n/z_n)_{n\geq 1} \in E$; in particular, c_z denotes the set of all sequences y such that y/z converges. By Δ we denote the operator of the first difference defined by $\Delta_n y = y_n - y_{n-1}$ for all sequences y and all $n \geq 1$, with the convention $y_0 = 0$. In this paper, we state some results on the (SSE) of the form $(\ell_a^p)_\Delta + F_x = F_b$ with $p > 1$, $a, b \in U^+$ and $F \in \{c_0, c, \ell_\infty\}$. We apply these results to the solvability of the (SSE) $(\ell_a^p)_\Delta + s_x = s_u$ for $u > 0$, $(\ell_r^p)_\Delta + s_x^0 = s_b^0$ with $r \neq 1$ and $(\ell_r^p)_\Delta + F_x = F_u$ for r, $u > 0$ and $F = c_0$, c, or ℓ_∞. These results extend those stated in [**14**].

1. Introduction

For any given set of sequences E and for any positive sequence a, we write $E_a = (1/a)^{-1} * E$ for the set of all sequences y for which $y/a = (y_n/a_n)_{n\geq 1} \in E$. In [**3**], by s_a, s_a^0 and $s_a^{(c)}$ the sets E_a for $E = \ell_\infty$, c_0 or c, respectively. Then in [**6**] we defined the sum $E_a + F_b$ and the product $E_a * F_b$ where E and F are any of the sets ℓ_∞, c_0 or c. Then in [**9**] we gave a solvability of sequences spaces inclusions $G_b \subset E_a + F_x$ where E, F, $G \in \{\ell_\infty, c_0, c\}$ and some applications to sequence spaces inclusions with operators. In the same way recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $(n^{-1}\sum_{k=1}^n |y_k|)_{n\geq 1}$ is bounded and tend to zero, respectively. These spaces were studied by Maddox [**1**] and Malkowsky [**24**]. In [**20**] some properties were given of well known operators defined on the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. The sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ are defined by $\sup_{n\geq 1}\left(|y_n|^{1/n}\right) < \infty$ and $\lim_{n\to\infty}\left(|y_n|^{1/n}\right) = 0$, respectively. In this paper, we deal with special *sequence spaces inclusion equations (SSIE), (resp. sequence spaces equations (SSE)),* which are determined by an inclusion (resp. identity), for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $(E_{f(x)})_T$ where f maps U^+ to itself, E is any linear space of

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sequences and T is a triangle. Some results on the (SSIE) and the (SSE) were stated in [**12, 23, 9, 7, 8, 21, 11, 19, 22**].

In [**19**], we determined the set of all positive sequences x for which the *(SSIE)* $\left(s_x^{(c)}\right)_{B(r,s)} \subset \left(s_x^{(c)}\right)_{B(r',s')}$ holds, where r, r', s' and s are real numbers, and $B(r,s)$ is the generalized operator of the first difference defined by $(B(r,s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r,s)y)_1 = ry_1$. In this way we determined the set of all positive sequences x for which $(ry_n + sy_{n-1})/x_n \to l$ implies $(r'y_n + s'y_{n-1})/x_n \to l$ $(n \to \infty)$ for all y and for some scalar l.

In this paper, we extend in a certain sense some results given in [**23, 9, 7, 8, 21, 11, 22**]. In [**21**], we show that for any given positive sequences a and b the solutions of the equations $\chi_a + s_x^0 = s_b^0$ where χ is any of the symbols s or $s^{(c)}$ are determined by $s_x = s_b$ if $a/b \in c_0$, and if $a/b \notin c_0$ each of these equations has no solution. We also determined the set of all positive sequences x for which $y_n/b_n \to l$ if and only if there are sequences u and v for which $y = u + v$ and $u_n/a_n \to 0$, $v_n/x_n \to l'$ $(n \to \infty)$ for all y and for some scalars l and l'. This statement is equivalent to the equation $s_a^0 + s_x^{(c)} = s_b^{(c)}$. In [**11**], we gave some properties of the sets of $a-$analytic and a-entire sequences denoted by $\mathbf{\Lambda}_a$ and $\mathbf{\Gamma}_a$ and defined by $\sup_{n\geq 1} \left(|y_n|/a_n\right)^{1/n} < \infty$ and $\lim_{n\to\infty} \left(|y_n|/a_n\right)^{1/n} = 0$, respectively. Then, we determined the set of all $x \in U^+$ such that for every sequence y, we have $y_n/b_n \to l$ if and only if there are sequences u and v with $y = u + v$, $\left(|u_n|/a_n\right)^{1/n} \to 0$ and $v_n/x_n \to l'$ $(n \to \infty)$ for some scalars l and l'. This statement means $\mathbf{\Gamma}_a + s_x^{(c)} = s_b^{(c)}$. In [**8**], we solved the (SSE) $E_a + \left(s_x^{(c)}\right)_{B(r,s)} = s_x^{(c)}$ where $E = \ell_\infty$, c_0 or c and x is the unknown. In [**7**], under some conditions we determined the solutions of (SSE) with operators of the form $(E_a * E_x + E_b)_\Delta = E_\eta$, $(E_a * (E_x)^2 + E_b * E_x)_\Delta = E_\eta$ and $E_a + (E_x)_\Delta = E_x$ where E is any of the sets ℓ_∞ or c_0. In [**21**] we determined the sets of all positive sequences x that satisfy the systems $s_a^0 + (s_x)_\Delta = s_b$, $s_x \supset s_b$ and $s_a + \left(s_x^{(c)}\right)_\Delta = s_b^{(c)}$, $s_x^{(c)} \supset s_b^{(c)}$. Then, we dealt with the (SSE) with operators defined by $(E_a)_{C(\lambda)D_\tau} + \left(s_x^{(c)}\right)_{C(\mu)D_\tau} = s_b^{(c)}$ where E is either ℓ_∞ or c_0. In [**22**] we dealt with the (SSE) $E_a + s_x = s_b$, where $E \in \{w_\infty, w_0, \ell_p\}$ where ℓ_p is the set of all sequences of p-absolute type. Then, we solved the (SSE) $E_a + s_x^{(c)} = s_b^{(c)}$, where $E \in \{w_0, \ell_p\}$ and a solvability can be found of the equation $E_a + s_x = s_b$, where $E \in \{c, \ell_\infty\}$. In [**10**], we studied the (SSE) with operators $(E_a)_{C(\lambda)C(\mu)} + (E_x)_{C(\lambda\sigma)C(\mu)} = E_b$, where $b \in \widehat{C}_1$ and E is any of the sets ℓ_∞ or c_0. More recently in [**12**], we dealt with the solvability of (SSE) of the form $E_T + F_x = F_b$ where T is either of the triangles Δ or Σ, where Δ is the operator of the first difference and Σ is the operator defined by $\Sigma_n y = \sum_{k=1}^n y_k$ for all sequences y. More precisely, we gave a solvability of the (SSE) $E_\Delta + F_x = F_b$, where E is any of the sets c_0, ℓ_p, $(p > 1)$, w_0 or $\mathbf{\Lambda}$ and $F = c$ or ℓ_∞. Then, there is a solvability of the (SSE) $E_\Sigma + F_x = F_b$ where E is any of the sets c_0, c, ℓ_∞, ℓ_p, $(p > 1)$, w_0, $\mathbf{\Gamma}$, $\mathbf{\Lambda}$ and $F = c$ or ℓ_∞. Finally, there is a solvability of the (SSE) with operator defined by $E_\Sigma + F_x = F_b$ where $E = \Gamma$ or $\mathbf{\Lambda}$, $F = c$ or ℓ_∞, and a solvability of the (SSE) $\Gamma_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$. In [**13**] for any given positive sequence a and b we solved the (SSE) defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$ or ℓ_p, $(p > 1)$ and the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ for $E = c$ or s_1, and we gave applications to particular classes of (SSE). In [**15**] we dealt with the solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where F' is either c_0 or ℓ_∞. Then, we solved each of the (SSIE) $c_0 \subset \mathcal{E} + s_x$, $c \subset \mathcal{E} + s_x^{(c)}$ and the (SSE) $\mathcal{E} + s_x^{(c)} = c$. Then, it can be found a resolution of the (SSE) of the form $(\ell_r^p)_\Delta + F_x = F$ with $p \geq 1$, $r > 0$ and $F \in \{c_0, c, \ell_\infty\}$. In this paper, we extend some of the previous results and we obtain a resolution of the next (SSE) $(\ell_a^p)_\Delta + s_x = s_u$ for $u > 0$, $(\ell_r^p)_\Delta + s_x^0 = s_b^0$ with $r \neq 1$ and $(\ell_r^p)_\Delta + F_x = F_u$ for r, $u > 0$ and $F = c_0$, c or ℓ_∞.

This paper is organized as follows. In Section 2, we recall some definitions and results on sequence spaces and matrix transformations. In Section 3, we state some results on the multiplier $M(E, F)$ of classical spaces In Section 4, we recall some results on the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ, $\widehat{C_1}$ and G_1. In Section 5, we recall some definitions and results on the (SSIE) and (SSE) and deal with the solvability of the (SSIE) $s_1 \subset \mathcal{E} + s_x$, $c_0 \subset \mathcal{E} + s_x^0$, $c \subset \mathcal{E} + s_x^{(c)} = c$ and on the (SSE) $\mathcal{E} + s_x = s_1$, $\mathcal{E} + s_x^0 = c_0$ and $\mathcal{E} + s_x^{(c)} = c$. In Section 6, we state some results on the (SSE) $(E_r)_\Delta + F_x = F_u$, where E and F are any of the sets c_0, c or ℓ_∞. In Section 7, we deal with the (SSE) $(\ell_a^p)_\Delta + F_x = F_b$, $(p > 1)$ and we solve the (SSE) $(\ell_a^p)_\Delta + s_x = s_u$ and $(\ell_r^p)_\Delta + s_x^0 = s_b^0$ for $r \neq 1$. Finally, in Section 8, we apply the results of Section 7 to the solvabilty of the (SSE) $(\ell_r^p)_\Delta + F_x = F_u$ for r, $u > 0$, where F are any of the spaces c_0, c or ℓ_∞.

2. Preliminary results

An FK space is a *complete linear metric space* for which convergence implies *coordinatewise convergence.* A *BK space* is a Banach space of sequences that is an *FK space.* A BK space E *is said to have AK if for every sequence* $y = (y_n)_{n\geq 1} \in E$, *then* $y = \lim_{n\to\infty} \sum_{k=1}^n y_k e^{(k)}$, *where* $e^{(k)} = (0, \dots, 0, 1, 0, \dots)$, 1 *being in the* k*-th position.*

For a given infinite matrix $\Lambda = (\lambda_{nk})_{n,k\geq 1}$ we define the operators Λ_n for any integer $n \geq 1$, by $\Lambda_n y = \sum_{k=1}^\infty \lambda_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator Λ defined by $\Lambda y = (\Lambda_n y)_{n\geq 1}$ mapping between sequence spaces. When Λ *maps* E *into* F, *where* E *and* F *are any sets of sequences, we write that* $\Lambda \in (E, F)$, (cf. [**1**]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0} (\|L(y)\|_E / \|y\|_E)$ satisfies the identity $\mathcal{B}(E) = (E, E)$. By ω, c_0, c and ℓ_∞ for the sets of all sequences, the sets of null, convergent and bounded sequences. By ℓ^p for $p \geq 1$ we denote the set of all sequences $y = (y_k)_{k\geq 1}$ such that $\sum_{k=1}^\infty |y_k|^p < \infty$. Let $U^+ \subset \omega$ be the set of all sequences $\mathbf{u} = (u_n)_{n\geq 1}$ with $u_n > 0$ for all n. Then, for any given sequence $\mathbf{u} = (u_n)_{n\geq 1} \in \omega$ we define the infinite diagonal matrix $D_\mathbf{u}$ by $[D_\mathbf{u}]_{nn} = u_n$ for all n. For $\mathbf{u} = (r^n)_{n\geq 1}$ we write D_r for $D_\mathbf{u}$. Let E be any subset of ω and $\mathbf{u}$ be any sequence with $u_n \neq 0$ for all n. Using Wilansky's notations [**27**] we have

$$(1/\mathbf{u})^{-1} * E = D_{\mathbf{u}} * E = \{y = (y_n)_{n\geq 1} \in \omega : y/\mathbf{u} \in E\}.$$

By $E_{\mathbf{u}}$, we can also denote the set $D_{\mathbf{u}} * E$. We use the sets s_a^0, $s_a^{(c)}$, s_a and ℓ_a^p defined as follows. For given $a \in U^+$ we let $D_a * c_0 = s_a^0$, $D_a * c = s_a^{(c)}$, also denoted by c_a, and $D_a * \ell^p = \ell_a^p$ for $p \geq 1$ and s_a for $p = \infty$, (cf. [**3, 5**]). Each of the spaces $D_a E$, where $E \in \{c_0, c, \ell_\infty, \ell^p\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_{n\geq 1} (|y_n| / a_n)$ and $\|y\|_{\ell_a^p} = \sum_{k=1}^{\infty} (|y_k| / a_k)^p$. Then, the spaces s_a^0 and ℓ_a^p *have AK*. If $a = (r^n)_{n\geq 1}$ with $r > 0$, we write s_r, s_r^0, $s_r^{(c)}$ and ℓ_r^p for the sets s_a, s_a^0, $s_a^{(c)}$ and ℓ_a^p, respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$, $s_1^{(c)} = c$ and $\ell_1^p = \ell^p$. Recall that $S_1 = (s_1, s_1)$ is a Banach algebra (cf. [**2**]) and $(c_0, s_1) = (c, s_1) = (s_1, s_1) = S_1$. We have $\Lambda \in S_1$ if and only if $\sup_{n\geq 1} (\sum_{k=1}^{\infty} |\lambda_{nk}|) < \infty$. For any subset F of ω, we write $F(\Lambda) = F_\Lambda = \{y \in \omega : \Lambda y \in F\}$ for the matrix domain of Λ in F. The infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. Throughout this paper we use the next well known statement. If T, T', and T'' are triangles, E and F are any sets of sequences, then we have $T \in (E_{T'}, F_{T''})$ if and only if $T'' T T'^{-1} \in (E, F)$, (cf. [**7**, Lemma 9, p. 45]). Then, for any given set E of sequences, we write $\Lambda E = \{y \in \omega : y = \Lambda x$ for some $x \in E\}$.

Finally, we recall the characterization of (ℓ^p, G) where $G = c_0$, c or ℓ_∞ which is used in the following. Throughout this paper we write $q = p/(p-1)$ for $p > 1$. We define $\mathcal{M}(\ell_p, \ell_\infty) = \sup_n (|\mathbf{a}_{nk}|)$ if $p = 1$, and $\mathcal{M}(\ell^p, \ell_\infty) = \sup_n (\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|^q)$ if $p > 1$. We obtain the following.

Lemma 1 ([**26**, Theorem 1.37, p. 161])**.** *Let $p \geq 1$. Then we have*

i) *$A \in (\ell^p, \ell_\infty)$ if and only if*

$$\mathcal{M}(\ell^p, \ell_\infty) < \infty. \tag{1}$$

ii) *$A \in (\ell^p, c_0)$ if and only if the condition in* (1) *holds and* $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ *for all k.*

iii) *$A \in (\ell^p, c)$ if and only if the condition in* (1) *holds and* $\lim_{n\to\infty} \mathbf{a}_{nk} = l_k$ *for some scalar l_k and for all k.*

3. The multipliers of some sets of sequences

First we need to recall some well-known results. Let y and z be sequences and let E and F be two subsets of ω, then we write $yz = (y_n z_n)_{n\geq 1}$, we denote by $M(E, F) = \{y \in \omega : yz \in F$ for all $z \in E\}$ the *multiplier space of E and F*. In this way, we recall the following results.

Lemma 2. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then*

(i) *$M(E, F) \subset M(\widetilde{E}, F)$ for all $\widetilde{E} \subset E$,*

(ii) *$M(E, F) \subset M(E, \widetilde{F})$ for all $F \subset \widetilde{F}$.*

Lemma 3. *Let $a \in \omega$, b be a nonzero sequence and E, $F \subset \omega$. Then $\Lambda \in (D_a * E, D_b * F)$ if and only if $D_{1/b} \Lambda D_a = (\lambda_{nk} a_k / b_n)_{n,k\geq 1} \in (E, F)$.*

We have $D_a * E \subset D_b * F$ if and only if $I \in (D_a * E, D_b * F)$, which is equivalent to $D_{a/b} \in (E, F)$ and to $a/b \in M(E, F)$. This gives the following lemma.

Lemma 4. *Let a, $b \in U^+$ and let E and F be two subsets of ω. Then $D_a * E \subset D_b * F$ if and only if $a/b \in M(E, F)$.*

In a similar way we obtain.

Lemma 5. *Let a, $b \in U^+$ and let E, F and G be subsets of ω that satisfy the condition $M(E, F) = G$. Then the following statements are equivalent:*

i) $a \in D_b * G$,
ii) $a/b \in M(E, F)$,
iii) $D_a * E \subset D_b * F$.

In the following we use the notation $E^+ = E \cap U^+$ for any subset E of ω. We have the next result.

Lemma 6. *Let E, F be linear spaces of sequences and assume F is of absolute type, that is,*

$$z \in F \iff |z| \in F \text{ for all } z \in \omega. \tag{2}$$

Then $M(E^+, F) = M(E, F)$.

By [**25**, Lemma 3.1, p. 648] and [**26**, Example 1.28, p. 157], we obtain the following result.

Lemma 7. *Let $p \geq 1$. We have:*

i) $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ *and* $M(c, c) = c$.
ii) $M(E, \ell_\infty) = M(c_0, F) = \ell_\infty$ *for* E, $F = c_0$, c *or* ℓ_∞.
iii) $M(c_0, \ell^p) = M(c, \ell^p) = M(\ell_\infty, \ell^p) = \ell^p$.
iv) $M(\ell^p, F) = \ell_\infty$ *for* $F \in \{c_0, c, s_1, \ell^p\}$.

4. The sets $\widehat{\Gamma}$, $\widehat{C}$, Γ, $\widehat{C_1}$ and G_1.

To solve the next equations we recall some definitions and results. Now let U be the set of all sequences $(u_n)_{n \geq 1} \in \omega$ with $u_n \neq 0$ for all n. The infinite matrix $C(a)$ with $a = (a_n)_n \in U$ is the triangle defined by $[C(a)]_{nk} = 1/a_n$ for $k \leq n$. It can be shown that the triangle $\Delta(a)$ whose the nonzero entries are defined by $[\Delta(a)]_{nn} = a_n$, and $[\Delta(a)]_{n,n-1} = -a_{n-1}$ for all $n \geq 2$ is the inverse of $C(a)$, that is, $C(a)(\Delta(a)y) = \Delta(a)(C(a)y)$ for all $y \in \omega$. If $a = e = (1, 1, \dots 1, \dots)$ then we obtain the well known operator of the first difference represented by $\Delta(e) = \Delta$. Then we have $\Delta_n y = y_n - y_{n-1}$ for all sequences y and for all $n \geq 1$, with the convention $y_0 = 0$. It is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Δ, $\Sigma \in S_R$ for any $R > 1$. By $\widehat{C_1}$ and $\widehat{C}$ we define the sets of all positive sequences a that satisfy the conditions $C(a)a \in \ell_\infty$, and $C(a)a \in c$, respectively. Then, we write $\widehat{\Gamma}$ and Γ for the sets of all positive sequences a that satisfy the conditions $\lim_{n\to\infty}(a_{n-1}/a_n) < 1$ and $\overline{\lim}_{n\to\infty}(a_{n-1}/a_n) < 1$, respectively (cf. [**3**]). Finally, by G_1 the set $G_1 = \{x \in U^+ : x_n \geq K\gamma^n \text{ for all } n \text{ for some } K > 0 \text{ and } \gamma > 1\}$. We obtain the following lemmas.

Lemma 8. *We have* $\widehat{\Gamma} = \widehat{C} \subset \Gamma \subset \widehat{C}_1 \subset G_1$.

Proof. The identity $\widehat{\Gamma} = \widehat{C}$ follows from [**18**, Proposition 2.2 p. 88], and the inclusions $\Gamma \subset \widehat{C}_1 \subset G_1$ follow from [**3**, Proposition 2.1, p. 1786]. □

Lemma 9. *Let* $a \in U^+$. *Then we have:*

i) *The following statements are equivalent*

   α) $a \in \widehat{C}_1$,

   β) $(s_a)_\Delta = s_a$,

   γ) $\left(s_a^0\right)_\Delta = s_a^0$.

ii) $a \in \widehat{\Gamma}$ *if and only if* $\left(s_a^{(c)}\right)_\Delta = s_a^{(c)}$.

iii) $a \in \widehat{\Gamma}$ *implies* $\left(\ell_a^p\right)_\Delta = \ell_a^p$.

Proof. The statement in i) follows from [**3**, Theorem 2.6, pp. 1789–1790]. ii) follows from [**3**, Theorem 2.6, pp. 1789–1790] and [**18**, Proposition 2.2 p. 88]. iii) follows from [**5**, Theorem 6.5, p. 3200]. □

5. On some (SSIE) and the (SSE)

5.1. The sets $\mathcal{I}(E,F)$, $\mathcal{S}(E,F)$ and the relation R_F.

The solvability of the (SSE) $E_a + F_x = F_b$ consists in determining the set of all positive sequences x that satisfy the statement $y/b \in F$ if and only if there are two sequences α, β such that $y = \alpha + \beta$, where $\alpha/a \in E$ and $\beta/x \in F$ for all y. For instance, the solvability of the equation $s_a + s_x^{(c)} = s_b^{(c)}$ for a, $b \in U^+$ consists in determining the set of all $x \in U^+$ that satisfy the next statement. For every sequence y, the condition $y_n/b_n \to l$ $(n \to \infty)$ if and only if there are two sequences α, β such that $y = \alpha+\beta$ for which $\sup_{n\geq 1}(|\alpha_n|/a_n) < \infty$ and $\beta_n/x_n \to l'$ $(n \to \infty)$ for some scalars l, l'. For any given linear spaces of sequences E and F, we put $\mathcal{I}(E,F) = \{x \in U^+ : F_b \subset E_a + F_x\}$ and $\mathcal{S}(E,F) = \{x \in U^+ : E_a + F_x = F_b\}$. To characterize the set $\mathcal{S}(E,F)$ we need to define the relation R_F as follows. For $b \in U^+$ and for any subset F of ω, by $\mathrm{cl}^F(b)$ the equivalent class for the equivalence relation R_F defined by

$$xR_Fy \text{ if } D_x * F = D_y * F \qquad \text{for } x, y \in U^+.$$

It can be easily seen that $\mathrm{cl}^F(b)$ is the set of all $x \in U^+$ such that $x/b \in M(F,F)$ and $b/x \in M(F,F)$, (cf. [**21**]). Then we have $\mathrm{cl}^F(b) = \mathrm{cl}^{M(F,F)}(b)$. For instance $\mathrm{cl}^c(b)$ is the set of all $x \in U^+$ such that $D_x * c = D_b * c$, that is, $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. In the following we write $\mathrm{cl}^\infty(b)$ for $\mathrm{cl}^{\ell_\infty}(b)$ for the set of all positive sequences such that $K_1 b_n \leq x_n \leq K_2 b_n$ for some K_1, $K_2 > 0$ and for all n, (cf. [**4**, Proposition 1, p. 244]). For $b = (r^n)_{n\geq 1}$ we write $\mathrm{cl}^F(r)$ for the set $\mathrm{cl}^F(b)$, to simplify.

Now, recall the next elementary result on the sum of linear spaces of sequences. Let E, F and G be linear subspaces of ω, then we have $E + F \subset G$ if and only if $E \subset G$ and $F \subset G$. For instance we have $c_0 + s_x \subset s_1$ if and only if $x \in s_1^+$ and there is no positive sequence x for which $s_1 + s_x^0 \subset$

c_0. Then, we let $\overline{s_{1/b}} = \{x \in U^+ : x_n \geq K b_n \text{ for some } K > 0 \text{ and for all } n\}$, and $\overline{s_{1/b}^{(c)}} = \{x \in U^+ : \lim_{n\to\infty} (x_n/b_n) = l \text{ for some } l \in]0, +\infty]\}$, (cf. [**21**]). To simplify, we write $\overline{s_{(1/r^n)_{n\geq 1}}} = \overline{s_{1/r}}$ and $\overline{s^{(c)}_{(r^{-n})_{n\geq 1}}} = \overline{s^{(c)}_{1/r}}$, (or $\overline{c_{1/r}}$) for $r > 0$. Notice that $\mathrm{cl}^{(c)}(b) = s_b^{(c)} \setminus s_b^0$. It can be easily seen that $\overline{s_{1/b}^{(c)}} = \overline{c_{1/b}} = \{x \in U^+ : s_b^{(c)} \subset s_x^{(c)}\}$, (cf. [**21**]).

5.2. On the (SSIE) $s_1 \subset \mathcal{E} + s_x$, $c_0 \subset \mathcal{E} + s_x^0$, $c \subset \mathcal{E} + s_x^{(c)} = c$ and the (SSE) $\mathcal{E} + s_x = s_1$, $\mathcal{E} + s_x^0 = c_0$ and $\mathcal{E} + s_x^{(c)} = c$.

5.2.1. On the (SSIE) $s_1 \subset \mathcal{E} + s_x$, $c_0 \subset \mathcal{E} + s_x^0$ and $c \subset \mathcal{E} + s_x^{(c)}$.

In this part, we recall some results on the solvability of the (SSIE) of the form $F \subset \mathcal{E} + F_x$ where $\mathcal{E}$ and F are linear spaces of sequences. These results may be applied to the (SSIE) $F_b \subset E_a + F_x$ which is equivalent to $F \subset E_{a/b} + F_y$ with $y = x/b$.

Proposition 10. *Let $\mathcal{E}$ be a linear space of sequences and let $p \geq 1$. Then we have*

i) *Let $\mathcal{E} \subset c$. Then, the set $\mathcal{I}(\mathcal{E}, s_1)$ of all positive sequences x such that $s_1 \subset \mathcal{E} + s_x$ is determined by*
$$\mathcal{I}(\mathcal{E}, s_1) = \overline{s_1}.$$

ii) *Let $\mathcal{E} \subset \ell^p \cup s_\alpha$ with $\alpha \in c_0$. Then, the set $\mathcal{I}(\mathcal{E}, c_0)$ of all the positive sequences x such that $c_0 \subset \mathcal{E} + s_x^0$ is determined by*
$$\mathcal{I}(\mathcal{E}, c_0) = \overline{s_1}.$$

iii) *Let $\mathcal{E} \subset c_0 \cup (s_\alpha)_\Delta$ with $\alpha \in cs$. Then, the set $\mathcal{I}(\mathcal{E}, c)$ of all the positive sequences x such that $c \subset \mathcal{E} + s_x^{(c)}$ is determined by*
$$\mathcal{I}(\mathcal{E}, c) = \overline{c}.$$

Proof. i) follows from [**15**, Theorem 2, p. 113]. ii) Let $x \in \mathcal{I}(\mathcal{E}, c_0)$ and assume $\mathcal{E} \subset s_\alpha$ with $\alpha \in c_0$. Then we have
$$c_0 \subset \mathcal{E} + s_x^0 \subset s_\alpha + s_x = s_{\alpha+x}$$
and $1/(\alpha + x) \in M(c_0, s_1)$. Since $M(c_0, s_1) = s_1$ we deduce there is $K > 0$ such that $\alpha_n + x_n \geq K$ and since $\alpha \in c_0$ there is $K' > 0$ such that $x_n \geq K'$ for all n and $x \in \overline{s_1}$. So, we have shown $\mathcal{I}(\mathcal{E}, c_0) \subset \overline{s_1}$ if $\mathcal{E} \subset s_\alpha$ with $\alpha \in c_0$. Now, we consider the case $\mathcal{E} \subset \ell^p$. Let $x \in \mathcal{I}(\mathcal{E}, c_0)$ with $\mathcal{E} \subset \ell^p$. Then we have $s_x^0 \subset \ell^p_{\lambda x}$ if and only if $1/\lambda \in \ell^p$ since $M(c_0, \ell^p) = \ell^p$. Then we have
$$c_0 \subset \mathcal{E} + s_x^0 \subset \ell^p + \ell^p_{\lambda x} = \ell^p_{e+\lambda x}$$
and $1/(e + \lambda x) \in M(c_0, \ell^p) = \ell^p$. Since $\ell^p \subset c_0$, we deduce $1 + \lambda_n x_n \to \infty$ $(n \to \infty)$ and $1/\lambda x \in c_0$ for all $\lambda \in U^+$ such that $1/\lambda \in \ell^p$. By Lemma 6 we have $1/x \in M(\ell^p, c_0)$ and since $M(\ell^p, c_0) = s_1$ we conclude $x \in \overline{s_1}$ and again $\mathcal{I}(\mathcal{E}, c_0) \subset \overline{s_1}$ if $\mathcal{E} \subset \ell^p$. Conversely, if $x \in \overline{s_1}$ then we have $c_0 \subset s_x^0$ and $c_0 \subset \mathcal{E} + s_x^0$ and we conclude $\mathcal{I}(\mathcal{E}, c_0) = \overline{s_1}$.

iii) The case $\mathcal{E} \subset c_0$. By [**16**] we have $c \subset \mathcal{E} + s_x^{(c)}$ if and only if $x \in \overline{c}$. The case $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\alpha \in cs$ follows from [**15**], Theorem 5, p. 119. This completes the proof. □

5.2.2. On the solvability of the (SSE) $\mathcal{E}+s_x=s_1$, $\mathcal{E}+s_x^0=c_0$ and $\mathcal{E}+s_x^{(c)}=c$. The previous results may be applied to the (SSE) $\mathcal{E} + s_x = s_1$ as follows.

Proposition 11.

i) *Let $\mathcal{E} \subset c$ be a linear space of sequences. Then, the set $\mathcal{S}(\mathcal{E}, s_1)$ of all positive sequences x such that $\mathcal{E} + s_x = s_1$ is determined by*

$$\mathcal{S}(\mathcal{E}, s_1) = \mathrm{cl}^\infty (e).$$

ii) *Let $p \geq 1$ and let $\mathcal{E}$ be a linear space of sequences such that $\mathcal{E} \subset \ell^p \cup s_\alpha$ with $\alpha \in c_0$. The set $\mathcal{S}(\mathcal{E}, c_0)$ of all the positive sequences x such that $\mathcal{E} + s_x^0 = c_0$ is determined by*

$$\mathcal{S}(\mathcal{E}, c_0) = \mathrm{cl}^\infty (e).$$

iii) *Let $\mathcal{E}$ be a linear space of sequences with $\mathcal{E} \subset c_0 \cup (s_\alpha)_\Delta$ with $\alpha \in cs$. Then, the set $\mathcal{S}(\mathcal{E}, c)$ of all positive sequences x such that $\mathcal{E} + s_x^{(c)} = c$ is determined by*

$$\mathcal{S}(\mathcal{E}, c) = \mathrm{cl}^c (e).$$

Proof. i) Since $\mathcal{E} \subset c$ the equation $\mathcal{E} + s_x = s_1$ is equivalent to the statements $x \in s_1$ and $x \in \mathcal{I}(\mathcal{E}, s_1)$, and by i) in Proposition 10 we have $\mathcal{I}(\mathcal{E}, s_1) = \overline{s_1}$ and conclude $\mathcal{S}(\mathcal{E}, s_1) = s_1 \cap \overline{s_1} = \mathrm{cl}^\infty (e)$.

ii) Let $x \in \mathcal{S}(\mathcal{E}, c_0)$. Then we have $s_x^0 \subset c_0$, that is, $x \in s_1$ and by ii) in Proposition 10 we obtain $x \in \mathcal{I}(\mathcal{E}, c_0) = \overline{s_1}$. So, we have shown $\mathcal{S}(\mathcal{E}, c_0) \subset \mathrm{cl}^\infty (e)$. Conversely, let $x \in \mathrm{cl}^\infty (e)$. Then we have $s_x^0 = c_0$ which implies $\mathcal{E} + s_x^0 = \mathcal{E} + c_0 = c_0$ since $\mathcal{E} \subset \ell^p \cup s_\alpha \subset c_0$. This concludes the proof of ii).

iii) Case $\mathcal{E} \subset c_0$. By iii) in Proposition 10 we have $\mathcal{I}(\mathcal{E}, c) = \overline{c}$. Then, the inclusion $\mathcal{E} + s_x^{(c)} \subset c$ holds if and only if $s_x^{(c)} \subset c$, that is, $x \in c$. So, we have shown that if $\mathcal{E} \subset c_0$ then we have $\mathcal{S}(\mathcal{E}, c) = \mathrm{cl}^c (e)$. The case $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\alpha \in cs$ follows from [**15**, Theorem 5, p. 119]. This concludes the proof. □

6. On the (SSE) with operator of the form $(E_a)_\Delta + F_x = F_b$

We need some additional definitions and notations. Let a, $b \in U^+$ and let E, F be linear spaces of sequences. By $\mathcal{S}((E_a)_\Delta, F)$ we denote the set of all $x \in U^+$ that satisfy the (SSE) with operator

$$(E_a)_\Delta + F_x = F_b.$$

To simplify we write $S_E^0 = \mathcal{S}((E_a)_\Delta, c_0)$, $S_E^c = \mathcal{S}((E_a)_\Delta, c)$ and $S_E^\infty = S((E_a)_\Delta, \ell_\infty) = \mathcal{S}((E_a)_\Delta, s_1)$. For instance S_E^0 is the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the next statement. The condition $y/b \in c_0$ holds if and only if there are u, $v \in \omega$ such that $y = u + v$, $D_{1/a}\Delta u \in E$ and $v/x \in c_0$ for all y (cf. [**12, 15, 14**]).

We may apply the previous results to the solvability of the (SSE) $(s_r)_\Delta + F_x = F_u$, $\left(s_r^0\right)_\Delta + F_x = F_u$ and $\left(s_r^{(c)}\right)_\Delta + F_x = F_u$ for $F = c_0$ or ℓ_∞ for r, $u > 0$. The solvability of the (SSE) $(s_r)_\Delta + s_x = s_u$, $\left(s_r^0\right)_\Delta + s_x = s_u$ was obtained in [**14**, Theorem 16, p. 235]. The solvability of the (SSE) $\left(s_r^{(c)}\right)_\Delta + s_x = s_u$ can be obtained from i) in Proposition 11. This (SSE) is equivalent to the next statement. The condition $y_n/u^n = O(1)$ $(n \to \infty)$ holds if and only if there are α, $\beta \in \omega$ with $y = \alpha + \beta$ such that $\lim_{n\to\infty} \Delta_n \alpha / r^n = l$ and $\beta_n / x_n = O(1)$ $(n \to \infty)$ for all y and for some scalar l. We are led to state the following results.

Proposition 12. *Let* r, $u > 0$ *and let* $E = s_1$, c *or* c_0.

i) *If* $r < 1$, *then* $S_E^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u \geq 1, \\ \emptyset & \text{if } u < 1. \end{cases}$

ii) *If* $r = 1$, *then* $S_E^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$

iii) *If* $r > 1$, *then*

a)

$$S_{s_1}^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } r < u, \\ s_u^+ & \text{if } r = u, \\ \emptyset & \text{if } r > u. \end{cases} \tag{3}$$

b)

$$S_{c_0}^\infty = S_c^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } r \leq u, \\ \emptyset & \text{if } r > u. \end{cases} \tag{4}$$

Concerning the solvability of the (SSE) $(E_r)_\Delta + s_x^0 = s_u^0$ with $E \in \{c_0, c, \ell_\infty\}$ we obtain the following proposition.

Proposition 13 ([**14**, Theorem 22, p. 237])**.** *Let* r, $u > 0$ *and let* $E \in \{c_0, c, s_1\}$.

i) *If* $r < 1$, *then* $S_E^0 = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$

ii) *If* $r = 1$, *then* $S_E^0 = S_E^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$

iii) *If* $r > 1$, *then*

a) $S_{c_0}^0 = S_{s_1}^\infty$, *where* $S_{s_1}^\infty$ *is determined by* (3) *in Proposition 12.*

b) $S_c^0 = S_{s_1}^0$ *and* $S_{s_1}^0 = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } r < u, \\ \emptyset & \text{if } r \geq u. \end{cases}$

7. On the (SSE) of the form $(\ell_a^p)_\Delta + F_x = F_b$ $(p > 1)$ and application to special cases

First, we state general results on the (SSE) $(\ell_a^p)_\Delta + F_x = F_b$ and apply them to the solvability of the special (SSE) of the form $(\ell_a^p)_\Delta + s_x = s_u$, $\left(\ell^p_{(1/n^\xi)_{n\geq 1}}\right)_\Delta + s_x = s_u$, $(\ell_r^p)_\Delta + s_x^0 = s_b^0$ for $r \neq 1$ and $(\ell_r^p)_\Delta + s_x^0 = s^0_{(n^\theta)_{n\geq 1}}$. For this study, we use the factorable matrix $D_{1/b}\Sigma D_a$ which is the triangle whose nonzero entries are defined by $\left(D_{1/b}\Sigma D_a\right)_{nk} = a_k/b_n$ with $k \leq n$.

7.1. On the (SSE) $(\ell_a^p)_\Delta + F_x = F_b$, $(p > 1)$, a, $b \in U^+$ and $F \in \{c_0, c, \ell_\infty\}$.

In this part we let $\sigma^{(q)} = (b_n^{-q}\sum_{k=1}^n a_k^q)_{n\geq 1}$ for a, $b \in U^+$. We can state the following result.

Proposition 14. *Let $p > 1$ and a, $b \in U^+$. Then we have:*

i) *The set $S^c_{\ell^p} = S\left((\ell_a^p)_\Delta, c\right)$ of all positive sequences $x = (x_n)_{n\geq 1}$ such that $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$ satisfies the next properties:*
  a) $S^c_{\ell^p} = \mathrm{cl}^c(b)$ *if* $\sigma^{(q)} \in \ell_\infty$ *and* $1/b \in c_0$.
  b) $S^c_{\ell^p} = \emptyset$ *if* $\sigma^{(q)} \notin \ell_\infty$ *or* $1/b \notin c$.

ii) *The set $S^\infty_{\ell^p} = S\left((\ell_a^p)_\Delta, \ell_\infty\right)$ of all positive sequences $x = (x_n)_{n\geq 1}$ such that $(\ell_a^p)_\Delta + s_x = s_b$ satisfies the next properties:*
  a) $S^\infty_{\ell^p} = \mathrm{cl}^\infty(b)$ *if* $\sigma^{(q)} \in \ell_\infty$ *and* $1/b \in c$.
  b) $S^\infty_{\ell^p} = \emptyset$ *if* $\sigma^{(q)} \notin \ell_\infty$.

iii) *The set $S^0_{\ell^p} = S\left((\ell_a^p)_\Delta, c_0\right)$ of all positive sequences $x = (x_n)_{n\geq 1}$ such that $(\ell_a^p)_\Delta + s_x^0 = s_b^0$ satisfies the next properties:*
  a) $S^0_{\ell^p} = \mathrm{cl}^\infty(b)$ *if there is* $\alpha \in c_0$ *such that* $\left((\alpha_n b_n)^{-q}\sum_{k=1}^n a_k^q\right)_{n\geq 1} \in \ell_\infty$.
  b) $S^0_{\ell^p} = \emptyset$ *if* $\sigma^{(q)} \notin \ell_\infty$ *or* $1/b \notin c_0$.

Proof. First, notice that since $\Delta^{-1} = \Sigma$ the (SSE) $(\ell_a^p)_\Delta + F_x = F_b$ where $F \in \{c_0, c, \ell_\infty\}$ is equivalent to the (SSE) $\left(D_{1/b}\Sigma D_a\right)\ell^p + F_{x/b} = F$. i) a) By Proposition 11 we have $S^c_{\ell^p} = \mathrm{cl}^c(b)$ if $\mathcal{E} = \left(D_{1/b}\Sigma D_a\right)\ell^p \subset c_0$, that is, $D_{1/b}\Sigma D_a \in (\ell^p, c_0)$ which is equivalent to $\sigma^{(q)} \in \ell_\infty$ and $1/b \in c_0$. b) We have $x \in S^c_{\ell^p}$, $D_{1/b} * (\ell_a^p)_\Delta \subset c$ and $D_{1/b}\Sigma D_a \in (\ell^p, c)$ which is equivalent to $\sigma^{(q)} \in \ell_\infty$ and $1/b \in c$. So if $\sigma^{(q)} \notin \ell_\infty$ or $1/b \notin c$ then we conclude $S^c_{\ell^p} = \emptyset$.

ii) a) By Proposition 11 we have $S^\infty_{\ell^p} = \mathrm{cl}^\infty(b)$ if $\mathcal{E} = \left(D_{1/b}\Sigma D_a\right)\ell^p \subset c$. This inclusion is equivalent to $\sigma^{(q)} \in \ell_\infty$ and $1/b \in c$. b) may be shown as above. iii) We have $S^0_{\ell^p} = \mathrm{cl}^\infty(b)$ if $\mathcal{E} = \left(D_{1/b}\Sigma D_a\right)\ell^p \subset s_\alpha$ with $\alpha \in c_0$. This inclusion is equivalent to $D_{1/\alpha b}\Sigma D_a \in (\ell^p, s_1)$ and we conclude by the characterization of (ℓ^p, s_1). b) can be shown using the same arguments as those above. □

Remark 15. The solvability of the (SSE) $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$ was given in [**13**, Theorem 1, pp. 117–118] and we will recall this result in Section 8. Nevertheless, the statement i) in Proposition 14 gives another point of view on the study of the (SSE) $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$.

As a direct consequence of Proposition 14 we obtain the following results.

Example 16. Assume $\sum_{k=1}^{n} a_k^q = O\left(t^{qn}\right)$ $(n \to \infty)$ for all $t > 1$. Then by Proposition 14 i) the set $\overline{S_u^c}$ of all the solutions of the (SSE) $(\ell_a^p)_\Delta + s_x^{(c)} = s_u^{(c)}$ with $u \neq 1$ is determined by $\overline{S_u^c} = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u > 1, \\ \emptyset & \text{if } u < 1. \end{cases}$ Since for any given real τ we have $\sum_{k=1}^{n} k^{\tau q} = O\left(t^{qn}\right)$ $(n \to \infty)$ for all $t > 1$ the set of all the solutions of the (SSE) $\left(\ell^p_{(n^\tau)_{n\geq 1}}\right)_\Delta + s_x^{(c)} = s_u^{(c)}$ is determined by $\overline{S_u^c}$.

Remark 17. Let a, $b \in U^+$ and assume $a^q \in cs$. By Proposition 14 iii) with $\alpha = 1/b$, we conclude that if $1/b \in c_0$ then $S^0_{\ell^p} = \mathrm{cl}^\infty(b)$. Again by Proposition 14 iii) the condition $1/b \notin c_0$ implies $S^0_{\ell^p} = \emptyset$. Then we have $S^0_{\ell^p} = \begin{cases} \mathrm{cl}^\infty(b) & \text{if } 1/b \in c_0, \\ \emptyset & \text{if } 1/b \notin c_0. \end{cases}$

7.2. On the solvability of the (SSE) $(\ell_a^p)_\Delta + s_x = s_u$ and $(\ell_r^p)_\Delta + s_x = s_b$.

In this part we consider the (SSE) $(\ell_a^p)_\Delta + s_x = s_u$ with $p > 1$ and $u > 0$, which is equivalent to the next statement. The condition $y_n/u^n = O(1)$ $(n \to \infty)$ holds if and only if there are α, $\beta \in \omega$ with $y = \alpha + \beta$ such that $\sum_{k=1}^{\infty} \left(\left|\alpha_k - \alpha_{k-1}\right|/a_k\right)^p < \infty$ and $\beta_n/x_n = O(1)$ $(n \to \infty)$ for all y and for some scalar l. Then, we study the (SSE) $(\ell_r^p)_\Delta + s_x = s_b$ for $r > 0$.

7.2.1. Solvability of the (SSE) $(\ell_a^p)_\Delta + s_x = s_u$, $u > 0$. The next result follows from Proposition 14 ii), where $b = (u^n)_{n\geq 1}$ and $\sigma^{(q)} = (u^{-qn} \sum_{k=1}^{n} a_k^q)_{n\geq 1}$.

Corollary 18. *The set $\overline{S_u}$ of all positive sequences x such that $(\ell_a^p)_\Delta + s_x = s_u$ is determined by*

$$\overline{S_u} = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } \sigma^{(q)} \in \ell_\infty, \\ \emptyset & \text{if } \sigma^{(q)} \notin \ell_\infty. \end{cases}$$

Proof. The condition $\sigma^{(q)} \in \ell_\infty$ with $b = (u^n)_{n\geq 1}$ implies there is $K > 0$ such that

$$\frac{a_1^q}{u^{qn}} \leq \frac{a_1^q + \cdots + a_n^q}{u^{qn}} \leq K \qquad \text{for all } n$$

and $u \geq 1$. So we have $1/b = (1/u^n)_{n\geq 1} \in c$ and by Proposition 14 ii) we conclude $\overline{S_u} = \mathrm{cl}^\infty(u)$ if $\sigma^{(q)} \in \ell_\infty$. Then, the condition $\sigma^{(q)} \notin \ell_\infty$ implies $\overline{S_u} = \emptyset$. This completes the proof. □

Remark 19. It can be easily seen that if $\sum_{k=1}^{\infty} a_k^q < \infty$ then we have $\overline{S_u} = \mathrm{cl}^\infty(b)$ if and only if $u \geq 1$. Then if $\sum_{k=1}^{\infty} a_k^q = \infty$ then the condition $\overline{S_u} = \mathrm{cl}^\infty(u)$ implies $u > 1$.

Example 20. Let $\xi \in \mathbb{R}$. The set $\overline{S_{\xi,u}}$ of all the solutions of the (SSE) $\left(\ell^p_{(1/n^\xi)_{n\geq 1}}\right)_\Delta + s_x = s_u$ is determined in the following way.

i) If $u < 1$ then $\overline{S_{\xi,u}} = \emptyset$. ii) If $u > 1$ then $\overline{S_{\xi,u}} = \mathrm{cl}^{\infty}(u)$. iii) If $u = 1$ then $\overline{S_{\xi,u}} = \begin{cases} \mathrm{cl}^{\infty}(u) & \text{if } \xi > 1/q \\ \emptyset & \text{if } \xi \leq 1/q. \end{cases}$ This result follows from the fact that the series $\sum_{k=1}^{\infty} 1/k^{\xi q}$ is convergent if and only if $\xi > 1/q$.

7.2.2. On the solvability of the (SSE) $(\ell_r^p)_\Delta + s_x = s_b$ with $r > 0$. In this part we state a result on the solvability of the (SSE) $(\ell_r^p)_\Delta + s_x = s_b$ which can be shown by Proposition 14 ii).

Corollary 21. *Let $r > 0$. The set $\overline{\overline{S_r}}$ of all positive sequences that satisfy the (SSE) $(\ell_r^p)_\Delta + s_x = s_b$ with $r > 0$ is determined in the following way*

i) *If $r < 1$ then $1/b \in c \Longrightarrow \overline{\overline{S_r}} = \mathrm{cl}^{\infty}(b)$ and $1/b \notin \ell_\infty \Longrightarrow \overline{\overline{S_r}} = \emptyset$.*

ii) *If $r = 1$ then* $\overline{\overline{S_r}} = \begin{cases} \mathrm{cl}^{\infty}(b) & \text{if } \left(n^{1/q}/b_n\right)_{n\geq 1} \in \ell_\infty, \\ \emptyset & \text{if } \left(n^{1/q}/b_n\right)_{n\geq 1} \notin \ell_\infty. \end{cases}$

iii) *If $r > 1$ then* $\overline{\overline{S_r}} = \begin{cases} \mathrm{cl}^{\infty}(b) & \text{if } (r^n/b_n)_{n\geq 1} \in \ell_\infty, \\ \emptyset & \text{if } (r^n/b_n)_{n\geq 1} \notin \ell_\infty. \end{cases}$

Remark 22. The (SSE) $(\ell_r^p)_\Delta + s_x = s_b$ is totally solved for all $r \geq 1$.

7.3. Solvability of the (SSE) $(\ell_r^p)_\Delta + s_x^0 = s_b^0$ for $p > 1$ and $r \neq 1$.

We state the next result where $a = (r^n)_{n\geq 1}$ and $b \in U^+$ and we solve the equation $(\ell_r^p)_\Delta + s_x^0 = s_b^0$ for $r \neq 1$.

Proposition 23. *Let $r \neq 1$ and $b \in U^+$. Let $\overline{S_r^0}$ be the set of all positive sequences $x \in U^+$ that satisfy the (SSE) $(\ell_r^p)_\Delta + s_x^0 = s_b^0$. Then we have*

i) *If $r < 1$ then*

$$\overline{S_r^0} = \begin{cases} \mathrm{cl}^{\infty}(b) & \text{if } 1/b \in c_0, \\ \emptyset & \text{if } 1/b \notin c_0. \end{cases}$$

ii) *If $r > 1$ then $\overline{S_r^0} = \overline{\overline{S_r}}$ defined by* iii) *in* Corollary 21.

Proof. i) follows from Remark 17 where $\sum_{k=1}^{n} r^{qk} < \infty$. ii) Since $r > 1$ by Lemma 9 iii) we have $(\ell_r^p)_\Delta = \ell_r^p$ and $x \in \overline{S_r^0}$ if and only if

$$\ell^p_{(r^n/b_n)_n} + s_y^0 = c_0 \text{ with } y = x/b.$$

We have $\ell^p_{(r^n/b_n)_n} \subset \ell^p$ if $(r^n/b_n)_{n\geq 1} \in M(\ell^p, \ell^p)$ and since $M(\ell^p, \ell^p) = \ell_\infty$ we conclude by Proposition 11 ii) with $\mathcal{E} = \ell^p_{(r^n/b_n)_n}$ that the condition $(r^n/b_n)_{n\geq 1} \in \ell_\infty$ implies $\overline{\overline{S_r^0}} = \mathrm{cl}^{\infty}(b)$. Now, the condition $\overline{S_r^0} \neq \emptyset$ implies $\ell^p_{(r^n/b_n)_n} \subset c_0$ and $(r^n/b_n)_{n\geq 1} \in M(\ell^p, c_0) = \ell_\infty$. So, the condition $(r^n/b_n)_{n\geq 1} \notin \ell_\infty$ implies $\overline{S_r^0} = \emptyset$. This concludes the proof. □

In a similar way we may solve the (SSE) $(\ell_r^p)_\Delta + s_x^0 = s^0_{(n^\theta)_{n\geq 1}}$ with $\theta \in \mathbb{R}$.

Corollary 24. *Let $r > 0$ and $\theta \in \mathbb{R}$. Let $\overline{S^0_{r,\theta}}$ be the set of all positive sequences $x \in U^+$ that satisfy the (SSE) $(\ell^p_r)_\Delta + s^0_x = s^0_{(n^\theta)_{n\geq 1}}$. Then we have*

i) *If $r < 1$ then* $\overline{S^0_{r,\theta}} = \begin{cases} \mathrm{cl}^\infty(b) & \text{if } \theta > 0, \\ \emptyset & \text{if } \theta \leq 0. \end{cases}$

ii) *If $r = 1$ then $\overline{S^0_{r,\theta}} = \mathrm{cl}^\infty(b)$ if $\theta > 1/q$, and $\overline{S^0_{r,\theta}} = \emptyset$ if $\theta < 1/q$.*

iii) *If $r > 1$ then $\overline{S^0_{r,\theta}} = \emptyset$.*

Proof. i) and iii) are direct consequences of Proposition 23 ii) Let α be a sequence defined by $\alpha_1 = 1$ and $\alpha_n = 1/\ln n$ for $n \geq 2$. We have

$$\mathcal{E} = D_{(1/n^\theta)_{n\geq 1}} * \ell^p_\Delta = D_{(1/n^\theta)_{n\geq 1}} * \Sigma(\ell^p) \subset s_\alpha$$

since

$$\left(\frac{\ln n}{n^\theta}\right)^q \sum_{k=1}^n 1^q = \frac{(\ln n)^q}{n^{\theta q - 1}} = O(1) \qquad (n \to \infty) \text{ for } \theta > 1/q.$$

By Proposition 11 ii) we conclude $\overline{S^0_{r,\theta}} = \mathrm{cl}^\infty(b)$ if $\theta > 1/q$. Now, the condition $\overline{S^0_{r,\theta}} \neq \emptyset$ implies $(\ell^p_r)_\Delta \subset s^0_{(n^\theta)_n}$ and $D_{(1/n^\theta)_n}\Sigma \in (\ell^p, c_0)$, that is, $1/n^{\theta q - 1} = O(1)$ $(n \to \infty)$ and $\theta \geq 1/q$. So, the condition $\theta < 1/q$ implies $\overline{S^0_{r,\theta}} = \emptyset$. This completes the proof. □

8. Solvability of the (SSE) $(\ell^p_r)_\Delta + F_x = F_u$ for $p > 1$.

In this section, we solve the next particular (SSE) $(\ell^p_r)_\Delta + c_x = c_u$, $(\ell^p_r)_\Delta + s_x = s_u$ and $(\ell^p_r)_\Delta + s^0_x = s^0_u$ with r, $u > 0$. First, we recall a general result on the resolution of the (SSE) $\left(s^0_a\right)_\Delta + s^{(c)}_x = s^{(c)}_b$ and $(\ell^p_a)_\Delta + s^{(c)}_x = s^{(c)}_b$ for a, $b \in U^+$.

8.1. Solvability of the (SSE) $(\ell^p_r)_\Delta + c_x = c_u$ with r, $u > 0$.

The solvability of the (SSE) $(\ell^p_r)_\Delta + c_x = c_u$ is a direct consequence of [**13**, Theorem 1]. We recall this result where we use the notation $\sigma_n = \sigma^{(1)}_n = \left(\sum_{k=1}^n a_k\right)/b_n$.

Lemma 25 ([**13**, Theorem 1, pp. 117–118])**.** *Let a, $b \in U^+$. Then we have:*

i) *The set $S\left(\left(s^0_a\right)_\Delta, c\right)$ of all the solutions of the (SSE) $\left(s^0_a\right)_\Delta + s^{(c)}_x = s^{(c)}_b$ is determined in the following way:*
   a) *If $a \notin cs$ (that is, $\sum_{k=1}^\infty a_k = \infty$), then we have* $S\left(\left(s^0_a\right)_\Delta, c\right) = \begin{cases} \mathrm{cl}^c(b) & \text{if } \sigma \in s_1, \\ \emptyset & \text{if } \sigma \notin s_1. \end{cases}$
   b) *If $a \in cs$, then we have*

$$S\left(\left(s^0_a\right)_\Delta, c\right) = \begin{cases} \mathrm{cl}^c(b) & \text{if } 1/b \in c_0, \\ \mathrm{cl}^c(e) & \text{if } 1/b \in c \setminus c_0, \\ \emptyset & \text{if } 1/b \notin c. \end{cases} \tag{5}$$

ii) *The set* $S\left(\left(\ell_a^p\right)_\Delta, c\right)$ *with* $p>1$*, of all the solutions of the (SSE)* $\left(\ell_a^p\right)_\Delta + s_x^{(c)} = s_b^{(c)}$ *is determined in the following way:*

a) *If* $a^q \notin cs$*, then* $S\left(\left(\ell_a^p\right)_\Delta, c\right) = \begin{cases} \mathrm{cl}^c(b) & \text{if } \sigma^{(q)} \in s_1, \\ \emptyset & \text{if } \sigma^{(q)} \notin s_1. \end{cases}$

b) *b) If* $a^q \in cs$*, then* $S\left(\left(\ell_a^p\right)_\Delta, c\right) = S\left(\left(s_a^0\right)_\Delta, c\right)$ *defined by* (5).

To state the following result we use the set

$$\mathcal{V}^q = \left\{(r,u) \in \mathbb{R}^{+2} : r < 1 \leq u \text{ or } r = 1 < u \text{ or } 1 < r \leq u\right\}.$$

Notice that we have $\left(\mathcal{V}^q\right)^c = \emptyset$ if and only if $r<1$ and $u<1$ or $u<r=1$ or $r > \max(1,u)$. By Lemma 25 ii) we obtain the following result.

Proposition 26. *Let* r*,* $u>0$ *and* $p>1$*. Then, the set* $S_p^c = S\left(\left(\ell_r^p\right)_\Delta, c\right)$ *of all the solutions of the (SSE)* $\left(\ell_r^p\right)_\Delta + s_x^{(c)} = s_u^{(c)}$ *is determined by*

$$S_p^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } (r,u) \in \mathcal{V}^q, \\ \emptyset & \text{if } (r,u) \notin \mathcal{V}^q. \end{cases}$$

Proof. The proof follows from Lemma 25 ii). If $r<1$, trivially we have $(r^n)_{n\geq 1} \in cs$ and $1/b = (u^{-n})_{n\geq 1} \in c$ if and only if $u \geq 1$. Let $r=1$. We have $(r^n)_{n\geq 1} \notin cs$ and $\sigma_n = nu^{-nq} = O(1)$ $(n\to\infty)$ which implies that $S_p^c = \mathrm{cl}^c(u)$ if and only if $u>1$. If $r>1$, we obtain

$$\sigma_n \sim \frac{r^q}{r^q-1}\left(\frac{r}{u}\right)^{nq} \quad (n\to\infty)$$

and $\sigma \in \ell_\infty$ which implies $S_p^c = \mathrm{cl}^c(u)$ if and only if $r \leq u$. This concludes the proof. □

This result leads to the next remark.

Remark 27. By [**14**, Theorem 24, p. 238], we have $S_p^c = S_{c_0}^c$. If $r \leq 1$ we have $S_p^c = S_E^c$ for $E = c_0$, c, ℓ_∞ or w_∞ by [**14**, Theorem 24, p. 238] and [**14**, Theorem 28, p. 239].

We may rewrite the set S_p^c as follows.

Corollary 28. *Let* $p>1$ *and* r*,* $u>0$*. Then we have*

i) *If* $r<1$*, then* $S_p^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } u \geq 1, \\ \emptyset & \text{if } u < 1. \end{cases}$

ii) *If* $r=1$*, then* $S_p^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1. \end{cases}$

iii) *If* $r>1$*, then* $S_p^c = \begin{cases} \mathrm{cl}^c(u) & \text{if } r \leq u, \\ \emptyset & \text{if } r > u. \end{cases}$

8.2. Solvability of the (SSE) $(\ell_r^p)_\Delta + s_x = s_u$ with $r,\ u > 0$.

Let $S_p^\infty = S\left((\ell_r^p)_\Delta, s_1\right)$ be the set of all the positive sequences x that satisfy the (SSE) $(\ell_r^p)_\Delta + s_x = s_u$ with $r,\ u > 0$ and $p > 1$. For $r,\ u > 0$ and $q = p/(p-1) > 1$ we write $\sigma_n^{(q)}(r,u) = u^{-qn}\sum_{k=1}^n r^{qk}$. It can be easily seen that $\mathcal{V}^q = \left\{(r,u) : \sigma^{(q)}(r,u) = \left(\sigma_n^{(q)}(r,u)\right)_{n\geq 1} \in \ell_\infty\right\}$. We obtain the following result.

Theorem 29. *Let $p > 1$ and $r,\ u > 0$. Then we have*

$$S_p^\infty = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } (r,u) \in \mathcal{V}^q, \\ \emptyset & \text{if } (r,u) \notin \mathcal{V}^q. \end{cases} \tag{6}$$

Proof. The condition $\sigma^{(q)}(r,u) \in \ell_\infty$ implies $1/b = (u^{-n})_{n\geq 1} \in c$ and $u \geq 1$. So by Proposition 14 ii) we have $S_p^\infty = \mathrm{cl}^\infty(u)$ if $(r,u) \in \mathcal{V}^q$. Again, by Proposition 14 we have $S_p^\infty = \emptyset$ if $(r,u) \notin \mathcal{V}^q$. □

Remark 30. The proof of Theorem 29 follows from the equivalence of $D_{1/u}\Sigma D_r \in (\ell^p, c)$ and $D_{1/u}\Sigma D_r \in (\ell^p, \ell_\infty)$.

Remark 31. It can be easily verified that $S_p^\infty = S_{c_0}^\infty$ for $r \geq 1$.

8.3. Solvability of the (SSE) $(\ell_r^p)_\Delta + s_x^0 = s_u^0$ with $r,\ u > 0$.

Let $S_p^0 = S\left((\ell_r^p)_\Delta, c_0\right)$ be the set of all the positive sequences x that satisfy the (SSE) $(\ell_r^p)_\Delta + s_x^0 = s_u^0$ with $r,\ u > 0$ and $p \geq 1$.

Theorem 32. *Let $p > 1$. If $r \leq 1$ then $S_p^0 = S_E^0$ where $E = c_0,\ c,\ \ell_\infty$ defined in* Lemma 13 *and if $r > 1$ then $S_p^0 = S_p^\infty$ where S_p^∞ is defined in* (6).

Proof. If $r \neq 1$, we may apply Proposition 23 with $b = (u^n)_{n\geq 1}$. The case $r = 1$ follows from Proposition 14 iii) where $\alpha = (1/n)_{n\geq 1}$. We have

$$n^q u^{-qn}\sum_{k=1}^n r^{qk} = n^{q+1}u^{-qn} = O(1) \qquad (n \to \infty)$$

if $u > 1$. This completes the proof. □

Remark 33. We may rewrite S_p^0 defined in Theorem 32 as follows:

i) If $r \leq 1$, then we have $S_p^0 = S_E^0 = \begin{cases} \mathrm{cl}^\infty(u) & \text{if } u > 1, \\ \emptyset & \text{if } u \leq 1 \end{cases}$,

where $E = c_0,\ c,\ \ell_\infty$ or w_∞ by [**14**, Theorem 24 p. 238] and [**14**, Theorem 28, p. 239].

ii) If $r > 1$ then, by [**14**, Theorem 16, p. 235], we have $S_p^0 = S_{c_0}^\infty = S_c^\infty$ where $S_{c_0}^\infty$ is defined by (4).

References

1. Maddox I. J., *Infinite matrices of operators,* Springer-Verlag, Berlin, Heidelberg and New York, 1980.

2. de Malafosse B., *Contribution à l'étude des systèmes infinis*, Thèse de Doctorat de 3è cycle, Université Paul Sabatier, Toulouse III, 1980.

3. de Malafosse B., *On some BK space*, Int. J. Math. Math. Sci. **28** (2003), 1783–1801.
4. de Malafosse B., *Sum and product of certain BK spaces and matrix transformations between these spaces*, Acta Math. Hung. **104**(3) (2004), 241–263.
5. de Malafosse B., *On the Banach algebra* $\mathcal{B}(l_p(\alpha))$, Int. J. Math. Math. Sci. **60** (2004), 3187–3203.
6. de Malafosse B., *Sum of sequence spaces and matrix transformations,* Acta Math. Hung. **113**(3) (2006), 289–313.
7. de Malafosse B., *Application of the infinite matrix theory to the solvability of certain sequence spaces equations with operators.* Mat. Vesnik **54**(1) (2012), 39–52.
8. de Malafosse B., *Applications of the summability theory to the solvability of certain sequence spaces equations with operators of the form* $B(r,s)$. Commun. Math. Anal. **13**(1) (2012), 35–53.
9. de Malafosse B., *Solvability of certain sequence spaces inclusion equations with operators,* Demonstratio Math. **46**(2) (2013), 299–314.
10. de Malafosse B., *Solvability of certain sequence spaces equations with operators*, Novi Sad. J. Math. **44**(1) (2014), 9–20.
11. de Malafosse B., *Solvability of sequence spaces equations using entire and analytic sequences and applications*, J. Indian Math. Soc. **81**(1–2) (2014), 97–114.
12. de Malafosse B., *On sequence spaces equations of the form* $E_T + F_x = F_b$ *for some triangle* T, Jordan J. Math. Stat. **8**(1) (2015), 79–105.
13. de Malafosse B., *Solvability of sequence spaces equations of the form* $(E_a)_\Delta + F_x = F_b$, Fasc. Math. **55** (2015), 109–131.
14. de Malafosse B., *New results on the sequence spaces equations using the operator of the first difference,* Acta Math. Univ. Comenian. **86**(2) (2017), 227–242.
15. de Malafosse B., *Extension of some results on the (SSIE) and the (SSE) of the form* $F \subset \mathcal{E} + F'_x$ *and* $\mathcal{E} + F_x = F$, Fasc. Math. **59** (2017), 107–123.
16. de Malafosse B., *Application of the infinite matrix theory to the solvability of sequence spaces inclusion equations with operators,* in press in Ukrainian Math. J.
17. de Malafosse B., *On new classes of sequence spaces inclusion equations involving the sets* c_0, c, ℓ_p, $(1 \le p \le \infty)$, w_0 *and* w_∞, J. Indian Math. Soc. **84**(3–4) (2017), 211–224.
18. de Malafosse B., Malkowsky E., *Matrix transformations in the sets* $\chi\left(\overline{N}_p\overline{N}_q\right)$ *where* χ *is in the form* s_ξ, *or* s_ξ°, *or* $s_\xi^{(c)}$. Filomat **17** (2003), 85–106.
19. de Malafosse B., Malkowsky E., *On the solvability of certain (SSIE) with operators of the form* $B(r,s)$, Math. J. Okayama. Univ. **56** (2014), 179–198.
20. de Malafosse B., Rakočević V., *Calculations in new sequence spaces and application to statistical convergence*, Cubo A **12**(3) (2010), 117–132.
21. de Malafosse B., Rakočević V., *Matrix transformations and sequence spaces equations*, Banach J. Math. Anal. **7**(2) (2013), 1–14.
22. de Malafosse B., Malkowsky E., *On sequence spaces equations using spaces of strongly bounded and summable sequences by the Cesàro method,* Antartica J. Math. **10**(6) (2013), 589–609.
23. Farés A., de Malafosse B., *Sequence spaces equations and application to matrix transformations*, Int. Math. Forum **3**(19) (2008), 911–927.
24. Malkowsky E., *The continuous duals of the spaces* $c_0(\Lambda)$ *and* $c(\Lambda)$ *for exponentially bounded sequences* Λ, Acta Sci. Math. (Szeged) **61** (1995), 241–250.
25. Malkowsky E., *Linear operators between some matrix domains*, Rend. del Circ. Mat. di Palermo. **68**(2) (2002), 641–655.
26. Malkowsky E., Rakočević V., *An introduction into the theory of sequence spaces and measure of noncompactness*, Zbornik radova, Matematički institut SANU **9** (17) (2000), 143–243.
27. Wilansky, A., *Summability through Functional Analysis,* North-Holland Mathematics Studies 85, 1984.

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