THE PRAEGER-XU GRAPHS: CYCLE STRUCTURES, MAPS, AND SEMITRANSITIVE ORIENTATIONS

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ABSTRACT. We consider tetravalent graphs within a family introduced by Praeger and Xu in 1989. These graphs have the property of having exceptionally large symmetry groups among all tetravalent graphs. This very property makes them unsuitable for the use of simple computer techniques. We apply techniques from coding theory to determine for which values of the parameters the graphs allow cycle structures, semitransitive orientations, or rotary maps; all without recourse to the use of computers.

1. INTRODUCTION

In [11], it was proved that a finite connected tetravalent arc-transitive graph either has a relatively small symmetry group (in a sense that can be made precise) or it belongs to a family of graphs, now known as the Praeger-Xu graphs.

Several standard methods for investigating structures in arc-transitive graphs can efficiently be used only when the graph has a relatively small symmetry group. Therefore, a family of graphs with large symmetry groups often needs to be analyzed with ad hoc techniques, tailored to the particular family.

The aim of this paper is to provide the techniques for analyzing *cycle structures*, *semitransitive orientations*, and *rotary maps* underlying the Praeger-Xu graphs.

We do this in the following way:

In Section 2, we give three definitions (or models) for the Praeger-Xu graphs.

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In Section 3, we talk about general graph symmetries, after which we define symmetries of the Praeger-Xu graphs and show that their groups of symmetries are almost always isomorphic to a semidirect product of a dihedral group with \mathbb{Z}_2^n .

In Section 4, we define three symmetric structures of particular interest with regard to highly symmetric graphs: *cycle structures* (dart-transitive cycle decompositions), *semitransitive orientations* (orientations whose group is transitive on both vertices and edges), and *rotary maps* (embeddings of the graph on a surface whose symmetry groups include all possible rotational symmetries).

In Section 5, we discuss a connection between symmetry groups of the Praeger-Xu graphs and *dihedral codes*. These are special cyclic codes which are invariant both under reversals as well as cyclic shifts. By invoking the usual correspondence between codewords and polynomials in a variable t, we show that dihedral codes correspond one-to-one to certain palindromic polynomials. Moreover, the degree of the polynomial determines the dimension of the code. We define sets $B_{k,i}$ of all codewords ending with k zeroes whose corresponding polynomials are palindromic and divide $\frac{t^n-1}{(t-1)^i}$. Finally, we apply the coding theory to the symmetry groups of the Praeger-Xu graphs.

In Section 6, we determine which Praeger-Xu graphs admit cycle structures. The classification is based on the membership of generators in the $B_{k,i}$ sets.

In Section 7, we similarly classify semitransitive orientations of Praeger-Xu graphs. Finally, in Section 8, we classify rotary maps of these graphs and show that they are all reflexible.

Section 9 is an appendix, in which we present the results for the special case n = 4.

2. Definitions and models

The Praeger-Xu graphs are the tetravalent case of the family that appeared in [13] under the name C(m, r, s), where m, r and s are integers such that $m \ge 2, r \ge 3$ and $s \ge 1$. The valence of C(m, r, s) is 2m. Due to their unique properties, these graphs appeared in several later works by a number of researchers in many different contexts [4, 10, 14, 16]. In the recent works, the notation PX(n, k) was introduced to denote the tetravalent graph C(2, n, k).

The graph PX(n, k) is also described in [3] in two different ways and [8] presents several different but equivalent definitions, which we will summarize here for completeness. Throughout the section, we assume that n and k are integers, $n \ge 3$ and $k \ge 1$.

2.1. The Bitstring Model

To give this construction of the graph, we need some notation about bitstrings. A *bitstring* of length k is a word of length k over the alphabet $\mathbb{Z}_2 = \{0, 1\}$. We also allow for the empty bitstring and say that it has length 0. For example x = 1011110 is a bitstring of length 7. If x is a bitstring of length k, then x_i is its *i*-th entry, and x^i is the string obtained from x by changing the *i*-th symbol x_i .

Strings begin at position 0, so if x = 001, then $x_0 = 0$, $x_1 = 0$, and $x_2 = 1$, while $x^0 = 101$, $x^1 = 011$ and $x^2 = 000$. Also, if x is of length k - 1, 1x is the string of length k formed from x by placing a '1' in front; similar definitions hold for the k-strings 0x, x0, x1. Finally, the string \bar{x} is the reversal of x, so if x = 001, then $\bar{x} = 100$.

The vertices of the graph PX(n, k) are the ordered pairs (i, x), where $i \in \mathbb{Z}_n$ and x is a bitstring of length k. Edges are the pairs of the form $\{(i, ax), (i+1, xb)\}$, where x is any bitstring of length k - 1 and a, b are elements of \mathbb{Z}_2 . Thus, for example in PX(5,3), the vertex (2,110) is adjacent to the four vertices (1,011), (1,111), (3,100), and (3,101). We will refer to the set of vertices (i, x), for a fixed i, as the i^{th} fibre.

Each edge $\{(i, ax), (i + 1, xb)\}$ of $\Gamma = PX(n, k)$ is associated with two directed edges that we will refer to as *darts*: the dart $(i, ax) \rightarrow (i + 1, xb)$ and the dart $(i + 1, xb) \rightarrow (i, ax)$. It will be helpful to refer to the dart $(i, ax) \rightarrow (i + 1, xb)$ as forward facing and the second dart as backward facing.

We note that even though the Praeger-Xu graphs PX(n, k) are defined for all integers $n \geq 3$ and $k \geq 1$, they are dart-transitive if and only if n > k [13, Theorem 2.10]. We will restrict our attention to this case.

2.2. The Sausage Graph Model

The *n*-sausage graph or doubled cycle is the multigraph obtained from the *n*cycle by doubling each of its edges. It has *n* vertices $u_0, u_1, u_2, \ldots, u_{n-1}$, and for each $i \in \mathbb{Z}_n$, there are two edges, called a_i and b_i , each joining u_i to u_{i+1} . The vertices of the graph PX(n, k) can then be identified with the *k*-paths in the *n*-sausage graph of the form $u_j c_j u_{j+1} c_{j+1} u_{j+2} \ldots c_{j+k-1} u_{j+k}$, $c_i \in \{a_i, b_i\}$, for $j \leq i \leq j + k - 1$ (with indices taken modulo *n*). For brevity, we denote such *k*-path by $c_j c_{j+1} \ldots c_{j+k-1}$. The *j*th fibre is the set of vertices corresponding to paths from u_j to u_{j+k} . A u_j -to- u_{j+k} path is adjacent to a u_{j+1} -to- u_{j+1+k} path if they coincide in the last k - 1 edges of the first path and the first k - 1 edges of the second path. For example, in PX(5,3), the four neighbors of the path $b_2 b_3 a_4$ (on vertices u_2, u_3, u_4, u_0) are the paths $a_1 b_2 b_3, b_1 b_2 b_3, b_3 a_4 a_0$, and $b_3 a_4 b_0$.

2.3. The Window Model

In this model, vertices are strings of length n over the alphabet $\{0, 1, *\}$ such that some window of k consecutive spaces are reserved for the symbols 0 and 1 while the rest of the string contains only the symbol *. The last character is followed by the first, so that the string is viewed as being circular, and the window can bridge this gap. Two vertices are adjacent when their windows begin in spaces jand j + 1 and agree on their overlap. The j^{th} fibre consists of all vertices whose windows begin at the j^{th} space. To use the same example as above, in PX(5,3), the 5-string [**110] is adjacent to the strings [*011*], [*111*], [0**10], and [1**10].



Figure 1. PX(5,3). Fibres are ordered clockwise, vertices within a fibre are in lexicographical order from the outside to the inside.

2.4. Isomorphisms

It is easy to see that all three models describe the same graph, with the isomorphisms mapping the pair (i, x) from the bitstring model to the sausage graph u_i -to- u_{i+k} path $c_i c_{i+1} \dots c_{i+k-1}$ where c_{i+j} is a_{i+j} if $x_j = 0$ and b_{i+j} otherwise, and mapping (i, x) to the *n*-string y in which y_{i+j} is x_j if $0 \le j \le k-1$ and $y_{i+j} = *$ otherwise. Thus, in PX(5,3), the path

$$(0, 110) - (1, 100) - (2, 001) - (3, 011)$$

in the bitstring model corresponds to the path

 $b_0b_1a_2 - b_1a_2a_3 - a_2a_3b_4 - a_3b_4b_0$

in the sausage graph model and to the path

$$[110**]-[*100*]-[**001]-[1**01]$$

in the window model.

2.5. Cycles

There are three families of cycles that frequently occur in our analysis of the Praeger-Xu graphs PX(n, k). First, there is the family of 4-cycles induced by the vertices (i, 0x), (i + 1, x0), (i, 1x), (i + 1, x1) and connecting two consecutive fibres, which we shall call the *standard* 4-cycles. Observe that, if $n \neq 4$ and $k \geq 2$, every 4-cycle in PX(n, k) is standard. The second family consists of the *n*-cycles that visit each fibre exactly once; one representative of this family is the 'zero' cycle Z induced by the *n* vertices of the form $(i, 0^k), i \in \mathbb{Z}_n$, where 0^k stands for the k-bitstring of all 0's. We will call these cycles *transversal n*-cycles. The third family consists of the 2*n*-cycles which wrap twice around the *n* fibres of PX(n, k); a representative of this family being the 2*n*-cycle Z' induced by the vertices $(0, 0^k), (1, 0^k), \ldots, (n - k, 0^k), (n - k + 1, 0^{k-1}1), (n - k + 2, 0^{k-2}1^2), \ldots, (n - 1, 01^{k-1}), (0, 1^k), (1, 1^k), \ldots, (n - k, 1^k), (n - k + 1, 1^{k-1}0), (n - k + 2, 1^{k-2}0^2), \ldots, (n - 1, 1(0^{k-1}))$. We will call these *transversal* 2*n*-cycles.

3. Symmetries

3.1. Graph symmetries

For any graph Γ , a symmetry (often called an *automorphism*) of Γ is a permutation of the vertices of Γ which preserves its set of edges. The collection of all symmetries of Γ , called Aut(Γ), is then a group under composition. The group Aut(Γ) might act transitively on the vertices of Γ , on its edges, or on its darts, in which case(s) we say that Γ itself is *vertex-transitive*, *edge-transitive* or *dart-transitive*, respectively.

Let $v_0v_1 \ldots v_{r-1}v_0$ be a cycle of a graph Γ , $r \geq 3$. If there exists a symmetry σ of Γ mapping v_i to v_{i+1} , for all $i \in \mathbb{Z}_r$, then the sequence $[v_0, v_1, v_2, \ldots, v_{r-1}]$ is called a *consistent cycle* and σ is a *shunt* for this consistent cycle. The long-overlooked result of Biggs and Conway [1] states that in every *d*-valent dart-transitive graph Γ the set of consistent cycles splits into precisely d-1 orbits under the action of Aut(Γ). Notice that direction is important; the sequence $[v_0, v_1, v_2, \ldots, v_{r-1}]$ and the reverse sequence $[v_{r-1}, v_{r-2}, \ldots, v_2, v_1, v_0]$ need not be in the same orbit. If they are in the same orbit, we refer to each of them as *reflexible*.

3.2. The symmetry groups of PX(n,k)

We define here some symmetries of PX(n, k) which preserve the set of fibres (map fibres to fibres). The first two such symmetries are the symmetries ρ and μ given in the bitstring model by $(j, x)\rho = (j + 1, x)$ and $(j, x)\mu = (-j, \bar{x})$. Note that the group $\langle \rho, \mu \rangle$ is isomorphic to the dihedral group D_n of order 2n, and its induced action on the fibres of PX(n, k) is permutation isomorphic to the 'usual' transitive action of D_n on n points.

In the sausage graph model $u_i \rho = u_{i+1}$, $a_i \rho = a_{i+1}$ and $b_i \rho = b_{i+1}$, while μ interchanges $u_i \leftrightarrow u_{k-i}$, $a_i \leftrightarrow a_{k-1-i}$, $b_i \leftrightarrow b_{k-1-i}$. In the window model, ρ consists of shifting each character one step to the right, while μ is defined for each string y, by: $(y\mu)_i = y_{k-1-i}$, where the indices are taken mod n.

We next define symmetries τ_i , $i \in \mathbb{Z}_n$, setwise fixing each fibre. These are most easily seen in the sausage and window models: In the sausage graph model, τ_i simply exchanges a_i with b_i , leaving all vertices and all other edges fixed. In the window model, τ_i simply exchanges 0's and 1's in position *i* of each string, leaving * unchanged.

In the bitstring model, for $i \in \mathbb{Z}_n$, we define the symmetry τ_i to be the permutation which interchanges (j, x) with (j, x^{i-j}) for $j = i - k + 1, i - k + 2, \ldots, i$, and leaves all other vertices fixed.

Referring to the example from PX(5,3), in subsection 2.4, applying τ_2 to the path shown in three ways there, the result is the path:

(0, 111) - (1, 110) - (2, 101) - (3, 011)

corresponding to

 $b_0b_1b_2 - b_1b_2a_3 - b_2a_3b_4 - a_3b_4b_0$

corresponding to

$$[111 * *] - [*110*] - [* * 101] - [1 * *01].$$

In general, τ_i permutes without fixed points the vertices inside each of the fibres $i - k + 1, i - k + 2, \ldots, i$, leaving all other vertices fixed. Notice how conjugation works on the τ_i 's: $\tau_i^{\rho} = \tau_{i+1}$ and $\tau_i^{\mu} = \tau_{k-1-i}$. Consequently, $\rho^{\tau_i} = \rho \tau_i \tau_{i+1}$. If n > k, the symmetries $\tau_0, \tau_1, \ldots, \tau_{n-1}$ commute with each other and thus

If n > k, the symmetries $\tau_0, \tau_1, \ldots, \tau_{n-1}$ commute with each other and thus generate a group K which is isomorphic to \mathbb{Z}_2^n . Hence, the group of symmetries of PX(n,k) contains the group A generated by ρ, μ, τ_0 , which is isomorphic to a semidirect product of D_n with K. Further, K is the kernel of the induced action of A on the fibres of PX(n,k). Hence K is normal in A. For $n \neq 4$, the group A is all of Aut(PX(n,k)) [13]. When n = 4 and k = 3, 2, 1, A has index 2, 3, 9, respectively, in Aut(PX(n,k)). Results for the special case n = 4 will be reported in the appendix, Section 9. In the rest of this paper, we will assume that n is not 4.

Now, A acts dart-transitively on the graph, and so, by the result of Biggs and Conway [1], A must have three orbits of consistent cycles. One orbit consists of the standard 4-cycles; the symmetry $\tau_0 \mu \rho$ is a shunt for the particular cycle $(0, 0^k), (1, 0^{k-1}1), (0, 10^{k-1}), (1, 0^k)$. The symmetry ρ is a shunt for the cycle Z, while $\rho' = \rho \tau_0$ is a shunt for the cycle Z'. These are cycles of lengths n and 2n, and represent the other two orbits of consistent cycles, and hence, all three orbits consist of reflexible consistent cycles. Notice that $(\rho')^n$ is the all-swapper $\alpha = \tau_0 \tau_1 \tau_2 \dots \tau_{n-1}$.

Lemma 3.1. Every element of the coset ρK in A is conjugate to either ρ or ρ' via an element in K.

Proof. Let $L = \langle \tau_i \tau_{i+1} | i \in \mathbb{Z}_n \rangle$ and observe that L is an index 2 subgroup of K, with the other coset of L in K being $\tau_0 L$. Since $\rho^{\tau_i} = \rho \tau_i \tau_{i+1}$, and since K is generated by the τ_i 's, it follows that $\rho^K = \rho L$, and $\rho'^K = (\rho \tau_0)^K = (\rho \tau_0) L$. Therefore, $\rho^K \cup \rho'^K = \rho K$.

The proof of the following lemma is left to the reader:

Lemma 3.2. The following hold in any PX(n,k) for n > k.

- 1. The subgroup $K_i = \langle \tau_i, \tau_{i+1}, \tau_{i+2}, \ldots, \tau_{i+k-1} \rangle$ acts regularly on the vertices of the *i*-th fibre.
- 2. The τ_i 's not in K_i generate the pointwise stabilizer of the i^{th} fibre.
- 3. Any element of K which fixes one vertex in the i^{th} fibre fixes every vertex in the i^{th} fibre.
- 4. Any non-trivial element of K moves vertices in at least k different fibres.
- 5. Any element of K fixing n k or more vertices from different fibres must be trivial.

4. Symmetric structures

Our aim is to investigate three ideas related to symmetry in the Praeger-Xu graphs. These are the ideas of a *cycle structure*, a *semitransitive orientation* and a *rotary* map.

4.1. Cycle structures

A cycle structure in a tetravalent graph Γ is a partition \mathcal{Y} of its edges into cycles such that the subgroup $\operatorname{Aut}(\mathcal{Y})$ of $\operatorname{Aut}(\Gamma)$ which preserves \mathcal{Y} is transitive on the darts of Γ . It follows that the cycles of \mathcal{Y} must be consistent and all of the same length; in fact, they must all be within the same orbit of consistent cycles under $\operatorname{Aut}(\Gamma)$. For example, in K_5 , any 5-cycle and its complement form a cycle structure. In the Octahedron, there is a cycle structure consisting of three 4-cycles, and there is another consisting of four edge-disjoint 3-cycles. Among tetravalent dart-transitive graphs, a surprising number admit cycle structures. See [9] for more details.

Two cycle structures \mathcal{Y} and \mathcal{Y}' in a graph Γ are said to be *isomorphic* if there exists a symmetry of Γ mapping the cycles in \mathcal{Y} to the cycles in \mathcal{Y}' . We will call a cycle structure *bipartite* provided that we can partition the cycles of \mathcal{Y} into two colors, red and green, so that each vertex belongs to one cycle of each color. In our examples, the structure on K_5 is bipartite, but neither of the above cycle structures on the Octahedron is.

4.2. Semitransitive orientations

An orientation is a digraph Δ such that for all vertices u and v, if the pair (u, v) is a dart of Δ , then (v, u) is not a dart of Δ . If (u, v) is a dart in Δ , we say that the vertex u is its tail and v is its head. A semitransitive orientation of a graph Γ is an orientation Δ such that the underlying graph of Δ is Γ and $\operatorname{Aut}(\Delta)$ is transitive on the vertices and on the darts of Δ (and, so also transitive on the edges of Γ). Any sequence of darts in an orientation Δ of a graph Γ that induces a cycle in Γ will be called a cycle in Δ . A cycle in Δ is said to be consistent if there exists a $g \in \operatorname{Aut}(\Delta)$ which is a shunt for its underlying cycle in Γ . Note that a consistent cycle C in Δ must have the property that each of its vertices must have one dart of C coming in and one dart of C coming out. We say that two semitransitive orientations are *isomorphic* if there exists an automorphism of Γ that maps the darts of the first one onto the darts of the second.

For example, in the Octahedron, if we choose the orientation of each edge so that arrows around each face all point clockwise or all counterclockwise, we get a semitransitive orientation. K_5 on the other hand, has no semitransitive orientation.

4.3. Rotary maps

A map is an embedding of a graph (or multigraph) Γ into a surface S so that each connected component of $S \setminus \Gamma$ is, topologically, a disk; these components are the faces of the map. If the surface S is orientable, we call the map orientable.

All the maps we will encounter in this paper are *polytopal* maps, which means that no face is incident more than once with any edge or any vertex. We can present a polytopal map as a pair (Γ, \mathcal{F}) where Γ is a graph and \mathcal{F} is a collection of cycles of Γ , called faces, satisfying these two properties: (1) each edge of Γ belongs to two faces, and (2) for any vertex v, the edges containing v can be put into a circular order, such that any two consecutive edges in this order share a face incident with v.

A symmetry of a map \mathcal{M} is a symmetry of Γ which preserves \mathcal{F} . The map \mathcal{M} is rotary provided that for some face f and some vertex v of f, $\operatorname{Aut}(\mathcal{M})$ has elements R and S where R is a shunt of f and S acts as a rotation one step about v. By replacing S with S^{-1} if necessary, we may assume that RS^{-1} fixes an edge incident with both f and v, and is therefore an involution. Further, the map \mathcal{M} is reflexible provided that it is rotary and also has a symmetry X which acts as a reflection fixing v and reversing f. We set $\operatorname{Aut}(\mathcal{M})$ to be the group of all symmetries of \mathcal{M} and $\operatorname{Aut}^+(\mathcal{M})$ to be the subgroup generated by R and S. If \mathcal{M} is orientable and reflexible, then $\operatorname{Aut}^+(\mathcal{M})$ has index 2 in $\operatorname{Aut}(\mathcal{M})$. Otherwise, the two groups are the same. The group of symmetries $\operatorname{Aut}(\mathcal{M})$ contains at most twice as many elements as does the set of darts of \mathcal{M} , and \mathcal{M} is reflexible if and only if $|\operatorname{Aut}(\mathcal{M})|$ is exactly equal to twice the number of darts of \mathcal{M} .

A Petrie path in the map \mathcal{M} is a cycle C of Γ such that each two consecutive edges of C are consecutive edges of a face, but no three consecutive edges are consecutive edges of a face. One can view a Petrie path as a 'zig-zag' or 'leftright' path: when following the path, one turns right at a vertex and then left at the following vertex and then right again and then left and so on, until the cycle closes. If \mathcal{M} is reflexible, it has a symmetry T which is a shunt for some Petrie path, namely $T = RS^{-1}X$. It is not difficult to see that if we denote by \mathcal{P} the collection of Petrie paths of $\mathcal{M} = (\Gamma, \mathcal{F})$, the pair (Γ, \mathcal{P}) also forms a map, called the Petrie of \mathcal{M} and denoted by $P(\mathcal{M})$. The surface underlying $P(\mathcal{M})$ is usually not the same as the one underlying \mathcal{M} itself. The map \mathcal{M} is reflexible if and only if $P(\mathcal{M})$ is reflexible. If the map \mathcal{M} has p-gonal faces and q of them meet at each vertex, we say the map is of type $\{p, q\}$. Further, if the map has Petrie paths of

length r, we say the map is of type $\{p,q\}_r$. Then $P(\mathcal{M})$ has type $\{r,q\}_p$. See [15] for more details.

Our primary results in this paper concern the above structures in the case of Praeger-Xu graphs. They include Theorem 6.4 which classifies cycle structures in these graphs, Theorems 7.1 and 7.2 which show a one-to-one correspondence between their bipartite semitransitive orientations and their cycle structures, Corollary 7.4, which uses Theorems 7.1 and 7.2 to classify their semitransitive orientations, and Theorem 8.5 which classifies rotary maps of Praeger-Xu graphs.

5. Cyclic and dihedral codes

We have remarked that the kernel K of the induced action of the symmetry group $\operatorname{Aut}(\operatorname{PX}(n,k))$ on the fibres of the graph is isomorphic to $V = \mathbb{Z}_2^n$. We wish to make that isomorphism explicit and use it to exploit known facts about cyclic codes in order to state and prove new facts about cycle structures, semitransitive orientations and maps of $\operatorname{PX}(n,k)$.

Writing the elements of V in the form of bitstrings, we will associate the bitstring $u = u_0 u_1 u_2 \ldots u_{n-1} \in V$ with the product $\hat{u} = \tau_0^{u_0} \tau_1^{u_1} \ldots \tau_{n-1}^{u_{n-1}} \in K$. In particular, the all-swapper α is $\widehat{1^n}$.

The function sending u to \hat{u} is an isomorphism between the additive group V and the group K. The group (or vector space) V has an automorphism (or a linear operator) named P, which maps $u = u_0 u_1 u_2 \dots u_{n-1}$ to $uP = u_{n-1} u_0 u_1 u_2 \dots u_{n-2}$. This P is referred as a *cyclic shift*. Clearly, the n identities of the form $\tau_i^{\rho} = \tau_{i+1}$ yield $\hat{u}^{\rho} = \hat{uP}$.

Similarly, we can define a linear operator M of V by letting $u_0u_1u_2...u_{n-1}M = u_{k-1}u_{k-2}...u_1u_0u_{n-1}...u_k$ (where k is the parameter of PX(n,k)), and the identities $\tau_i^{\mu} = \tau_{k-1-i}$ yield $\hat{u}^{\mu} = \widehat{uM}$.

If F is any subgroup of K invariant under conjugation by ρ , then $F = \widehat{C}$ for some subspace C of V invariant under P. A subspace C of V invariant under the cyclic shift P is called a *cyclic code* (cyclic codes are not necessarily binary; we shall, however, be concerned only with binary cyclic codes). Cyclic codes are wellknown objects introduced in coding theory while also of interest in other branches of mathematics. If r is the largest number of consecutive 0's in a non-identity element of a cyclic code C then C contains a unique bitstring u which ends with exactly r 0's. We will call this u the generator bitstring of C and the first n - rsymbols of u the head of u. Then C is generated by $u, uP, uP^2, \ldots, uP^r$, which are linearly independent, and so C consists of 2^{r+1} vectors.

On the other hand, for any $x \in V$, the subspace $\langle x \langle P \rangle \rangle = \langle x, xP, \dots, xP^{n-1} \rangle$ is a cyclic code. It is quite likely that the generator bitstring u of $\langle x \langle P \rangle \rangle$ is not equal to any of the vectors x, xP, \dots, xP^{n-1} . To find u given x, we rely on the theory of cyclic codes. If we let each bitstring $a = a_0a_1a_2 \dots a_{n-1}$ correspond to the polynomial $a(t) = a_0 + a_1t + a_2t^2 + \dots + a_{n-1}t^{n-1}$ in $\mathbb{Z}_2[t]$, then an element u of $\langle x \langle P \rangle \rangle$ is a generator bitstring for $\langle x \langle P \rangle \rangle$ if its corresponding polynomial is the unique polynomial of the smallest degree among all polynomials corresponding to the elements of $\langle x \langle P \rangle \rangle$. One of the fundamental observations of the theory of cyclic codes asserts that this smallest degree polynomial u(t) is a divisor of $t^n - 1$ in $\mathbb{Z}_2[t]$. The polynomial u(t) is called the *generator polynomial* of $\langle x \langle P \rangle \rangle$. If uends with r 0's, then the degree of u(t) is n - r - 1 and the dimension of $\langle x \langle P \rangle \rangle$ is r + 1; see for example [5]. It is then easy to see that the generator bitstring ufor $\langle x \langle P \rangle \rangle$ corresponds to the greatest common divisor of $t^n - 1$ and x(t).

Lemma 5.1. If C_1 and C_2 are cyclic codes in \mathbb{Z}_2^n having generator bitstrings u and v respectively, such that C_2 is a subroup of C_1 of index 2, then v = u + uP or, equivalently, v(t) = (t+1)u(t).

Proof. Suppose that the order of C_2 is 2^k ; then the order of C_1 is 2^{k+1} , u ends in k 0's and v ends in k-1 0's. Note that v is the unique bitstring in C_2 ending in k-1 0's, while u + uP also ends in k-1 0's, hence v = u + uP.

If C is invariant under both P and M, we call it a *dihedral code*. In this case, the head of u must be a palindrome. Conversely, if x is any bitstring with a palindromic 'head', the subspace $\langle x \langle P \rangle \rangle$ will be a dihedral code; its generator bitstring need not be x. More detailed information on dihedral codes may be found in [7].

5.1. The sets $B_{k,i}$

Define $B_{k,i}$ to be the set of all bitstrings u of any length satisfying the following properties: (1) u ends with exactly k 0's (the remainder of u is a bitstring, which we shall call the *head* of u; it begins and ends with 1), (2) the head of u is palindromic, and (3) $(t-1)^{i}u(t)$ divides $t^{n}-1$ in $\mathbb{Z}_{2}[t]$, where n is the length of u.

For example, $B_{1,0}$ consists of bitstrings of length $n \ge 2$ that have a 0 in the last position while the first n-1 positions correspond to the coefficients of palindromic divisors of $t^n - 1$ of degree n - 2 (which can be obtained by dividing $t^n - 1$ by a palindromic divisor of degree 2). Thus, $B_{1,0}$ contains 10, 110 as well as 1010, 101010, 10101010, ..., and 110110, 110110110, etc. It is not hard to see that all of these bitstrings are formed by repeating the bitstring 10 or repeating the bitstring 110. We will abbreviate and say that $B_{1,0} = \{10\}^+ \cup \{110\}^+$. Similarly, $B_{1,1} = B_{1,2} = \{10\}^+$. In what follows, we will only need the sets $B_{k,0}$, $B_{k,1}$ and $B_{k,2}$. Table 1 shows the representative bitstrings for these sets for $1 \le k \le 5$.

6. Cycle structures in PX(n,k)

We have mentioned that in PX(n, k), the standard 4-cycles are consistent. As they are edge-disjoint, and cover all the edges of PX(n, k), they form a partition of the edge set. Since we assume $n \neq 4$, it is easy to see that this partition is invariant under A = Aut(PX(n, k)), and hence it is a cycle structure. We denote this cycle structure by \mathcal{Y}^* .

Table 1. The representative bitstrings for $B_{k,0}$, $B_{k,1}$, and $B_{k,2}$, for $k \in \{1, 2, 3, 4, 5\}$.

k	$B_{k,0}$	$B_{k,1}$	$B_{k,2}$
1	10, 110	10	10
2	100, 1100	100, 1100	1100
3	1000, 11000,	1000,	1000,
	101000, 111000	111000	111000
4	10000, 110000,	10000, 110000,	11110000,
	11110000, 101111010000	11110000, 101111010000	101111010000
5	100000, 1100000,	100000,	100000,
	10100000, 100100000,	10100000,	10100000,
	1111100000, 111011100000,	1111100000,	1111100000,
	110101100000, 101100110100000	110101100000	110101100000

Let Z be the 'zero' cycle in PX(n, k) defined in Section 2.5, i.e., the cycle induced by the vertices of the form $(i, 0^k)$, $i \in \mathbb{Z}_n$, where 0^k stands for the k-bitstring of all 0's. This is a consistent cycle with shunt ρ . Similarly, let Z' be the transversal 2n-cycle defined in Section 2.5, whose vertices form the orbit of $(0, 0^k)$ under the group generated by $\rho' = \rho \tau_0$. Notice that, by Lemma 3.2, the pointwise stabilizer in A of Z or of Z' must be trivial. Then Z is reversed by the symmetry μ , and Z' is reversed by the symmetry

$$\mu' = \mu \tau_{k+1} \tau_{k+2} \dots \tau_{n-1}.$$

Construction 6.1. For any u in \mathbb{Z}_2^n , let $C = \langle u \langle P \rangle \rangle$, and let \mathcal{Y}_u be the orbit of Z under \widehat{C} .

Construction 6.2. For any u in \mathbb{Z}_2^n , let $C = \langle u \langle P \rangle \rangle$, and let \mathcal{Y}'_u be the orbit of Z' under \widehat{C} .

We will state and prove criteria for each of these to be a cycle structure, and to be bipartite. We will then show that every cycle structure on PX(n,k) is equal to \mathcal{Y}^* or it is isomorphic to some \mathcal{Y}_u or \mathcal{Y}'_u .

We begin by showing which of the cycle sets coming from Constructions 6.1 and 6.2 are cycle structures.

Theorem 6.3. The following hold:

- 1. If u is a bitstring in $B_{k,0} \cap \mathbb{Z}_2^n$, then \mathcal{Y}_u is a cycle structure in PX(n,k).
- 2. For u in $B_{k,0} \cap \mathbb{Z}_2^n$, \mathcal{Y}_u is a bipartite cycle structure in PX(n,k) if and only if $u \in B_{k,1}$.
- 3. If u is a bitstring in $B_{k,1} \cap \mathbb{Z}_2^n$, then \mathcal{Y}'_u is a cycle structure in $\mathrm{PX}(n,k)$.
- 4. For u in $B_{k,1} \cap \mathbb{Z}_2^n$, \mathcal{Y}'_u is a bipartite cycle structure in PX(n,k) if and only if $u \in B_{k,2}$.

Proof. (1) The fact that u is a bitstring in $B_{k,0} \cap \mathbb{Z}_2^n$ tells us a lot about $C = \langle u \langle P \rangle \rangle$. First, C is a dihedral code of order 2^{k+1} . Second, no non-trivial element

of C has any more than k consecutive 0's. While an element of $F = \widehat{C}$ might fix a vertex (and hence every vertex in its fibre), no element of F other than the identity can fix, pointwise, an edge of the graph. Because the stabilizer of Z under F is trivial, the orbit \mathcal{Y}_u of Z under F consists of 2^{k+1} edge-disjoint *n*-cycles. This shows that \mathcal{Y}_u is a cycle decomposition.

Since C is a dihedral code, we know that $F = \hat{C}$ is normalized by ρ and μ , and, so $G = \langle \rho, \mu, \hat{u} \rangle$ is a group of order $n2^{k+2}$ which acts on \mathcal{Y}_u and its kernel $G \cap K$ is just F. Because $\langle \rho, \mu \rangle$ acts transitively on the darts of Z, G acts transitively on the darts of \mathcal{Y}_u , making \mathcal{Y}_u a cycle structure.

(2) For u in $B_{k,0} \cap \mathbb{Z}_2^n$, \mathcal{Y}_u is a cycle structure by part (1). Suppose that $u \in B_{k,1}$, and let v = u + uP. Then v ends in a 1 followed by k - 1 consecutive 0's, and has a palindromic head, so $D = \langle v \langle P \rangle \rangle$ is a dihedral code of index 2 in C. Thus $E = \widehat{D}$ has index 2 in F and no non-trivial element of E fixes any vertex. We denote by \mathcal{R} the orbit of Z under E and by \mathcal{G} its complement in \mathcal{Y}_u . Then \mathcal{R} is a collection vertex-disjoint cycles, as is \mathcal{G} , making \mathcal{Y}_u bipartite.

Conversely, for $u \in B_{k,0} \cap \mathbb{Z}_2^n$, if \mathcal{Y}_u is bipartite, then the color-preserving subgroup E of F must have index 2 in F. As ρ and μ preserve colors, they must conjugate E to itself. Then $E = \widehat{D}$ for some dihedral code D. If v is the generating bitstring for D, then v(t) must divide $t^n - 1$, and v(t) = (1+t)u(t) by Lemma 5.1. That implies that u is in $B_{k,1}$.

(3) If u is a bitstring in $B_{k,1} \cap \mathbb{Z}_2^n$, then what do we know about $C = \langle u \langle P \rangle \rangle$? First, C is a dihedral code of order 2^{k+1} . Second, no non-trivial element of C has any more than k consecutive 0's. While an element of $F = \widehat{C}$ might fix a vertex (and hence every vertex in its fibre), no element of F other than the identity can fix, pointwise, an edge of the graph. Now, the stabilizer of Z' under F is non-trivial, as it contains (and is generated by) α . Thus, the orbit \mathcal{Y}'_u of Z' under F consists of $2^{k+1}/2 = 2^k$ edge-disjoint 2n-cycles. This shows that \mathcal{Y}'_u is a cycle decomposition.

Since C is a dihedral code, we know that $F = \widehat{C}$ is normalized by ρ' and μ' . Because $F \cap \langle \rho', \mu' \rangle = \langle \alpha \rangle$, of order 2, $G = \langle \rho', \mu', \widehat{u} \rangle$ is a group of order $(2^{k+1})(4n)/2 = n2^{k+2}$ which acts on \mathcal{Y}'_u , and its kernel $G \cap K$ is just F. Because $\langle \rho', \mu' \rangle$ acts transitively on the darts of Z, G acts transitively on the darts of \mathcal{Y}'_u , making \mathcal{Y}'_u a cycle structure.

(4) Given that $u \in B_{k,1} \cap \mathbb{Z}_2^n$, let $v = u + uP, C = \langle u \langle P \rangle \rangle, D = \langle v \langle P \rangle \rangle, F = \widehat{C}, E = \widehat{D}$. Then C and D are dihedral codes, with D of index 2 in C.

If we now require that u be in $B_{k,2}$, then $1^n \in D$, forcing $\alpha \in E$. Then the stabilizer of Z' in F is the same as in E. By the Orbit-Stabilizer Theorem, this implies that the orbit Z'E is half the size of $\mathcal{Y}_u = Z'F$. Color the cycles in Z'E red and those not in Z'E green. Some vertex must meet a green and a red cycle, and since E is transitive on vertices, every vertex meets both red and green cycles. Thus \mathcal{Y}_u is bipartite.

Conversely, if \mathcal{Y}_u is bipartite, then the color-preserving subgroup E' of F must have index 2 in F. As ρ' and μ' preserve Z', they must preserve colors, and so they must conjugate E' to itself. Then $E' = \widehat{D}'$ for some dihedral code D'. If v' is

the generating bitstring for D, then v'(t) must divide $t^n - 1$, and v'(t) = (1+t)u(t)by Lemma 5.1. Thus v' = u + uP = v and so D' = D and E' = E. Now $\alpha = \rho'^n$ preserves colors, so that $\alpha \in E'$ which forces $a = 1^n \in D$. Then v(t) divides $a(t) = \frac{t^n - 1}{t - 1}$. That implies that u is in $B_{k,2}$.

Theorem 6.4. If \mathcal{Y} is a cycle structure in $\Gamma = PX(n,k)$, $n \neq 4$, then exactly one of these three things must happen:

1. $\mathcal{Y} = \mathcal{Y}^*$.

- 2. \mathcal{Y} is isomorphic to \mathcal{Y}_u , where u is a bitstring in $B_{k,0} \cap \mathbb{Z}_2^n$.
- 3. \mathcal{Y} is isomorphic to \mathcal{Y}'_u , where u is a bitstring in $B_{k,1} \cap \mathbb{Z}_2^n$.

Proof. The cycles in \mathcal{Y} must be 4-cycles, *n*-cycles or 2*n*-cycles. If they are 4-cycles, since *n* is not 4, there are no consistent 4-cycles other than those in \mathcal{Y}^* and so case (1) must hold.

If the cycles in \mathcal{Y} are *n*-cycles, then \mathcal{Y} is isomorphic to a cycle structure that contains Z. Hence, we assume that Z is in \mathcal{Y} , and under that assumption, we will show that \mathcal{Y} is \mathcal{Y}_u for some $u \in B_{k,0}$.

Let $G = \operatorname{Aut}(\mathcal{Y})$. Set $F = G \cap K$ and observe that F is normal in G. Then $F = \widehat{C}$ for some code C. Consider the 2^{k+1} forward-facing darts leading from a vertex in the 0^{th} fibre to the 1^{st} fibre. The group $G = \operatorname{Aut}(\mathcal{Y})$ is transitive on this set. Moreover, any symmetry sending one dart in this set to another must preserve the 0^{th} and 1^{st} fibres; it must then preserve all fibres and so be in F. Thus F is transitive on these darts. Because each such dart belongs to exactly one element of \mathcal{Y} , F is transitive on the cycles of \mathcal{Y} . Because the stabilizer in F of any transversal cycle is trivial, F acts regularly on the cycles and so its order is 2^{k+1} . Now, \mathcal{Y} is a cycle structure and Z is one of its cycles. Therefore $\operatorname{Aut}(\mathcal{Y})$ must contain a shunt for Z. As ρ and ρ^{-1} are the only shunts for Z, ρ must be in $\operatorname{Aut}(\mathcal{Y})$. Similarly, $\mu \in \operatorname{Aut}(\mathcal{Y})$. Then F is normalized by both ρ and μ , and so C is a dihedral code of order 2^{k+1} . If u is the generating bitstring for C, then u must end in k zeroes and thus must be in $B_{k,0}$. As \mathcal{Y} is the orbit of Z under $F = \widehat{C}$, \mathcal{Y} must be identical to \mathcal{Y}_u .

Finally, suppose that the cycle structure \mathcal{Y} on $\mathrm{PX}(n,k)$ has cycles of length 2n. Without loss of generality, we may assume that one of the cycles is Z'. Then the cycles of \mathcal{Y} use up the $n2^{k+1}$ edges of Γ , and so there are 2^k such cycles. Again, let G be $\mathrm{Aut}(\mathcal{Y})$ and set $F = G \cap K$. Then $F = \widehat{C}$ for some code C. Consider the 2^{k+1} forward-facing darts leading from a vertex in the 0^{th} fibre to one in the 1^{st} fibre. The group $G = \mathrm{Aut}(\mathcal{Y})$ is transitive on this set. Moreover, any symmetry sending one dart in this set to another must preserve the 0^{th} and 1^{st} fibres; it must then preserve all fibres and so be in F. Thus F is transitive on these darts. Because each such dart belongs to exactly one element of \mathcal{Y} , F is transitive on the cycles of \mathcal{Y} . Because the stabilizer in F of any transversal cycle is $\langle \alpha \rangle$, the order of F is 2^{k+1} .

Now, \mathcal{Y} is a cycle structure and Z' is one of its cycles. Therefore $\operatorname{Aut}(\mathcal{Y})$ must contain a shunt for Z'. As ρ' and its inverse are the only shunts for Z', ρ' must be in $\operatorname{Aut}(\mathcal{Y})$. Similarly, $\mu' \in \operatorname{Aut}(\mathcal{Y})$. Then F is normalized by both ρ' and μ' , and

so C is a dihedral code of order 2^{k+1} . If u is the generating bitstring for C, then u must end in k zeroes and thus must be in $B_{k,0}$. As \mathcal{Y} is the orbit of Z' under $F = \hat{C}, \mathcal{Y}$ must be identical to \mathcal{Y}'_u .

Since $\rho' \in G$, so is $(\rho')^n$, which is the all-swapper $\alpha = \tau_0 \tau_1 \tau_2 \dots \tau_{n-1} = \hat{a}$, where $a = 1^n$ and so $a(t) = \frac{t^n - 1}{t - 1}$. Thus *a* must belong to *C*, and so u(t) must divide $\frac{t^n - 1}{t - 1}$, putting *u* in $B_{k,1}$.

7. Semitransitive orientations in PX(n, k)

Recall that each edge $\{(i, ax), (i + 1, xb)\}$ of $\Gamma = PX(n, k)$ is associated with two darts: the dart $(i, ax) \rightarrow (i + 1, xb)$ and the dart $(i + 1, xb) \rightarrow (i, ax)$; which we call forward facing and backward facing, respectively.

We begin this section with two orientations, one consisting of the set of all forward facing darts of Γ , called the *Flow*, and the other consisting of the set of all backward facing darts of Γ , called the *Reverse Flow*. As μ interchanges these two, we see that they are isomorphic. The group generated by ρ and K preserves the Flow and is transitive on edges, and so the Flow and the Reverse Flow are clearly semitransitive orientations. In the paper [2], the authors show that in a 4-valent semitransitive orientation Δ , there are exactly two Aut(Δ)-orbits of consistent cycles. In the Flow and in the Reverse Flow, the two orbits of consistent cycles consist of the transversal *n*-cycles and the transversal 2*n*-cycles. Furthermore, it is not hard to see that if Δ is a semitransitive orientation of Γ in which a standard 4-cycle is not consistent, all darts in Δ must be forward facing (or all darts must be backward facing), and so Δ must be the Flow (or the Reverse Flow).

Conversely, if a semitransitive orientation Δ contains a 4-cycle that *is* consistent, it must be a standard 4-cycle. Therefore, Δ is neither the Flow nor the Reverse Flow, and there exists a vertex v that is the head of one forward facing and one backward facing dart in Δ as well as the tail of one forward facing and one backward facing dart in Δ . By the vertex-transitivity of Aut(Δ), this must be true for every vertex of Γ . Furthermore, for any vertex v of Γ , the symmetry σ sending the forward facing dart in Δ the head of which is v to the forward facing dart in Δ the tail of which is v is a shunt for some cycle at v, and that cycle must be a transversal cycle of length n or 2n.

The following theorem establishes a link between the concepts of cycle structure and semitransitive orientation:

Theorem 7.1. Let $\Gamma = PX(n, k)$ and suppose that \mathcal{Y} is a bipartite cycle structure on Γ with cycles of length n or 2n. Then the orientation Δ that consists of the forward facing green darts and the backward facing red darts is a semitransitive orientation of Γ .

Proof. Let $G = \operatorname{Aut}(\mathcal{Y})$, let H_1 be the subgroup of G preserving the Flow and let H_2 be the subgroup of G fixing the set of green cycles (and hence the set of red cycles). As these are normal subgroups of index 2 in G, the set $H_3 =$ $(H_1 \cap H_2) \cup ((G \setminus H_1) \cap (G \setminus H_2))$ is also a subgroup of G, also of index 2. Any

element of H_3 which preserves the colors also preserves the Flow orientation, while any element of H_3 which interchanges colors also reverses the Flow. Hence every element of H_3 preserves the orientation Δ defined in the theorem. Now, H_1 is transitive on all of the darts of the Flow, and so $H_1 \cap H_2$ is transitive on all of the green darts of the Flow, and thus is transitive on all the green darts of Δ . Its complement in H_3 sends those darts to the red darts of Δ , and hence H_3 is transitive on the darts of Δ , making Δ a semitransitive orientation of Γ .

In fact, the converse of Theorem 7.1 also holds:

Theorem 7.2. If Δ is a semitransitive orientation of $\Gamma = PX(n, k)$ in which a 4-cycle is consistent, then the standard 4-cycles form an Aut(Δ)-orbit of consistent cycles. The other Aut(Δ)-orbit of consistent cycles forms a bipartite cycle structure in Γ .

Proof. Suppose that Δ is a semitransitive orientation of $\Gamma = PX(n, k)$ containing a consistent 4-cycle, and let $G = Aut(\Delta)$. Then by the transitivity of G on edges, every 4-cycle of Δ is consistent, and all 4-cycles of Γ are in the same orbit, proving the first claim of the theorem.

As explained in the paragraphs preceding Theorem 7.1, the set of forward facing darts of Δ forms a collection of vertex-disjoint consistent transversal cycles (that we consider green), and so does the set of backward facing darts (which we consider red). Let \mathcal{Y} be the collection of these green and red (undirected) consistent cycles. The set \mathcal{Y} is clearly a bipartite cycle decomposition of Γ . As G is contained in the group Aut(\mathcal{Y}) of symmetries of Γ that preserve \mathcal{Y} , G acts transitively on the vertices and edges of \mathcal{Y} .

To complete the proof of the theorem, it remains to show that some $\gamma \in \operatorname{Aut}(\mathcal{Y})$ sends some dart in Δ to a dart not in Δ . We will do this in two steps: we will first argue that we can assume that either Z is a green cycle in \mathcal{Y} , or that Z' is. We will then show that under either the first or the second assumption, μ , or μ' respectively, is in $\operatorname{Aut}(\mathcal{Y})$.

To argue that we may assume that either Z or Z' is green, note that all cycles in \mathcal{Y} are in the same Aut(Γ)-orbit and that must be the orbit of Z or of Z'. Because all consistent cycles in Γ are reflexible, for any cycle D of \mathcal{Y} , there must be a σ in Aut(Γ) sending the darts of D in Δ to the forward-facing darts of Z or Z'. Replacing Δ by $\Delta \sigma$, we can assume that Z or Z' is a green cycle in \mathcal{Y} .

To show that either μ or μ' belong to Aut(\mathcal{Y}) (depending on whether Z or Z' is assumed to be green), note first that because ρ is the only shunt for Z which sends $(0, 0^k)$ to $(1, 0^k)$, ρ must be in G if $Z \in \mathcal{Y}$. Similarly, $\rho' \in G$ if $Z' \in \mathcal{Y}$. We claim that in the first case Aut(\mathcal{Y}) contains μ , and in the second case it contains μ' (with μ reversing Z and μ' reversing Z'). Arguing this claim will complete the proof.

Note that, in either case, $F = G \cap K$ is transitive on each fibre. To see this, consider any pair of vertices of the same fibre. Each of them is the tail of a unique green dart, and since G acts transitively on the darts of Δ , G contains an element that maps the first green dart to the second. This element necessarily fixes two

consecutive fibres, and thus it fixes setwise every fibre. Hence, it belongs to F, which proves that F acts transitively on each fibre.

We claim further that F acts transitively on the green cycles as well as on the red cycles. This is due to the facts that every green cycle and every red cycle intersects every fibre, while F acts transitively on each fibre and preserves colors. Since the pointwise stabilizer of the cycle Z and the pointwise stabilizer of the cycle Z' are trivial in A, the action of F on each fibre is regular. This means, in particular, that the order of F is 2^k . Recall that F is invariant under the conjugation by ρ , and thus, $F = \hat{C}$ for some cyclic code C whose generator bitstring v ends with exactly k - 1 0's.

Having shown the above two claims, which hold regardless of whether Z or Z' is green, we can now return to the main line of the proof. As argued already, we have two possibilities to consider.

First, suppose that Z is a green cycle in \mathcal{Y} and ρ belongs to G. Since G is transitive on the darts of Δ , there exists an element of G fixing $(0, 0^k)$ and sending Z to the unique red cycle through $(0, 0^k)$. If there were two such elements, their product would fix Z pointwise and would have to be trivial. This says that the symmetry must be unique and that its square is trivial. This symmetry must then be of the form $\mu \hat{u}$ for some $u \in V$. Then, $1_A = (\mu \hat{u})^2 = \mu \hat{u} \mu \hat{u} = \hat{u}^{\mu} \hat{u}$. Thus, $\hat{u}^{\mu} = \hat{u}$, and the bitstring uM must be u itself. It also follows that conjugation by μ fixes $\mu \hat{u}$. Furthermore, the symmetry $\mu \hat{u} \hat{v}$ fixes, setwise, the fibre containing $(0, 0^k)$ and reverses orientation, so its square must be in F. Thus F contains $(\mu \hat{u} \hat{v})^2 = \hat{u}^{\mu} \hat{v}^{\mu} \hat{u} \hat{v} = \hat{u} \hat{v}^{\mu} \hat{u} \hat{v} = \hat{v}^{\mu} \hat{v}$, and \hat{v}^{μ} belongs to F, and so vM belongs to C. It is obvious that the longest block of consecutive 0's in vM must be of the same length as the longest block of consecutive 0's in v. Since v is the generator bitstring of C, vM must be equal to vP^i , for some $i \in \mathbb{Z}_n$. It follows that G is generated by $\rho, \mu \hat{u}$ and \hat{v} , and thus is normalized by μ . Since \mathcal{Y} is the orbit of Z under G, and since μ normalizes G and preserves Z, it preserves \mathcal{Y} , as claimed.

Next suppose that Z' is a green cycle in \mathcal{Y} and ρ' is in G. As in the previous case, G contains an element fixing $(0, 0^k)$ and sending Z' to the unique red cycle through $(0, 0^k)$. As above, this symmetry must be an involution, and it also must be of the form $\mu'\hat{u}$ for some $u \in V$. Then, $\hat{u}^{\mu'} = \hat{u}$. The symmetry $\mu'\hat{u}\hat{v}$ fixes the fibre containing $(0, 0^k)$ and reverses orientation, and therefore its square must belong to F. Thus, F contains $(\mu'\hat{u}\hat{v})^2 = \hat{v}^{\mu'}\hat{v}$, and thus $\hat{v}^{\mu'}$ must be in F. But $\hat{v}^{\mu'} = \tau_{n-1}\tau_{n-2}\ldots\tau_{k+1}\mu\hat{v}\mu\tau_{k+1}\ldots\tau_{n-2}\tau_{n-1}$. Since $\mu\hat{v}\mu \in K$, and K is abelian, this product equals $\mu\hat{v}\mu = \hat{v}^{\mu} = \hat{v}M$, and hence vM is in C. Again, the longest block of consecutive 0's in vM is of the same length as the longest block of consecutive 0's in the generator bitstring of C, vM must be equal to vP^i , for some $i \in \mathbb{Z}_n$. Similarly as in the previous case, G is generated by $\rho', \mu'\hat{u}$ and \hat{v} , and thus is normalized by μ' . Since \mathcal{Y} is the orbit of Z' under G, and since μ' normalizes G and preserves Z', it preserves \mathcal{Y} , as claimed.

Thus, every semitransitive orientation in PX(n, k) other than the Flow and the Reverse Flow arises from a bipartite cycle structure, all of which are classified in Theorem 6.4. Applying Theorem 6.4, we see that every semitransitive orientation

of PX(n, k) other than the Flow and the Reverse is isomorphic to one given by the following construction:

Construction 7.3. For $u \in B_{k,1} \cap \mathbb{Z}_2^n$, let v = u + uP, and let Δ_u be the digraph whose dart-set is the orbit of the dart $d = ((0, 0^k), (1, 0^k))$ under $\langle \rho, \hat{v}, \mu \hat{u} \rangle$. If, in addition, $u \in B_{k,2}$, define Δ'_u to be the orbit of the dart d under $\langle \rho', \hat{v}, \mu' \hat{u} \rangle$.

The following corollary summarizes the discussion of this section:

Corollary 7.4. Each Δ_u and Δ'_u given by Construction 7.3 is a semitransitive orientation of PX(n,k), and every semitransitive orientation of PX(n,k) is either isomorphic to the Flow, the Reverse Flow, or to one of the orientations Δ_u, Δ'_u given by Construction 7.3.

Finally, it is natural to ask whether or when are the orientations Δ_u, Δ'_u isomorphic. Notice that if u_1 and u_2 are distinct elements of $B_{k,1} \cap \mathbb{Z}_2^n$, then Δ_{u_1} has consistent cycles of lengths 4 and n, while Δ'_{u_2} has consistent cycles of lengths 4 and n. Thus, the two orientations are never isomorphic.

Furthermore, when considering the consistent forward-facing cycles in Δ_{u_1} which share a vertex with Z, the ones that share the most vertices are those of the form $Z\widehat{u_1}\rho^j$. Thus Δ_{u_1} and Δ_{u_2} with distinct u_1 and u_2 cannot be isomorphic.

8. Rotary maps for PX(n, k)

We first present two constructions for maps with underlying graphs $\Gamma = PX(n, k)$. We then provide conditions under which these constructions yield rotary maps. We show those maps to be reflexible and determine conditions for the maps to be orientable and determine the lengths of their Petrie paths. Finally, we show that any rotary map on Γ must be isomorphic to one arising from these constructions.

Construction 8.1. For integers $n \ge 3$ and $n > k \ge 1$, and for a bitstring u of length n, let $G = \langle \rho, \mu, \hat{u} \rangle \le \operatorname{Aut}(\operatorname{PX}(n, k))$ and let \mathcal{F} be the orbit of the cycle Z under G. Let $\mathcal{M}_{n,k,u}$ denote the pair ($\operatorname{PX}(n,k),\mathcal{F}$).

Construction 8.2. For integers $n \geq 3$ and $n > k \geq 1$, and for a bitstring u of length n, let $G' = \langle \rho, '\mu', \widehat{u} \rangle \leq \operatorname{Aut}(\operatorname{PX}(n, k))$ and let \mathcal{F}' be the orbit of the cycle Z' under G'. Let $\mathcal{M}'_{n,k,u}$ denote the pair $(\operatorname{PX}(n,k),\mathcal{F}')$.

8.1. Properties of maps $\mathcal{M}_{n,k,u}$ and $\mathcal{M}'_{n,k,u}$

Theorem 8.3. If $u \in B_{k+1,0} \cap \mathbb{Z}_2^n$, then $\mathcal{M}_{n,k,u}$ is a reflexible map of PX(n,k), and has Petrie paths of length n or 2n, depending on whether the sum of the digits in u is even or odd, respectively. The map $\mathcal{M}_{n,k,u}$ is orientable if and only if $u \in B_{k+1,1}$.

Proof. Our proof uses the vertex $z = (0, 0^k)$ in PX(n, k) and its four neighbors. We write each of these five vertices in the bitstring as well as in the window notation:

x =	$(-1, 10^{k-1})$	$= [0 \dots 00 * * * *1],$
y =	$(-1, 0^k)$	$= [0 \dots 00 * * * * 0],$
z =	$(0, 0^k)$	$= [00 \dots 00 * * * *],$
a =	$(1, 0^k)$	$= [*00 \dots 00 * **],$
b =	$(1, 0^{k-1}1)$	$= [*00 \dots 01 * **].$

In the bitstring notation, " 0^{j} " stands for a string of j 0's. In the window notation we adopt the convention that the dots stand for a string of 0's, the dashes stand for a string of asterisks, with a total of k 0's and 1's, and n - k asterisks.

Let $v = uP^k$, so that v begins with k 0's followed by a 1, and ends in 10, and let w = vP, so that it begins with k + 1 0's followed by a 1 and ends in a 1. The symmetries \hat{v}, \hat{w} and μ permute the five vertices x, y, z, a, b among themselves, and on those vertices, have the following action:

(1)
$$\widehat{v} = (a \ b), \ \widehat{w} = (x \ y), \ \mu = (a \ y)(b \ x).$$

To prove the theorem, we first must show that Construction 8.1 actually yields a map, i.e., we must show that each edge of $\mathcal{M}_{n,k,u}$ belongs to exactly two faces, and that the faces at each vertex form a cycle.

Lemma 5.1 and our choice of u yield that the subspace generated by u and its images under $\langle P \rangle$ constitutes a dihedral code C of order 2^{k+2} . Thus, \hat{C} is a normal subgroup of the group $G = \langle \rho, \mu, \hat{u} \rangle$, and therefore $|G| = n2^{k+3}$. The stabilizer in G of Z is $\langle \rho, \mu \rangle$ of order 2n, and so the G-orbit \mathcal{F} of Z consists of $\frac{n2^{k+3}}{2n} = 2^{k+2}$ faces of face length n. Furthermore, G is transitive on edges, and so each edge must belong to the same number, m, of faces. Counting in two ways the number of edge-face incidences implies $n|\mathcal{F}| = m|\mathcal{E}|$, i.e., $n(2^{k+2}) = m(2n2^k)$. Thus, m = 2, as required.

Since the face Z contains consecutively the vertices $y, y\rho = z$, and $z\rho = a$, its image $A = Z\hat{v}$, contains the vertices $y\hat{v} = y, z\hat{v} = z, a\hat{v} = b$, its image $B = Z\hat{w}$ contains the vertices x, z, a, and its image $C = B\hat{v} = A\hat{w}$ contains the vertices x, z, b; in that order. It follows that the four faces $A, B, C, Z \in \mathcal{F}$ share the vertex z, and contain vertices as shown in Figure 2. Since G is transitive on the vertices of PX(n, k), and the faces in \mathcal{F} are all images of Z under G, the cyclic order of the faces A, B, C, Z around the vertex z determines the cyclic order of faces around each vertex of PX(n, k), and $\mathcal{M}_{n,k,u}$ is indeed a map.

It is easy to see that all symmetries in G preserve the vertex-edge-face incidences of $\mathcal{M}_{n,k,u}$, and are therefore map symmetries of $\mathcal{M}_{n,k,u}$ as well. Hence, the symmetry group of $\mathcal{M}_{n,k,u}$ consists of at least $|G| = n2^{k+3}$ elements. Since the number of darts in $\mathcal{M}_{n,k,u}$ is $n2^{k+2}$, which is one half of the order of G, G is the full symmetry group of $\mathcal{M}_{n,k,u}$, and $\mathcal{M}_{n,k,u}$ is a reflexible map. In particular, note that the symmetry \hat{w} fixes the directed edge (z, a) and moves the face Zto the adjacent face B. The symmetry ρ acts as rotation R about face Z, while $S = \mu \hat{w}$ fixes z but not a, sending Z to A, and so acts as rotation about z (these symmetries show again that the map is rotary). Further, μ acts as a reflection Xfixing both Z (setwise) and z (showing again that the map is reflexible).



Figure 2. The neighborhood of $z = (0, 0^k)$ in $\mathcal{M}_{n,k,u}$.

We now consider Petrie paths in $\mathcal{M}_{n,k,u}$. Let us denote the length of a Petrie path by r, and let $s \in \mathbb{Z}_2$ be the sum of the entries in $u \mod 2$. Since $v = uP^k$, s is also the sum of the entries in v. The symmetry which moves the map one step along the Petrie path $\ldots y \ z \ a \ldots$ is $T = RS^{-1}X = \rho \widehat{w} \mu \mu = \rho \widehat{w}$, and r is the order of T. Note that $T^2 = \rho \widehat{w} \rho \widehat{w} = \rho^2 \widehat{w}^{\rho^{-1}} \widehat{w}$, and similarly, $T^3 = \rho^3 \widehat{w}^{\rho^{-2}} \widehat{w}^{\rho^{-1}} \widehat{w}$. Due to the power of ρ at the front of this expression, the order of T is necessarily a multiple of the order n of ρ . Hence, the smallest positive power of T elligible to be the identity is the n-th power, i.e., $T^n = \rho^n \widehat{w}^\rho \widehat{w}^{\rho^2} \ldots \widehat{w}^{\rho^{-1}} \widehat{w}$. If we denote the product $\widehat{w}^\rho \widehat{w}^\rho^2 \ldots \widehat{w}^{\rho^{-2}} \widehat{w}^{\rho^{-1}} \widehat{w}$ by \widehat{h} , then $h = wP + wP^2 + \cdots + wP^{n-2} + wP^{n-1} + w$ (with the addition in \mathbb{Z}_2^n). Each entry in w (and thus each entry in u) appears once in each columnar position in that sum and so $h = s(1^n)$, and $T^n = \alpha^s$. Clearly, the order of T^n is 1 if s = 0 and is 2 if $s = 1 \pmod{2}$. The claim about Petrie paths follows directly from this.

If we assign to each edge its forward facing dart, then when viewed from within each face, darts point consistently around that face. In adjacent faces, the darts point in different directions (clockwise when viewed from inside of one of the faces, and counterclockwise from inside the other). If the map is orientable, half of the faces are oriented clockwise and half counterclockwise, implying that the map is face-bipartite. Conversely, if the map is face-bipartite, we can define 'clockwise' globally by having it agree with the Flow on white faces and disagree on black faces. Thus $\mathcal{M} = \mathcal{M}_{n,k,u}$ is orientable if and only if it is face-bipartite.

This can happen if and only if G (the full symmetry group of \mathcal{M}) has a subgroup H (the face-color preserving group) of index 2 containing ρ and μ but not \hat{w} . The kernel L of H must be of index 2 in F, and since H contains ρ and μ , L must be \hat{D} for some dihedral code D of index 2 in C. The generator bistring g of D must end in k 0's and have a palindromic head. By Lemma 5.1, g must be equal to uP + u. Since g is a generator, g(t) must divide $t^n - 1$. And since g(t) = u(t) + tu(t) = (1 + t)u(t), the final claim of the theorem follows.

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Theorem 8.4. If $u \in B_{k+1,1}$, then $\mathcal{M}'_{n,k,u}$ is a reflexible map of PX(n,k), and has Petrie paths of length 2n or n, depending on whether the sum of the digits in u is even or odd, respectively. The map $\mathcal{M}'_{n,k,u}$ is orientable if and only if $u \in B_{k+1,2}$.

The proof of this claim is very similar to the previous one and it is left to the reader.

8.2. Rotary maps on PX(n,k) must be isomorphic to $\mathcal{M}_{n,k,u}$ or $\mathcal{M}'_{n,k,u}$

We prove in this section that the rotary maps constructed using Constructions 8.1 and 8.2 are the only rotary embeddings of the Praeger-Xu graphs.

Suppose that \mathcal{M} is a rotary map with underlying graph $\Gamma = PX(n,k)$. As observed in Section 4.3, the faces of \mathcal{M} must be consistent cycles in Γ , and so they must all be of length 4, all be of length *n*, or all be of length 2*n*. However, each edge of Γ belongs to exactly one standard 4-cycle, which are the only consistent 4-cycles in Γ (recall that we assume $n \neq 4$), and thus the faces of \mathcal{M} cannot consist of 4-cycles.

Theorem 8.5. If \mathcal{M} is a rotary map with underlying graph PX(n,k), $n \neq 4$, then \mathcal{M} must be isomorphic to $\mathcal{M}_{n,k,u}$ or $\mathcal{M}'_{n,k,u}$ for some n, k, u satisfing the hypotheses of Theorems 8.3 or 8.4, respectively.

Proof. Let \mathcal{M} be a rotary map for $\Gamma = PX(n, k)$ with automorphisms R and S as defined in Section 4.3, and suppose first that the faces of \mathcal{M} are consistent n-cycles. Since consistent n-cycles constitute a single orbit under the action of the automorphism group of Γ , we can assume with no loss of generality that Z is a face of \mathcal{M} and that the shunt R of Z is equal to ρ . Choosing z, x, y, a, b as in (1) at the beginning of Section 8.1, we see that the faces around z must be as in Figure 2. Now S must reverse the Flow. Since it also fixes z, and hence the 0^{th} fibre, it must be of the form $S = \mu \hat{w}$, for some $w \in \mathbb{Z}_2^n$. Note that S and μ act locally as $(x \ a \ y \ b)$ and $(x \ a)(y \ b)$, respectively, and therefore \hat{w} acts on these vertices as $(x \ y)$. Because \hat{w} fixes a, b, z, the bitstring w must have 0's in positions $0, 1, 2, \ldots, k - 1, k$. Because \hat{w} interchanges a and y, w_{n-1} must be 1.

Since \mathcal{M} is assumed to be rotary, RS^{-1} is an involution, and so the identity $1_{\langle R,S\rangle} = (RS^{-1})^2 = (\rho \widehat{w} \mu)^2 = \rho \widehat{w} \mu \rho \widehat{w} \mu = \rho \widehat{w} \rho^{\mu} \widehat{w}^{\mu} = \rho \widehat{w} \rho^{-1} \widehat{w}^{\mu} = \widehat{w}^{\rho^{-1}} \widehat{w}^{\mu}$, and hence $\widehat{w}^{\rho^{-1}} = \widehat{w}^{\mu}$. It follows that the substring $w_{k+1}w_{k+2}\dots w_{n-2}w_{n-1}$ must be palindromic, and $u = wP^{-k-1}$ is a bitstring with palindromic head beginning with 1 and ending with 1 followed by k + 1 0's.

Let $G^+ = \operatorname{Aut}^+(\mathcal{M}) = \langle R, S \rangle$ and let $F = G^+ \cap K$ be the kernel of the action of G^+ on the fibres. Let $W = \{\omega \in K | \mu \omega \in G^+\}$. Clearly, $\widehat{w} \in W$, and hence the coset $\widehat{w}F$ is contained in W. If we let S_0 stand for the stabilizer in G^+ of the 0^{th} fibre, then F must be $S_0 \cap K$, and so has index 2 in S_0 . This implies that the coset of F in S_0 has size |F| and must contain μW , of size at least |F|. Thus, $\mu W = \mu \widehat{w}F$, and so $W = \widehat{w}F$.

It follows that $S_0 = F \cup \mu W = F \cup \mu \widehat{w}F$, and therefore every element g of G^+ must have a unique expression of the form $g = \rho^j(\mu \widehat{w})^{\epsilon}f$ for some $j \in \mathbb{Z}_n, \epsilon \in \mathbb{Z}_2$,

and $f \in F$. Consider, in particular, $g = \rho^2 \mu \widehat{w} \in G^+$. Then $g^{\rho} = \rho^2 (\mu^{\rho} \widehat{w}^{\rho})$ is also in G^+ . Recall that $\langle \rho, \mu \rangle$ is isomorphic to D_n , and hence $\mu^{\rho} = \rho^{-2} \mu$. Then g^{ρ} is $\mu \widehat{w}^{\rho}$, and so $\widehat{w}^{\rho} \in W$. Since F is invariant under conjugation by ρ , W is invariant under conjugation by ρ . In addition, $\widehat{w}^{\mu} = \widehat{w}^{\rho^{-1}}$, and so W is also invariant under conjugation by μ . Further, S_0 is invariant under conjugation by μ . Thus $G^+ = \langle \rho, S_0 \rangle$ is invariant under conjugation by μ .

To finish the proof, we first need to prove the following lemma:

Lemma 8.6. If \mathcal{M} is a rotary map with underlying graph PX(n,k) and having *n*-gonal faces, then \mathcal{M} is reflexible.

Proof. Assume, as above, that Z is a face of \mathcal{M} , that the shunt R of Z is equal to ρ , that z, x, y, a, b are as in (1) from the beginning of Section 8.1, and that the faces around z are as in Figure 2. We first claim that μ acts as a symmetry of \mathcal{M} . We know that μ is a symmetry of PX(n, k) and so we need only to show that μ sends faces to faces. Consider any face f of \mathcal{M} . Since G^+ is transitive on faces, there is some $g \in G^+$ such that f = Zg. Then $f\mu = Zg\mu = Z\mu\mu g\mu = Z\mu(g^{\mu})$, and since $Z\mu = Z$, this is Zg' for some $g' \in G^+$, which is a face of \mathcal{M} because $g' \in G^+$. So μ acts as a symmetry of \mathcal{M} , and it acts on Z as a reflection about the diameter through x. Thus \mathcal{M} is reflexible. \Box

Because G^+ is generated by $R = \rho$ and $S = \mu \hat{w}$, $G = \operatorname{Aut}(\mathcal{M})$ is generated by $\rho, \hat{u}, \text{ and } \mu$. Since \mathcal{M} is reflexible, the order of G is four times the number of edges of $\operatorname{PX}(n,k)$, namely $n2^{k+3}$. Let \hat{H} be the kernel of the action of G on the fibres. Then $\hat{H} = G \cap K$, and G/\hat{H} is isomorphic to D_n , which yields that the order of H is 2^{k+2} . The bitstring u belongs to H and ends in k + 1 0's, and thus must be the generator bitstring for H. Then u(t) must divide $t^n - 1$, and we conclude that the parameters n, k, u satisfy the hypotheses of Theorem 8.3 and \mathcal{M} is isomorphic to $\mathcal{M}_{n,k,u}$.

To complete the proof of the theorem, a very similar proof can be devised to show that if the faces of \mathcal{M} are consistent 2*n*-cycles, then n, k, u satisfy the hypotheses of Theorem 8.4 and \mathcal{M} is isomorphic to $\mathcal{M}'_{n,k,u}$.

8.3. Open questions about rotary maps for PX(n,k)

Two interesting questions about the maps $\mathcal{M}_{n,k,u}$ and $\mathcal{M}'_{n,k,u}$ remain open and would be worth considering:

1. It is well-known that for any map \mathcal{M} , the maps \mathcal{M} and its Petrie $\mathcal{P}(\mathcal{M})$ have the same underlying graph, and $\mathcal{P}(\mathcal{M})$ is reflexible if and only if \mathcal{M} is. Thus, the Petrie $\mathcal{P}(\mathcal{M}_{n,k,u})$ is a reflexible map for PX(n,k) again, and is therefore isomorphic to $\mathcal{M}_{n,k,v}$ or $\mathcal{M}'_{n,k,v}$ for some v. How can we determine the map for given n, k, u?

Some indication may be deduced by considering the sum s of the entries in u. If s is even then $\mathcal{M}_{n,k,u}$ and $\mathcal{M}'_{n,k,u}$ are of types $\{n,4\}_n$ and $\{2n,4\}_{2n}$, while if s is odd, they are of types $\{n,4\}_{2n}$ and $\{2n,4\}_n$. In the first case it might be reasonable to guess that both maps are self-Petrie, while in the second case,

we might assume that each map is the Petrie dual of the other. But we cannot know this for sure.

2. If $u_1 \neq u_2$, can \mathcal{M}_{n,k,u_1} be isomorphic to \mathcal{M}_{n,k,u_2} or can \mathcal{M}'_{n,k,u_1} be isomorphic to \mathcal{M}'_{n,k,u_2} ? Can we classify the maps $\mathcal{M}_{n,k,u}$ and the maps $\mathcal{M}'_{n,k,u}$ up to isomorphism?

9. Appendix: The special case n = 4

For the sake of completeness, we give here the results for the Praeger-Xu graphs $\Gamma = PX(n, k)$ in which the order of $Aut(\Gamma)$ is not $n2^{n+1}$. These are the graphs PX(4, k) for $1 \leq k < 4$, which are different because when n = 4, there are more 4-cycles than in the standard case. To be more precise, when n = 4, every edge belongs to at least two consistent 4-cycles and the graph underlies a toroidal map of type $\{4, 4\}$.

9.1. Case k = 1.

The graph $\Gamma_1 = PX(4, 1)$ is isomorphic to $K_{4,4}$, to the toroidal graph $\{4, 4\}_{2,2}$, and to the circulant graph $C_8(1, 3)$. Its symmetry group has order 2^73^2 . Its consistent cycles are of length 4, 6, 8. It has two (isomorphism classes of) cycle structures, one of four 4-gons and one of two 8-gons. Both of these are bipartite. It has two non-isomorphic semitransitive orientations, both circulant. It has only one rotary embedding. That embedding is on the torus as the reflexible self-Petrie map $\{4, 4\}_{2,2}$.

9.2. Case k = 2.

The graph $\Gamma_2 = PX(4, 2)$ is isomorphic to the Rose Window graph $R_8(6, 5)$ [6], to Q_4 (the skeleton of the 4-cube), and to the toroidal graph $\{4, 4\}_{4,0}$. Its symmetry group has order 2⁷3. Its consistent cycles are of length 4, 6, 8. It has two cycle structures, one of eight 4-gons and one of four 8-gons. Both of these are bipartite. It has two non-isomorphic semitransitive orientations. It has two rotary embeddings, one on the torus as $\{4, 4\}_{4,0}$. The second embedding is the Petrie of the first and is of type $\{8, 4\}_4$ on the orientable surface of genus 5.

9.3. Case k = 3.

The graph $\Gamma_3 = PX(4, 3)$ is isomorphic to the toroidal graph $\{4, 4\}_{4,4}$. Its symmetry group has order 2^8 . It has consistent cycles of length 4 and two orbits of cycles of length 8. It has two cycle structures, one of sixteen 4-gons and one of eight 8-gons. Both of these are bipartite. It has three non-isomorphic semitransitive orientations. It has two rotary embeddings, one on the torus as $\{4, 4\}_{4,4}$. The second embedding is the Petrie of the first, and is of type $\{8, 4\}_4$ on the orientable surface of genus 9.

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