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REPDIGITS AS PRODUCT OF TWO PELL OR PELL-LUCAS NUMBERS

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ABSTRACT. Let P_n, Q_n, B_n, and C_n denote, respectively Pell, Pell-Lucas, balancing, and Lucas-balancing numbers. In this study, we show that if P_mP_n is a repdigit, then $P_mP_n \in \{0, 1, 2, 4, 5\}$ and that if Q_mQ_n is a repdigit, then $Q_mQ_n = 4$. Moreover, we show that if B_mB_n is a repdigit, then $B_mB_n \in \{0, 1, 6\}$ and that if C_mC_n is a repdigit, then $C_mC_n \in \{1, 3, 9, 99\}$.

1. Introduction

Let $(\lambda, \delta) = (1/2\sqrt{2}, -1/2\sqrt{2}), (1, 1)$ and $(\alpha, \beta) = (1 + \sqrt{2}, 1 - \sqrt{2})$. For $n \geq 0$, we define E_n by

$$E_n = \lambda\alpha^n + \delta\beta^n \ .$$

It is clear that $E_n = P_n$, n-th Pell number for $(\lambda, \delta) = \left(1/2\sqrt{2}, -1/2\sqrt{2}\right)$ and that $E_n = Q_n$, n-th Pell-Lucas number for $(\lambda, \delta) = (1, 1)$. Let $B_n = P_{2n}/2$ and $C_n = Q_{2n}/2$. Then B_n is n-th balancing number and C_n is n-th Lucas balancing number. For more information about sequences of balancing and Lucas-balancing numbers, see [**7**], [**11**], and [**12**]. Actually, $E_n \neq 0$ for $n \geq 1$. It can be seen that $2 < \alpha < 3$ and $-1 < \beta < 0$. The inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{1}$$

and

$$Q_n < 2\alpha^n, \tag{2}$$

are well known, where $n \geq 1$. Thus, the inequality

$$\alpha^{n-2} \leq E_n < 2\alpha^n \tag{3}$$

is always true for $n \geq 1$. A repdigit is a non-negative integer whose digits are all equal. Investigation of the repdigits in the second-order linear recurrence sequences has been of interest to mathematicians. In [**9**], the authors found all Fibonacci and Lucas numbers which are repdigits. The largest repdigits in the Fibonacci and Lucas sequences are $F_5 = 55$ and $L_5 = 11$. After that, in [**1**], the authors

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showed that the largest Fibonacci number which is a sum of two repdigits is $F_{20} = 6765 = 6666 + 99$. In [**6**], the authors found all Pell and Pell-Lucas numbers which are repdigits. The largest repdigits in the Pell and Pell Lucas sequences are $P_3 = 5$ and $Q_2 = 6$. In this paper, we consider the Diophantine equation

$$E_m E_n = \frac{d(10^k - 1)}{9} \tag{4}$$

with $d \in \{1, 2, 3, \ldots, 9, 10, 12, 14, 16, 18, 20, 24, 28, 32, 36\}$. And thus, we solve the equations

$$P_m P_n = \frac{d(10^k - 1)}{9}, \tag{5}$$

$$Q_m Q_n = \frac{d(10^k - 1)}{9}, \tag{6}$$

$$P_m Q_n = \frac{d(10^k - 1)}{9}. \tag{7}$$

Furthermore, since $B_n = P_{2n}/2$ and $C_n = Q_{2n}/2$, we find that if $B_m B_n$ is a repdigit, then $B_m B_n = 0, 1, 6$, and if $C_m C_n$ is a repdigit, then $C_m C_n = 1, 3, 9, 99$. In Section 2, we introduce necessary lemmas and theorems. Then we prove our main theorems in Section 3.

2. Auxiliary results

In [**4**], in order to solve Diophantine equations of the form (4), the authors used Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving of Diophantine equations of this form, we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \left(\max \left\{ |\eta^{(i)}|, 1 \right\} \right) \right) \tag{8}$$

is called the logarithmic height of η. In particularly, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 1$, then $h(\eta) = \log\left(\max\left\{|a|, b\right\}\right)$.

The following properties of logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \le h(\eta) + h(\gamma) + \log 2, \tag{9}$$

$$h(\eta\gamma^{\pm 1}) \le h(\eta) + h(\gamma), \tag{10}$$

$$h(\eta^m) = |m| h(\eta). \tag{11}$$

The following theorem is deduced from Corollary 2.3 of Matveev [**10**] and provides a large upper bound for the subscripts n and m in the equation (4)(also see [**3**, Theorem 9.4]).

Theorem 1. *Assume that $\gamma_1, \gamma_2, \dots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \dots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \dots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 A_2 \dots A_t\right),$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, \dots, t$.

The following lemma was proved by Dujella and Pethő [**5**] and is a variation of a lemma of Baker and Davenport [**2**]. This lemma is used to reduce the upper bound for the subscripts n and m in the equation (4). Let the function $\|\cdot\|$ denote the distance from x to the nearest integer. That is, $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$ for any real number x. Then we have the following lemma.

Lemma 2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\gamma q\|$. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following lemma can be found in [**13**].

Lemma 3. *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1 + x)| < \frac{-\log(1 - a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1 - \mathrm{e}^{-a}} \cdot |\mathrm{e}^x - 1|.$$

The following two lemmas are given in [**6**] and the third one in [**8**].

Lemma 4. *The only repdigits in the Pell sequence are* $0, 1, 2, 5$.

Lemma 5. *The only repdigits in the Pell-Lucas sequence are* $2, 6$.

Lemma 6. *The only repdigits in the Balancing sequence are* $0, 1, 6$.

The following lemma is useful for the proof of Theorem 8.

Lemma 7. *Let $d \in \{10, 12, 14, \dots, 36\}$. Then all nonnegative integer solutions (n, d, k, E_n) of the equation $E_n = \dfrac{d(10^k - 1)}{9}$ are given by*

$$(n, d, k, E_n) \in \{(0, d, 0, 0), (4, 12, 1, 12), (3, 14, 1, 14), (4, 34, 1, 34), (6, 18, 2, 198)\}.$$

Proof. Assume that $d \in \{10, 12, 14, \ldots, 36\}$ and $E_n = \dfrac{d \cdot (10^k - 1)}{9}$. If $k = 0$, then $E_n = 0$, which implies that $n = 0$. Let $k \geq 1$. Then $n \geq 1$. Since

$$10^{k-1} < \frac{d(10^k - 1)}{9} = E_n < 2\alpha^n$$

by (3), it follows that $k \leq n$. With the help of Mathematica program, we obtain other solutions stated in Lemma 7 for $n \leq 60$. Now we assume that $n \geq 61$. Rewriting the equation $E_n = \dfrac{d(10^k - 1)}{9}$ as

$$\lambda\alpha^n - \frac{d \cdot 10^k}{9} = -\frac{d}{9} - \delta\beta^n,$$

and taking absolute values of both sides of this equality, we get

$$\left|\lambda\alpha^n - \frac{d \cdot 10^k}{9}\right| \leq \frac{d}{9} + |\delta|\,|\beta|^n. \tag{12}$$

Dividing both sides of (12) by $\lambda\alpha^n$, we obtain

$$\left|1 - \frac{d \cdot \alpha^{-n} 10^k}{9\lambda}\right| \leq \frac{d}{9\lambda\alpha^n} + \frac{|\delta|\,|\beta|^n}{\lambda\alpha^n} \leq \frac{8\sqrt{2}}{\alpha^n} + \frac{1}{\alpha^n} < \frac{12.35}{\alpha^n}. \tag{13}$$

Put $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := d/9\lambda$, and $b_1 := -n$, $b_2 := k$, $b_3 := 1$. Note that the numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = Q(\sqrt{2})$. The degree of the field $\mathbb{K}$ is 2. So $D = 2$. Let

$$\Lambda = 1 - \frac{d \cdot \alpha^{-n} 10^k}{9\lambda}.$$

If $\Lambda = 0$, then it follows that $\lambda\alpha^n = \frac{d \cdot 10^k}{9}$. Conjugating this relation in $\mathbb{Q}(\sqrt{2})$, we get $\delta\beta^n = \frac{d \cdot 10^k}{9}$. Thus we have

$$E_n = \lambda\alpha^n + \delta\beta^n = \frac{2d \cdot 10^k}{9} > \frac{d \cdot (10^k - 1)}{9} = E_n,$$

a contradiction. Therefore, $\Lambda \neq 0$. Since

$$h(\gamma_1) = h(\alpha) = \frac{\log\alpha}{2} = \frac{0.8813\ldots}{2}, \qquad h(\gamma_2) = h(10) = \log 10 < 2.31,$$

and

$$h(\gamma_3) = h(d/9\lambda) \leq h(9) + h(d) + h(\lambda) \leq \log 9 + \log 36 + \frac{\log 8}{2} \leq 6.83$$

by (10), we can take $A_1 := 1$, $A_2 := 4.7$, and $A_3 := 14$. Also, since $k \leq n$ and $B \geq \max\{|-n|, |k|, |1|\}$, we can take $B := n$. Thus, taking into account the inequality (13) and using Theorem 1, we obtain

$$n\log\alpha - \log(12.35) < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)(1 + \log n) \cdot 4.7 \cdot 14.$$

A computer calculation with Mathematica gives that $n < 2.65 \cdot 10^{15}$.

Let

$$z := k\log 10 - n\log\alpha + \log(d/9\lambda)$$

and $x := \mathrm{e}^z - 1$. Then

$$|x| = |1 - \mathrm{e}^z| < \frac{12.35}{\alpha^n} < \frac{1}{2}$$

by (13). Choosing $a := \frac{1}{2}$, we get the inequality

$$|z| = |\log(x+1)| < \frac{\log 2}{(1/2)} \cdot \frac{12.35}{\alpha^n}$$

by Lemma 3. Thus, it follows that

$$0 < |k \log 10 - n \log \alpha + \log(d/9\lambda)| < (17.13) \cdot \alpha^{-n}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < \left| k \frac{\log 10}{\log \alpha} - n + \frac{\log(d/9\lambda)}{\log \alpha} \right| < 19.5 \cdot \alpha^{-n}. \tag{14}$$

Take $\gamma := \frac{\log 10}{\log \alpha} \notin \mathbb{Q}$ and $M := 2.65 \cdot 10^{15}$. Then $q_{42} = 920197043232024959$, the denominator of the 42^{th} convergent of γ exceeds $6M$. Now put

$$\mu := \frac{\log(d/9\lambda)}{\log \alpha} \qquad \text{with } \lambda \in \left\{1/2\sqrt{2}, 1\right\}.$$

In this case, a quick computation with Mathematica gives us the inequality $0 < \varepsilon = \varepsilon(\lambda) := \|\mu q_{42}\| - M\|\gamma q_{42}\|$ for $\lambda \in \left\{1/2\sqrt{2}, 1\right\}$. Let $A := 19.5$, $B := \alpha$, and $w := n$ in Lemma 2. Thus, with the help of Mathematica, we can say that if the inequality (14) has a solution, then

$$n = w \leq \frac{\log(Aq_{42}/\varepsilon)}{\log B} \leq 55.88.$$

This contradicts our assumption that $n \geq 61$. Thus the proof is completed. □

3. Main Theorems

Theorem 8. *Let $d \in \{1, 2, 3, \ldots, 9, 10, 12, 14, 16, 18, 20, 24, 28, 32, 36\}$. Then all nonnegative integer solutions $(n, m, d, k, E_m E_n)$ of the equation (4) are the elements of the set*

$$\begin{aligned}
&\{(0,m,d,0,0), (n,0,d,0,0), (0,0,4,1,4), (1,0,4,1,4), (1,0,2,1,2),\\
&(1,1,1,1,1), (1,1,2,1,2), (1,1,4,1,4), (2,0,4,1,4), (2,0,12,1,12),\\
&(2,1,2,1,2), (2,1,12,1,12), (2,1,6,1,6), (2,2,12,1,12), (2,2,4,1,4),\\
&(2,2,36,1,36), (3,0,10,1,10), (3,0,28,1,28), (3,1,10,1,10), (3,1,28,1,28),\\
&(3,1,5,1,5), (3,1,14,1,14), (3,2,28,1,28), (3,2,10,1,10), (4,0,24,1,24),\\
&(4,1,12,1,12), (4,1,24,1,24), (4,2,24,1,24), (6,0,36,2,396), (6,1,18,2,198),\\
&(6,1,36,2,396), (6,2,36,2,396), (0,1,2,1,2), (0,1,4,1,4), (0,2,12,1,12),\\
&(0,3,28,1,28), (0,6,36,2,396), (0,2,4,1,4), (0,3,10,1,10), (0,4,24,1,24),\\
&(1,2,4,1,4), (1,2,2,1,2), (1,2,6,1,6), (1,2,12,1,12), (1,3,5,1,5),\\
&(1,3,10,1,10), (1,3,14,1,14), (1,3,28,1,28), (1,4,12,1,12), (1,4,24,1,24),\\
&(1,6,18,2,198), (1,6,36,2,396), (2,3,10,1,10), (2,4,24,1,24), (2,3,28,1,28),\\
&(2,6,36,2,396)\}.
\end{aligned}$$

Proof. Let $d \in \{1,2,3,\ldots,9,10,12,14,16,18,20,24,28,32,36\}$ and the equation (4) holds. If $k=0$, then $E_mE_n=0$. Assume that $k\geq 1$. Then $E_mE_n\neq 0$. If $m=0$, then we get $E_n=\frac{r(10^k-1)}{9}$, where $r=d/2\in\{1,2,3,\ldots,9,10,12,14,16,18\}$. This implies that $E_mE_n\in\{2,4,10,12,24,28,68,396\}$ by Lemmas 4, 5, and 7. If $m=1$ or $m=2$ and $E_m=P_2$, then we have $E_n=\frac{d(10^k-1)}{9}$ or $E_n=\frac{r(10^k-1)}{9}$, where $r=d/2\in\{1,2,3,\ldots,9,10,12,14,16,18\}$. Similarly, we obtain $E_mE_n\in\{1,2,4,5,6,10,12,14,24,28,34,68,198,396\}$ by Lemmas 4, 5, and 7. Since E_m and E_n are symmetric, E_mE_n has the same values for the cases $n=0, n=1$, or $n=2$, and $E_n=P_2$. Now assume that $m,n\geq 2$. Note that we consider only the case $E_m=Q_2$ or $E_n=Q_2$ when $m=2$ or $n=2$. From (3), we obtain

$$10^{k-1}\leq \frac{d(10^k-1)}{9}=E_mE_n<4\cdot\alpha^{n+m}.$$

Making necessary calculations, we see that $k<m+n$. On the other hand, we can rewrite equation (4) as

$$(\lambda_1\alpha^m+\delta_1\beta^m)(\lambda_2\alpha^n+\delta_2\beta^n)=\frac{d(10^k-1)}{9},$$

where $(\lambda_i,\delta_i)\in\left\{\left(1/2\sqrt{2},-1/2\sqrt{2}\right),(1,1)\right\}$ for $i=1,2$. Thus

$$\lambda_1\lambda_2\alpha^{m+n}-\frac{d\cdot 10^k}{9}=-\frac{d}{9}-\lambda_1\delta_2\alpha^m\beta^n-\delta_1\lambda_2\beta^m\alpha^n-\delta_1\delta_2\beta^{m+n}. \tag{15}$$

Taking absolute values of both sides of (15), we get

$$\left|\lambda_1\lambda_2\alpha^{m+n}-\frac{d\cdot 10^k}{9}\right|\leq\frac{d}{9}+\lambda_1\alpha^m\left|\delta_2\right|\left|\beta\right|^n+\lambda_2\left|\delta_1\right|\left|\beta\right|^m\alpha^n+\left|\delta_1\delta_2\right|\left|\beta\right|^{m+n}.$$

Dividing both sides of this inequality by $\lambda_1\lambda_2\alpha^{m+n}$, we obtain

$$\begin{aligned}\left|1-\frac{d\cdot 10^k}{9\lambda_1\lambda_2\cdot\alpha^{n+m}}\right|&\leq\frac{d}{9\lambda_1\lambda_2\cdot\alpha^{n+m}}+\frac{|\delta_2|\,|\beta|^n}{\lambda_2\alpha^n}+\frac{|\delta_1|\,|\beta|^m}{\lambda_1\alpha^m}+\frac{|\delta_1\delta_2|\,|\beta|^{m+n}}{\lambda_1\lambda_2\alpha^{n+m}}\\&<\frac{d}{9\lambda_1\lambda_2\cdot\alpha^{n+m}}+\frac{1}{\alpha^{2n}}+\frac{1}{\alpha^{2m}}+\frac{1}{\alpha^{m+n}}.\end{aligned}$$

It follows that

$$\left|1-\frac{\alpha^{-(n+m)}\cdot 10^k d}{9\lambda_1\lambda_2}\right|<\begin{cases}35\cdot\max\{\alpha^{-2m},\alpha^{-2n}\} & \text{if both } E_m \text{ and } E_n \text{ are Pell numbers,}\\ 15\cdot\max\{\alpha^{-2m},\alpha^{-2n}\} & \text{otherwise.}\end{cases} \tag{16}$$

Now, let us apply Theorem 1 with $\gamma_1:=\alpha$, $\gamma_2:=10$, $\gamma_3:=d/9\lambda_1\lambda_2$, and $b_1:=-(n+m)$, $b_2:=k$, $b_3:=1$. Note that the numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K}=Q(\sqrt{2})$. The degree of the field $\mathbb{K}$ is 2. So $D=2$. Now, we show that

$$\Lambda_1:=1-\frac{\alpha^{-(n+m)}\cdot 10^k d}{9\lambda_1\lambda_2}$$

is nonzero. On the contrary, assume that $\Lambda_1=0$. Then $\alpha^{n+m}=d\cdot 10^k/9\lambda_1\lambda_2$. Conjugating this in $\mathbb{Q}(\sqrt{2})$, we get $\beta^{n+m}=d\cdot 10^k/9\delta_1\delta_2$. This implies that

$10/9 \le d \cdot 10^k/9\,|\delta_1\delta_2| = |\beta|^{n+m} \le 1$, which is impossible. Since

$$h(\gamma_1) = h(\alpha) = \frac{\log\alpha}{2} = \frac{0.8813\ldots}{2}, \qquad h(\gamma_2) = h(10) = \log 10,$$

and

$$\begin{aligned} h(\gamma_3) &= h(d/9\lambda_1\lambda_2) \le h(9) + h(d) + h(\lambda_1) + h(\lambda_2) \\ &\le \log 9 + \log 36 + \frac{\log 8}{2} + \frac{\log 8}{2} \le 7.87 \end{aligned}$$

by (10), we can take $A_1 := 1$, $A_2 := 4.7$, and $A_3 := 16$. Also, since $k < n+m$ and $B \ge \max\{|-(n+m)|, |k|, |1|\}$, we can take $B := n+m$. Thus, taking into account the inequality (16) and using Theorem 1, we obtain

$$\begin{aligned} &35 \cdot \max\left\{\alpha^{-2m}, \alpha^{-2n}\right\} \\ &> |\Lambda_1| > \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1+\log 2)(1+\log(n+m))(4.7)(16)\right). \end{aligned}$$

By a simple computation, it follows that

$$\min\{2m\log\alpha, 2n\log\alpha\} < 7.3 \cdot 10^{13} \cdot (1+\log(n+m)). \tag{17}$$

Rearranging the equation (4) as

$$\lambda_2\alpha^n - \frac{d\cdot 10^k}{9E_m} = \delta_2\beta^n - \frac{d}{9E_m} \tag{18}$$

and taking absolute values of both sides of (18), we get

$$\left|\lambda_2\alpha^n - \frac{d\cdot 10^k}{9E_m}\right| \le |\delta_2|\,|\beta|^n + \frac{d}{9E_m}. \tag{19}$$

Dividing both sides of (19) by $\lambda_2\alpha^n$, we obtain

$$\left|1 - \frac{\alpha^{-n}10^k d}{9\lambda_2 E_m}\right| \le \frac{|\delta_2|\,|\beta|^n}{\lambda_2\alpha^n} + \frac{d}{9\lambda_2 E_m\alpha^n} \le \frac{1}{\alpha^{2n}} + \frac{8\sqrt{2}}{\alpha^n} < \frac{12.35}{\alpha^n}. \tag{20}$$

Taking $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := d/9\lambda_2E_m$, and $b_1 := -n$, $b_2 := k$, $b_3 := 1$, we can apply Theorem 1. The numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = Q(\sqrt{2})$, and so $D = 2$. Now, we show that

$$\Lambda_2 := 1 - \frac{\alpha^{-n}10^k d}{9\lambda_2 E_m}$$

is nonzero. Indeed, if $\Lambda_2 = 0$, then $\alpha^n = 10^k d/9\lambda_2 E_m$. Conjugating in $Q(\sqrt{2})$ gives us $\beta^n = 10^k d/9\delta_2 E_m$, and so $E_nE_m = 2d\cdot 10^k/9 > \frac{d(10^k-1)}{9}$, a contradiction. Since $h(\gamma_1) = \dfrac{\log\alpha}{2} = \dfrac{0.8813\ldots}{2}$, $h(\ \gamma_2) = \log 10$, and

$$\begin{aligned} h(\gamma_3) &= h\left(d/9\lambda_2E_m\right) \le h(d) + h(9) + h(\lambda_2) + h(E_m) \\ &\le \log(36) + \log(9) + \frac{\log 8}{2} + \log 2 + m\log\alpha < 7.52 + m\log\alpha \end{aligned}$$

by (10), we can take $A_1 := 1$, $A_2 := 4.7$, and $A_3 := 16.1 + 2m\log\alpha$. Since $k < n+m$ and $B \ge \max\{|-n|, |k|, |-1|\}$, we can take $B := n+m$. Thus, taking into account the inequality (20) and using Theorem 1, we obtain

$$(12.35)\cdot\alpha^{-n} > |\Lambda_2| > \exp\left(C\cdot(1+\log 2)(1+\log(n+m))\,(4.7)\,(16.1+2m\log\alpha)\right),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. So we get

$$n \log \alpha < 4.6 \cdot 10^{12} \cdot (1 + \log(n+m))\,(16.1 + 2m \log \alpha)\,. \tag{21}$$

In a similar way, it can be easily seen that

$$m \log \alpha < 4.6 \cdot 10^{12} \cdot (1 + \log(n+m))\,(16.1 + 2n \log \alpha)\,. \tag{22}$$

Using the inequalities (17), (21), and (22), a computer search with Mathematica gives us that $n < 1.95 \cdot 10^{30}$ if $n \geq m$, and that $m < 1.95 \cdot 10^{30}$ if $m \geq n$. From now on, we divide the proof into two cases.

Case 1. Assume that $n \geq m$. So, $n < 1.95 \cdot 10^{30}$. Let $2 \leq m \leq n \leq 60$ and let $(n, m, d, k, E_m E_n)$ be a solution of the equation (4). Then by using Mathematica program, it can be seen that this equation has only solutions for $m = 2$. Note that $E_m = Q_2 = 6$ when $m = 2$. In this case, $E_m E_n = 12, 36$. We may assume that $n > 60$.

Now, let us try to reduce the upper bound on n by applying Lemma 2. Let

$$z_1 := k \log 10 - (n+m) \log \alpha + \log(d/9\lambda_1\lambda_2).$$

Put $x := \mathrm{e}^{z_1} - 1$. Firstly, let $m = 2$. In this case, since $E_m = Q_2$, it follows that $|x| = |1 - \mathrm{e}^{z_1}| < 15 \cdot \alpha^{-2m} < \dfrac{1}{2}$ by (16). If $m \geq 3$, then $|x| = |1 - \mathrm{e}^{z_1}| < 35 \cdot \alpha^{-2m} < \dfrac{1}{4}$ by (16). Hence, inequality $|x| < \dfrac{1}{2}$ is always true. Choosing $a := \dfrac{1}{2}$, we get the inequality

$$|z_1| < |\log(x+1)| < \frac{-\log(1/2)}{1/2}\,|x| < \frac{-\log(1/2)}{1/2} \cdot 35\alpha^{-2m} < 49 \cdot \alpha^{-2m}$$

by (16) and Lemma 3. That is,

$$0 < |k \log 10 - (n+m) \log \alpha + \log(d/9\lambda_1\lambda_2)| < 49 \cdot \alpha^{-2m}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < \left| k\Big(\frac{\log 10}{\log \alpha}\Big) - (n+m) + \Big(\frac{\log(d/9\lambda_1\lambda_2)}{\log \alpha}\Big) \right| < 56 \cdot \alpha^{-2m}. \tag{23}$$

Take $\gamma := \dfrac{\log 10}{\log \alpha} \notin \mathbb{Q}$ and $M := 3.9 \cdot 10^{30}$. Then q_{73}, the denominator of the 73-th convergent of γ exceeds $6M$. Now take

$$\mu := \frac{\log(d/9\lambda_1\lambda_2)}{\log \alpha}.$$

In this case, a quick computation with Mathematica gives us the inequality $0 < \varepsilon = \varepsilon(\mu) := \|\mu q_{73}\| - M\|\gamma q_{73}\|$ for all values of d. Let $A := 56$, $B := \alpha$, and $w := 2m$ in Lemma 2. Thus, with the help of Mathematica, we can say that if the inequality (23) has a solution, then

$$2m = w \leq \frac{\log(A q_{73}/\varepsilon)}{\log B} \leq 96.26.$$

So, if the inequality (23) has a solution, then

$$m \leq 48.$$

Substituting this upper bound for m into (21), we obtain $n < (2.1) \cdot 10^{16}$.

Now, let

$$z_2 := k\log 10 - n\log\alpha + \log\left(d/9\lambda_2 E_m\right)$$

and $x := \mathrm{e}^{z_2} - 1$. Then we can see that

$$|x| = |1 - \mathrm{e}^{z_2}| < \frac{12.35}{\alpha^n} < 0.07$$

by (20) as $n > 60$. Put $a := 0.07$. Then, by Lemma 3, we get

$$|z_2| = |\log(x+1)| < \frac{-\log(0.93)}{0.07} \cdot \frac{12.35}{\alpha^n} < \frac{12.81}{\alpha^n}.$$

Therefore,

$$0 < |k\log 10 - n\log\alpha + \log\left(d/9\lambda_2 E_m\right)| < (12.81)\cdot\alpha^{-n}.$$

Dividing both sides of the above inequality by $\log\alpha$, we get

$$0 < \left|k\left(\frac{\log 10}{\log\alpha}\right) - n + \frac{\log\left(d/9\lambda_2 E_m\right)}{\log\alpha}\right| < (14.55)\cdot\alpha^{-n}. \tag{24}$$

Let $\gamma := \dfrac{\log 10}{\log\alpha}$ and $M := (4.2)\cdot 10^{16}$. Then the denominator of the 45-th convergent of γ exceeds $6M$. Taking

$$\mu := \frac{\log\left(d/9\lambda_2 E_m\right)}{\log\alpha}$$

and considering the fact that $m \le 48$, a quick computation with Mathematica gives us the inequality $0 < \varepsilon = \varepsilon(\mu) = \|\mu q_{45}\| - M\|\gamma q_{45}\|$ for all values of μ. Let $A := 14.55$, $B := \alpha$, and $w := n$ in Lemma 2. Thus, with the help of Mathematica, we can say that if the inequality (24) has a solution, then

$$n = w \le \frac{\log(Aq_{45}/\varepsilon)}{\log B} \le 57.15.$$

This contradicts our assumption that $n > 60$.

Case 2. A similar proof can be done for the case $m \ge n$, and therefore, we omit it. This completes the proof. □

Now we can give the following results.

Corollary 9. *If $P_m P_n$ is a repdigit, then $P_m P_n \in \{0, 1, 2, 4, 5\}$.*

Corollary 10. *If $Q_m Q_n$ is a repdigit, then $Q_m Q_n = 4$.*

Corollary 11. *If $P_m Q_n$ is a repdigit, then $P_m Q_n \in \{0, 2, 4, 6\}$.*

Since $B_n = P_{2n}/2$ and $C_n = Q_{2n}/2$, if $B_m B_n$ or $C_m C_n$ is a repdigit, then d given in Theorem 8 must be a multiple of 4. Thus, we can deduce the following results from Theorem 8.

Corollary 12. *If $B_m B_n$ is a repdigit, then $B_m B_n \in \{0, 1, 6\}$.*

Corollary 13. *If $C_m C_n$ is a repdigit, then $C_m C_n \in \{1, 3, 9, 99\}$.*

Corollary 14. *If $C_m B_n$ is a repdigit, then $C_m B_n \in \{1, 3, 6, 99\}$.*

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