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SOME ALGEBRAIC ASPECTS OF ENHANCED JOHNSON GRAPHS

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Abstract. For any given $n, m \in \mathbb{N}$ with $m < n$, the Johnson graph $J(n, m)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq I = \{1, \ldots, n\}, |v| = m\}$, where two vertices v,w are adjacent if and only if $|v \cap w| = m - 1$. Let $n = 2m$. The enhanced Johnson graph $EJ(2m, m)$ is the graph whose vertex set is the vertex set of $J(2m, m)$ and the edge set is $E_2 = E \cup E_1$ where E is the edge set of $J(2m, m)$ and $E_1 = \{\{v, v^c\} | v \subseteq I, |v| = m\}$; v^c is the complement of the subset v in the set I. In this paper, we show that the diameter of $EJ(2m, m)$ is $\lceil \frac{m}{2} \rceil$ (whereas the diameter of $J(2m, m)$ is m). Also, we determine the automorphism group of $EJ(2m, m)$, and we show that $EJ(2m, m)$ is an integral graph, namely, each of its eigenvalues is an integer. Although, some of our results are special cases of Jonse [**9**], unlike his proof that used some deep group-theoretical facts, ours uses no heavy group-theoretical facts.

1. Introduction

Johnson graphs arise from the association schemes of the same name. They are defined as follows.

Given $n, m \in \mathbb{N}$ with $m < n$, the Johnson graph $J(n, m)$ is defined by:

(1) The vertex set is the set of all subsets of $I = \{1, 2, \ldots, n\}$ with cardinality exactly m.
(2) Two vertices are adjacent if and only if the symmetric difference of the corresponding sets is two.

The Johnson graph $J(n, m)$ is a vertex-transitive graph [**7**]. It follows from the definition that for $m = 1$, the Johnson graph $J(n, 1)$ is the complete graph K_n. For $m = 2$ the Johnson graph $J(n, 2)$ is the line graph of the complete graph on n vertices, also known as the triangular graph $T(n)$. For instance, $J(5, 2)$ is the complement of the Petersen graph, displayed in Figure 1, and in general, $J(n, 2)$ is the complement of the Kneser graph $K(n, 2)$.

Johnson graphs have been studied by various authors and some of the recent papers include [**3**], [**9**], [**10**], [**18**].

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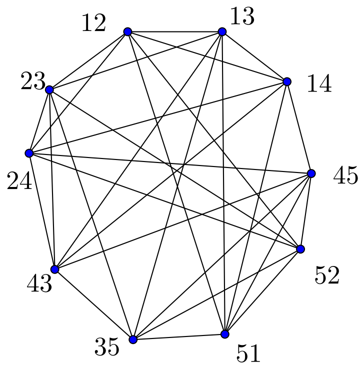

Figure 1. The Johnson graph $J(5,2)$.

Given a nonempty subset $S \subseteq I = \{1,\dots,n\}$, the *merged Johnson graph* $J(n,m)_S$, as is defined in [**9**], is the graph whose vertex set is the vertex set of $J(n,m)$, two m-element subsets are adjacent in $J(n,m)_S$ if their intersection has $m-i$ elements for some $i \in S$. So if $S=\{1\}$, then $J(n,m)_S = J(n,m)$.

Let $n=2m$. We define the *enhanced Johnson graph* $EJ(2m,m)$, to be the graph whose vertex set is the vertex set of $J(2m,m)$ and the edge set is $E_2 = E \cup E_1$ where E is the edge set of $J(2m,m)$ and $E_1 = \{\{v,v^c\} | v \subseteq I, |v| = m\}$, where v^c is the complement of the subset v in the set I. Hence, the enhanced Johnson graph $EJ(2m,m)$ is the merged Johnson graph $J(n,m)_S$, when $n=2m$ and $S=\{1,m\}$. This graph is of order $\binom{2m}{m} = \frac{2m!}{m!m!}$ and the degree of each vertex in it is m^2+1, whereas the degree of each vertex in $J(2m,m)$ is m^2. It is an easy task to show that the enhanced Johnson graph $EJ(2m,m)$ is also vertex-transitive. The enhanced Johnson graphs have some interesting properties, for example, we will see that the diameter of the enhanced Johnson graph $EJ(2m,m)$ is almost half of the diameter of the Johnson graph $J(2m,m)$. We will see that the automorphism group of the Johnson graph $J(2m,m)$ and the enhanced Johnson graph $EJ(2m,m)$

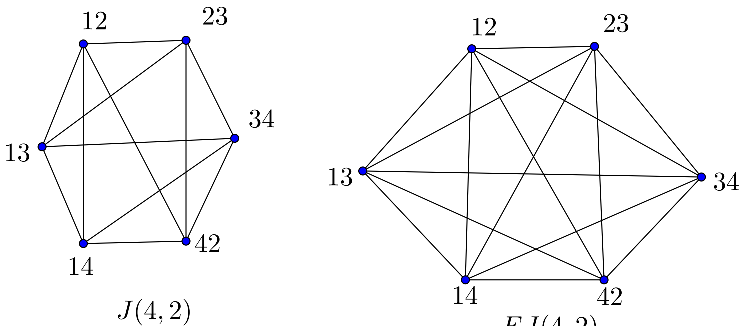

Figure 2.

are identical. Also, we will see that the enhanced Johnson graph $EJ(2m, m)$ is an integral graph, namely, each of its eigenvalues is an integer. Figure 2 displays $J(4, 2)$ and $EJ(4, 2)$ in the plane.

2. Preliminaries

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph, where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For the terminology and notation not defined here, we follow [**4**], [**7**], [**19**].

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\alpha\colon V_1 \to V_2$ such that $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an *isomorphism*. An *automorphism* of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group, called the automorphism group of Γ and denoted by $\mathrm{Aut}(\Gamma)$.

The group of all permutations of a set V is denoted by $\mathrm{Sym}(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $\mathrm{Sym}(V)$. In this case we say that G *acts* on V. If G acts on V, we say that G is *transitive* on V (or G acts *transitively* on V), when there is just one orbit. This means that given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$. If X is a graph with vertex-set V, then we can view each automorphism of X as a permutation on V, and so $\mathrm{Aut}(X) = G$ is a permutation group on V.

A graph Γ is called *vertex-transitive*, if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. For $v \in V(\Gamma)$ and $G = \mathrm{Aut}(\Gamma)$, the *stabilizer subgroup* G_v is the subgroup of G consisting of all automorphisms which fix v. In the vertex transitive case all stabilizer subgroups G_v are conjugate in G, and consequently are isomorphic, in this case, the index of G_v in G is given by the equation $|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|$.

Although, in most situations it is difficult to determine the automorphism group of a graph Γ, there are various papers in the literature, and some of the recent works include [**9**], [**11**]–[**16**], [**18**], [**20**].

3. Main results

If M is a m-subset of the set $I = \{1, \dots, n\}$, then the complementation of subsets $M \mapsto M^c$ induces an isomorphism $J(n, m) \cong J(n, n - m)$, hence in the sequel, we assume without loss of generality that $m \leq \frac{n}{2}$.

3.1. Diameter

Let Γ be a graph, $v, w \in V(\Gamma)$ and let $d_\Gamma(v, w) = d(v, w)$ denotes the distance between the vertices v and w in the graph Γ. It is an easy task to show that for any two vertices v, w in $\Gamma = J(n, m)$, $d_\Gamma(v, w) = t$ if and only if $|v \cap w| = m - t$ ([**5**], [**7**]). Therefore, if D is the diameter of the graph $\Gamma = J(n, m)$, then $D = m$.

We now show that the diameter of the enhanced Johnson graph $EJ(2m, m)$ is almost half of the Johnson graph $J(2m, m)$.

Proposition 3.1. *Let $m > 1$ be an integer. Then, the diameter of $EJ(2m, m)$ is $\lfloor \frac{m}{2} \rfloor + 1$ if m is odd, and $\lfloor \frac{m}{2} \rfloor$ if m is even.*

Proof. Let $\Gamma = J(2m, m)$ and $\Gamma_1 = EJ(2m, m)$. Let $v, w \in \Gamma_1$ and $|v \cap w| = t$. If $t \geq \lfloor \frac{m}{2} \rfloor$, then $d_\Gamma(v, w) = m - t \leq m - \lfloor \frac{m}{2} \rfloor \leq \lfloor \frac{m}{2} \rfloor + 1$, and since Γ is a subgraph of Γ_1, we have $d_{\Gamma_1(v,w)} \leq \lfloor \frac{m}{2} \rfloor + 1$.

Now let $|v \cap w| = t < \lfloor \frac{m}{2} \rfloor$. Suppose $v = \{x_1, \dots, x_t, y_1, \dots, y_{m-t}\}$ and $w = \{x_1, \dots, x_t, z_1, \dots, z_{m-t}\}$, where $z_1, \dots, z_{m-t} \in v^c$ (v^c is the complement of v in the set $I = \{1, 2, 3, \dots, m, \dots, 2m\}$). Thus, we have $|v^c \cap w| = m - t$, and hence $d_\Gamma(w, v^c) = m - (m - t) = t < \lfloor \frac{m}{2} \rfloor$. Now if $P\colon w, u_1, \dots, u_t = v^c$ is a path from w to v^c in Γ, then $Q\colon w, u_1, \dots, u_{t-1}\ , u_t = v^c, v$ is a path from w to v in Γ_1. It follows that $d_{\Gamma_1(v,w)} \leq t + 1 \leq \lfloor \frac{m}{2} \rfloor$.

On the other hand, if m is an even integer and $t = \lfloor \frac{m}{2} \rfloor$, then for $v = \{x_1, \dots, x_t, y_1, \dots, y_{m-t}\}$ and $w = \{x_1, \dots, x_t, z_1, \dots, z_{m-t}\}$, we have $d_{\Gamma_1(v,w)} = m - t = \frac{m}{2} = \lfloor \frac{m}{2} \rfloor$, whereas if m is an odd integer, then $d_{\Gamma_1(v,w)} = m - (m-t) + 1 = t + 1 = \lfloor \frac{m}{2} \rfloor + 1$. □

3.2. Automorphism group

Next, we determine the automorphism groups of enhanced Johnson graphs. To find the automorphism group of the enhanced Johnson graph $EJ(2m, m)$, we need the automorphism group of the Johnson graph $J(2m, m)$. The automorphism group of the Johnson graph $J(n, m)$ is already known ([**9**], and later [**18**] for all but the case $n = 2m$), but since our proof is different from those, we offer our proof. We determine the automorphism group of the Johnson graph $J(n, m)$ by an analysis of the structure of the subgraph induced by the vertices adjacent to a vertex. Although, our result is a special cases of Jones [**9**], unlike his proof that uses deep results of group theory, ours uses no heavy group-theoretic facts. We obtain our result by using some relatively elementary facts of graph theory and group theory. Our method for determining the automorphism group of $J(n, m)$ is even different from Ramras and Donovan [**18**].

Let Γ be a connected graph with diameter D and x be a vertex in Γ. Let $\Gamma_i = \Gamma_i(x)$ be the set of vertices in Γ at distance i from x. Thus, $\Gamma_0 = \{x\}$ and $\Gamma_1 = N(x)$, the set of vertices which are adjacent to the vertex x, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(x), \dots, \Gamma_D(x)$. In the first step, we need the following fact which is one of the key statements that the main proof is based on.

Proposition 3.2 ([**18**]). *Let $\Gamma = J(n, m)$, $n \geq 4$, $2 \leq m \leq n-1$. Let $x \in V(\Gamma)$, $\Gamma_i = \Gamma_i(x)$ and $v \in \Gamma_i$. Then for each i we have*

$$\bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i) = \{v\}.$$

Let Γ be a graph. The line graph $L(\Gamma)$ of the graph Γ is constructed by taking the edges of Γ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in Γ have a common vertex. There is an important relation between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(L(\Gamma))$. Indeed, we have the following fact [**4**, Chapter 15].

Theorem 3.3. *Let Γ be a connected graph. The mapping $\theta\colon \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(L(\Gamma))$ defined by;*

$$\theta(g)(\{u, v\}) = \{g(u), g(v)\}, \qquad g \in \mathrm{Aut}(\Gamma),\ \{u, v\} \in V(L(\Gamma))$$

is a group homomorphism and in fact we have

(i) *θ is a monomorphism provided $\Gamma \neq K_2$;*
(ii) *θ is an epimorphism provided Γ is not K_4, K_4 with one edge deleted, or K_4 with two incident edges deleted.*

Proposition 3.4. *Let v be a vertex of the Johnson graph $J(n, m)$. Then, $\Gamma_1 = \langle N(v)\rangle$, the subgraph of $J(n, m)$ induced by $N(v)$ is isomorphic to $L(K_{m,n-m})$, where $K_{m,n-m}$ is the complete bipartite graph with partitions of orders m and $m - n$.*

Proof. Let $I = \{1, 2, \ldots, n\}$, $v = \{x_1, \ldots, x_m\}$ and $w = v^c = \{y_1, \ldots, y_{n-m}\}$ be the complement of the subset v in I. Let $x_{ij} = (v - \{x_i\}) \cup \{y_j\}$, $1 \leq i \leq m$, $1 \leq j \leq n - m$. Then

$$N(v) = \{x_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n - m\}.$$

In $\Gamma_1 =< N(v) >$ two vertices x_{ij}, x_{rs} are adjacent if and only if $i = r$ or $j = s$. In fact, $\{x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_m\}$ and $\{x_1, \ldots, x_{r-1}, y_s, x_{r+1}, \ldots, x_m\}$ have $m - 1$ element(s) in common if and only if $x_i = x_r$ or $y_j = y_s$.

Let $X = \{v_1, \ldots, v_m\}$ and $Y = \{w_1, \ldots, w_{n-m}\}$ where $X \cap Y = \emptyset$, the empty set. We know that the complete bipartite graph $K_{m,n-m}$ is the graph with vertex set $X \cup Y$, and edge set $E = \{\{v_i, w_j\},\ 1 \leq i \leq m,\ 1 \leq j \leq n - m\}$. Then $L(K_{m,n-m})$ is the graph with vertex set $V(L(K_{m,n-m})) = E$ in which vertices $\{v_i, w_j\}$ and $\{v_r, w_s\}$ are adjacent, if and only if $v_i = v_r$ or $w_j = w_s$. Now it is an easy task to show that the mapping

$$\beta\colon L(K_{m,n-m}) \to \Gamma_1 = \langle N(v)\rangle, \qquad \beta(\{v_i, w_j\}) = x_{ij},$$

is a graph isomorphism. □

Theorem 3.5. *Let $\Gamma = J(n, m)$, $n \geq 4$, $2 \leq m \leq \frac{n}{2}$. If $n \neq 2m$, then $\mathrm{Aut}(\Gamma) \cong \mathrm{Sym}(n)$. If $n = 2m$, then $\mathrm{Aut}(\Gamma) \cong \mathrm{Sym}(n) \times \mathbb{Z}_2$.*

Proof. Let $G = \mathrm{Aut}(\Gamma)$. Let $x \in V = V(\Gamma)$, and $G_x = \{f \in G | f(x) = x\}$ be the stabilizer subgroup of the vertex x in Γ. Let $\langle N(x)\rangle = \Gamma_1$ be the induced subgraph of $N(x)$ in Γ. If $f \in G_x$ then $f_{|N(x)}$, the restriction of f to $N(x)$, is an automorphism of Γ_1. We define $\varphi\colon G_x \to \mathrm{Aut}(\Gamma_1)$ by the rule $\varphi(f) = f_{|N(x)}$. It is an easy task to show that φ is a group homomorphism. We show that $\ker(\varphi)$ is the identity group. If $f \in \ker(\varphi)$, then $f(x) = x$ and $f(w) = w$ for every $w \in N(x)$. Let D be the diameter of $\Gamma = J(n, m)$ and Γ_i be the set of vertices of Γ at distance i from the vertex x. Then $V = V(\Gamma) = \cup_{i=0}^{D} \Gamma_i$. We prove by induction on i that $f(u) = u$ for every $u \in \Gamma_i$. Let $d(u, x)$ be the distance of the vertex u from x. If $d(u, x) = 1$, then $u \in \Gamma_1$ and we have $f(u) = u$.

Assume $f(u) = u$ when $d(u, x) = i - 1$. If $d(u, x) = i$, then by Proposition 3.2, $\{u\} = \bigcap_{w \in \Gamma_{i-1} \bigcap N(u)} N(w)$, and therefore $f(u) = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} N(f(w))$. Since, $w \in \Gamma_{i-1}$, then $d(w, x) = i - 1$, and hence $f(w) = w$. Thus

$$f(u) = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} N(f(w)) = \bigcap_{w \in \Gamma_{i-1} \cap N(u)} N(w) = u.$$

Therefore, $\ker(\varphi) = \{1\}$. On the other hand

$$\frac{G_v}{\ker(\varphi)} \cong \varphi(G_v) \leq \operatorname{Aut}(\Gamma_1) = \operatorname{Aut}(\langle N(x) \rangle)$$

and thus $G_v \cong \varphi(G_v) \leq \operatorname{Aut}(\Gamma_1)$, and hence $|G_v| \leq |\operatorname{Aut}(\Gamma_1)|$.
By Proposition 3.4, $\Gamma_1 \cong L(K_{m,n-m})$, thus $\operatorname{Aut}(\Gamma_1) \cong \operatorname{Aut}(L(K_{m,n-m}))$. By Theorem 3.3, $\operatorname{Aut}(L(K_{m,n-m})) \cong \operatorname{Aut}(K_{m,n-m})$, hence $|\operatorname{Aut}(\Gamma_1)| = |\operatorname{Aut}(K_{m,n-m})|$, and therefore $|G_v| \leq |\operatorname{Aut}(K_{m,n-m})|$. Note that if $P = K_{s,t}$ is a complete bipartite graph, then for $t \neq s$ we have $|\operatorname{Aut}(P)| = s!t!$, and for $t = s$ we have $|\operatorname{Aut}(P)| = 2(s!)^2$ [**4**, Chapter17].

Since $\Gamma = J(n, m)$ is a vertex transitive graph, we have $|V(\Gamma)| = \frac{|G|}{|G_v|}$, thus $|G| = |G_v||V(\Gamma)| \leq |\operatorname{Aut}(K_{m,n-m})|\binom{n}{m}$. Now, if $n \neq 2m$, then we have

$$|G| \leq (m!)(n-m)! \frac{n!}{(m!)(n-m)!} = n!$$

and if $n = 2m$, then we have

$$(*) \qquad |G| \leq |\operatorname{Aut}(K_{m,m})| \binom{2m}{m}, \quad \text{and hence} \quad |G| \leq 2(m!)^2 \frac{(2m)!}{(m!)(m!)} = 2(2m)!$$

We know that if $\theta \in \operatorname{Sym}(I)$ where $I = \{1, 2, \ldots, n\}$, then

$$f_\theta \colon V(\Gamma) \to V(\Gamma), \qquad f_\theta(\{x_1, \ldots, x_m\}) = \{\theta(x_1), \ldots, \theta(x_m)\}$$

is an automorphism of Γ and the mapping $\psi\colon \operatorname{Sym}(I) \to \operatorname{Aut}(\Gamma)$, defined by the rule $\psi(\theta) = f_\theta$ is a group homomorphism. In fact ψ is an injection.

Now if $n \neq 2m$, since $|G| = |\operatorname{Aut}(\Gamma)| \leq n!$, we conclude that ψ is a bijection, and hence $\operatorname{Aut}(\Gamma) \cong \operatorname{Sym}(I)$.

If $n = 2m$, then $\Gamma = J(n, m) = J(2m, m)$ and the set $\{f_\theta | \ \theta \in \operatorname{Sym}(2m)\} = H$, is a subgroup of $\operatorname{Aut}(\Gamma)$. It is an easy task to show that the mapping $\alpha\colon V(\Gamma) \to V(\Gamma)$, $\alpha(v) = v^c$ where v^c is the complement of the set v in I, is also an automorphism of Γ, namely, $\alpha \in G = \operatorname{Aut}(\Gamma)$.

We show that $\alpha \notin H$. If $\alpha \in H$, then there is a $\theta \in \operatorname{Sym}(I)$ such that $f_\theta = \alpha$. Since $o(\alpha) = 2$, where $o(\alpha)$ is the order of α, then $o(f_\theta) = o(\theta) = 2$. We assert that θ has no fixed points, that is, $\theta(x) \neq x$, for every $x \in I$. In fact, if $x \in I$, and $\theta(x) = x$, then for the m-set $v = \{x, y_1, \ldots, y_{m-1}\} \subseteq I$, we have

$$f_\theta(v) = \{\theta(x), \theta(y_1), \ldots, \theta(y_{m-1})\} = \{x, \theta(y_1), \ldots, \theta(y_{m-1})\},$$

hence $x \in f_\theta(v) \cap v$, and therefore $f_\theta(v) \neq v^c = \alpha(v)$; which is a contradiction. Therefore, θ takes the form $\theta = (x_1, y_1) \dots (x_m, y_m)$ where (x_i, y_i) is a transposition of $\mathrm{Sym}(I)$. Now, for the m-set $v = \{x_1, y_1, x_2, \dots, x_{m-1}\}$ we have,

$$\alpha(v) = f_\theta(v) = \{\theta(x_1), \theta(y_1), \theta(x_{m-1})\} = \{y_1, x_1, \dots, \theta(x_{m-1})\}$$

and thus $x_1, y_1 \in f_\theta(v) \cap v$, hence $f_\theta(v) \neq v^c = \alpha(v)$; which is a contradiction.

It can be shown that for every $\theta \in \mathrm{Sym}(I)$, we have $f_\theta\alpha = \alpha f_\theta$. We now conclude that $H < \alpha >$ is a subgroup of G. Since $\alpha \notin H$ and $o(\alpha) = 2$, then $H\langle\alpha\rangle$ is a subgroup of G of order;

$$\frac{|H||\langle\alpha\rangle|}{|H \cap \langle\alpha\rangle|} = 2|H| = 2((2m))!.$$

Now, since by (*) $|G| \leq 2((2m)!)$, then $G = H\langle\alpha\rangle$. On the other hand, since $f_\theta\alpha = \alpha f_\theta$, for every $\theta \in \mathrm{Sym}(I)$, then H and $\langle\alpha\rangle$ are normal subgroups of G. Thus, G is a direct product of two groups H and $\langle\alpha\rangle$, namely, we have $G = H \times \langle\alpha\rangle \cong \mathrm{Sym}(2m) \times \mathbb{Z}_2$. □

We now determine the automorphism group of the enhanced Johnson graph $EJ(2m, m)$.

Lemma 3.6. *Let $m > 2$. Let x be a vertex of $EJ(2m, m)$ and $S = S_x = \langle N(x)\rangle$ be the induced subgraph of $EJ(2m, m)$ generated by $N(x)$. Then, x^c is the unique isolated vertex in S_x, namely, $\deg_S(x^c) = 0$. Furthermore, for all $z \in S_x - \{x^c\}$, $\deg_S(z) \geq m - 1$.*

Proof. Let $x = \{x_1, \dots, x_m\}$, $x^c = \{y_1, \dots, y_m\}$, where $\{1, \dots, m, m+1, \dots, 2m\} = I = \{x_1, \dots, x_m, y_1, \dots, y_m\}$. We define the sets $C_{x_i} = \bigcup_{j=1}^{m}(x - \{x_i\}) \cup \{y_j\}$ $(1 \leq i \leq m)$. Let $S_i = \langle C_{x_i}\rangle$ be the induced subgraph of C_{x_i}. Now it is obvious that S_i is a clique in S_x and $\{y_1, \dots, y_m\} = x^c \notin C_{x_i}$. On the other hand, we have $N(x) - x^c = \bigcup_{i=1}^{m} C_{x_i}$. Now it is obvious that x^c is not adjacent to any vertex of S_x. In fact, if $v \in N(x) - x^c$, then $\{v, x\}$ is an edge in $J(2m, m)$, therefore $|v \cap x| = m - 1$, hence there are i and j such that $v = \{x_1, \dots, x_{i-1}, y_j, x_{i+1}, \dots, y_m\}$, namely, $v \in C_{x_i}$. Since C_{x_i} is a m-clique in S_x, then the degree of every vertex of S_x which is in C_{x_i} is at least $m - 1$. □

Lemma 3.7. *Let $G_1 = \mathrm{Aut}(\Gamma_1)$ where $\Gamma_1 = EJ(2m, m)$, and $f \in G_1$. If $e = \{v, w\} \in E$, where E is the edge set of $\Gamma = J(2m, m)$, then $f(e) = \{f(v), f(w)\} \in E$. In other words, if $|v \cap w| = m - 1$, then $|f(v) \cap f(w)| = m - 1$.*

Proof. Let $S_v = \langle N(v)\rangle$ be the induced subgraph in Γ_1 generated by $N(v)$. Then, we have $\langle N(f(v))\rangle = f(\langle N(v)\rangle) = S_{f(v)}$. Since $w \in N(v)$ then $f(w) \in S_{f(v)}$, and since by Lemma 3.6, $deg(w)$ in S_v is at least $m - 1$, thus $deg(f(w))$ in $S_{f(v)}$ is at least $m - 1$, therefore $f(w) \neq f(v)^c$. We now conclude that $\{f(v), f(w)\} \notin E_1$ (where $E_1 = \{\{v, v^c\} \mid v \subseteq I = \{1, \dots, 2m\}, |v| = m\}$), and therefore we have $\{f(v), f(w)\} \in E$. □

Theorem 3.8. *The automorphism group of the enhanced Johnson graph $\Gamma_1 = EJ(2m, m)$ is identical with the automorphism group of the Johnson graph $\Gamma = J(2m, m)$.*

Proof. Let $I = \{1, \ldots, 2m\}$, $G = \mathrm{Aut}(\Gamma)$, $G_1 = \mathrm{Aut}(\Gamma_1)$, $E_1 = E(\Gamma_1)$ and $E = E(\Gamma)$. Note that $E \cap E_1 = \emptyset$. In the first step we show that $G_1 \leq G$. Let $f \in G_1$ and $e = \{v, w\}$ be an edge in Γ, then e is an edge in Γ_1, therefore $f(e) = \{f(v), f(w)\}$ is an edge in Γ_1. Now by Lemma 3.7, we have $f(w) \neq f(v)^c$, hence $|f(v) \cap f(w)| = m - 1$, thus $\{f(v), f(w)\}$ is an edge in Γ, and therefore f is an automorphism of Γ, namely, $f \in G$.

In this step, we show that $G \leq G_1$. Let $f \in G$. Then, by Theorem 3.5. $f = f_\theta \alpha^i$, $(i \in \{0, 1\})$, where $\theta \in \mathrm{Sym}(I)$, $\alpha \colon V \to V$, $\alpha(v) = v^c$ for any $v \in V(\Gamma)$ and $f_\theta(\{j_1, \ldots, j_m\}) = \{\theta(j_1), \ldots, \theta(j_m)\}$. If e is an edge of Γ_1, then $e \in E_2 = E \cup E_1$. If $e \in E$ then $f(e) \in E$, because $f \in G$. Note that if $e \in E_1$, then $e = \{v, v^c\}$, where v is an m-subset of I. If $e \in E_1$ and $i = 0$ then $f = f_\theta$, therefore $f(e) = f_\theta\{v, v^c\}$ $= \{f_\theta(v), f_\theta(v^c)\}$. Since $|v \cap v^c| = 0$, then we have

$$|f(v) \cap f(v^c)| = |f_\theta(v) \cap f_\theta(v^c)| = |\theta(v) \cap \theta(v^c)| = 0$$

and hence $\{f(v), f(v^c)\} \in E_1$. If $e \in E_1$ and $i = 1$, then $f = f_\theta\alpha$, therefore

$$f(e) = \{f(v), f(v^c)\} = \{f_\theta\alpha(v), f_\theta\alpha(v^c)\} = \{f_\theta(v^c), f_\theta(v)\}$$

Now since $\{f_\theta(v^c) \cap f_\theta(v)\} = \emptyset$, it follows that $f(e) \in E_1$, and thus $f(e) \in E_2$. Therefore, in any case, f is an automorphism of G_1. $\square$

We know that that the Johnson graph $J(2m, m)$ is a vertex-transitive graph, now it follows from Theorem 3.8, that the enhanced Johnson graph $EJ(2m, m)$ is also vertex-transitive.

3.3. Integrality

Let Γ be a graph with vertex set $V(\Gamma) = V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(\Gamma)$. The adjacency matrix $A = A(\Gamma) = [a_{ij}]$ of Γ is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The characteristic polynomial of Γ is the polynomial $P(G) = P(G, x) = \det(xI_n - A)$, where I_n denotes the $n \times n$ identity matrix. The spectrum of $A(\Gamma)$ is also called the spectrum of Γ. If the distinct eigenvalues are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we write

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_r \\ m_1, m_2, \ldots, m_r \end{pmatrix} \quad \text{or} \quad \mathrm{Spec}(\Gamma) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}\}.$$

If all the eigenvalues of the adjacency matrix of the graph Γ are integers, then we say that Γ is an integral graph. The notion of integral graphs was first introduced by F. Harary and A. J. Schwenk in 1974 (see [**8**]). In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area (for example [**6**]) and some of the recent works in this scope of research in algebraic graph theory include [**1**], [**2**], [**17**]. In the sequel, we show that the enhanced Johnson graph $EJ(2m, m)$ is an integral graph.

Let Γ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix A, and the rows and columns of A are labeled by the set V. Let π be a permutation of the set V. We know that π can be represented by a permutation matrix $P_\pi = P =$

(p_{ij}), where $p_{ij} = 1$ if $v_i = \pi(v_j)$, and $p_{ij} = 0$ otherwise. It is a well known fact that π is an automorphism of the graph Γ if and only if $AP = PA$ [**2**, Chapter 15].

Theorem 3.9. *If* $\Gamma = EJ(2m, m)$, *then* Γ *is an integral graph.*

Proof. We know that if $\Gamma = J(2m, m)$, then the permutation $\alpha : V(\Gamma) \to V(\Gamma)$, $\alpha(v) = v^c$, where v^c is the complement of the set v, is an automorphism of the graph Γ. Thus, if P is the permutation matrix of α, then we have $AP = PA$ where A is the adjacency matrix of the graph $J(2m, m)$.

Note that the adjacency matrix of $EJ(2m, m)$ is of the form $A + P$. Since α is of order 2, then $P^2 = E$ where $E = I_h$ is the identity matrix of size h ($h = \binom{2m}{m}$). Hence if $p(x) = x^2 - 1$, then $p(P) = 0$. Thus, if μ is an eigenvalue of the matrix P, then $p(\mu) = 0$, namely, $\mu \in \{1, -1\}$. Since $AP = PA$ and the matrices A and P are symmetric, hence the matrices A and P are diagonalizable, and therefore there is a basis $B = \{u_1, \ldots, u_h\}$ of $\mathbb{R}^h$ such that each u_i is an eigenvector of the matrices A and P [**5**], [**7**]. Therefore, if $Au_i = \lambda_i u_i$, then $(A+P)u_i = \lambda_i u_i + t_i u_i = (\lambda_i + t_i)u_i$, where $t_i \in \{1, -1\}$. The spectrum of Johnson graph $J(n, m)$, $n, m \in \mathbb{N}$, $m < n$, is

$$\left\{\lambda_i^{m_i} \mid \lambda_i = (m-i)(n-m-i) - i,\ m_i = \binom{n}{i} - \binom{n}{i-1},\ 0 \le i \le d\right\},$$

where $d = \min(m, n-m)$ [**5**, page 179]. Note that each λ_i is an integer, hence the Johnson graph $J(n, m)$ is an integral graph. Now, since each eigenvalue of the enhanced Johnson graph $EJ(2m, m)$ is of the form $\lambda + t$, $t \in \{1, -1\}$, where λ is an eigenvalue of the Johnson graph $J(2m, m)$, hence $EJ(2m, m)$ is an integral graph. □

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