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STABLE EMBEDDINGS ON CLOSED SURFACES WITH RESPECT TO MINIMUM LENGTH

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Abstract. An embedding of a graph on a closed surface with suitable metric is said to be *minimum-length embedding* if the total sum of lengths of its edges measured by the metric is the minimum among all embeddings isotopic to it and is said to be *stable* with respect to minimum length if the limit of any convergent sequence of minimum-length embeddings isotopic to it is an embedding of the graph. We shall discuss these notions and shall decide which 4-regular quadrangulations and which 6-regular triangulations on the torus have minimum-length embeddings and are stable with respect to minimum length.

1. Introduction

Research on how to draw or represent a graph is one of streams in geometric graph theory. The most classical theorem along this stream is Fáry's theorem [**2**], which shows that any plane graph can be deformed so that each edge becomes a straight line segment and there have been many studies on geometric representations of graphs, listed in [**13**]. Also Hubard and et al. [**3**] has discussed the existence of Riemannian metrics over closed surfaces for which embedded graphs have a special property, motivated by Negami's conjecture [**11**] on joint crossing numbers. In this paper, we shall discuss a kind of "stableness" of embeddings of graphs on closed surfaces with respect to metrics, formulated below.

Suppose that a graph G has been embedded on a closed surface F^2 which has a suitable metric, by an embedding map $f\colon G \to F^2$. Then we can measure the length of each edge $f(e)$ according to the metric over F^2. We denote the length of $f(e)$ by $|f(e)|$ and define the *total length* $\|f\|$ of the embedding of G by:

$$\|f\| = \|f(G)\| = \sum_{e \in E(G)} |f(e)|$$

We would like to consider what happens when we minimize the total length of an embedding over the surface; this question has appeared in [**7**]. For example, Figure 1 shows two different embeddings of the complete graph K_5 with five

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vertices on the torus. In the left one, four triangular faces incident to the vertex placed at the center form a rhombus. The rhombus will shrink to a point when we minimize the total length of this K_5. Thus, the final form of this embedding will not be an embedding of K_5 at all. On the other hand, we can show that the right one is an embedding of K_5 which attains the minimum of its total length, using our later arguments.

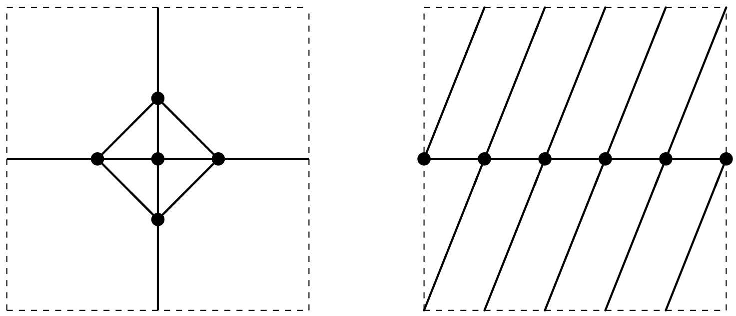

Figure 1. Two embeddings of K_5 on the torus.

We should prepare many things about topology and geometry to establish a rigorous theory on this subject. However, we dare to carry out slightly intuitive arguments meanwhile below.

Let F^2 be a closed surface which has a suitable metric and let G be a graph embedded on F^2 by an embedding map $f\colon G \to F^2$. Two embeddings f_1 and $f_2\colon G \to F^2$ are said to be *isotopic* if a continuous deformation over F^2 modifies $f_1(G)$ to $f_2(G)$, keeping it being an embedding of G. An embedding $f_{\min}\colon G \to F^2$ is said to be *minimum-length* if it attains the infimum of the total lengths $\|f'\|$ taken over all embeddings $f'\colon G \to F^2$ isotopic to f. We say simply that an embedding of G or G itself has a minimum-length embedding if it is isotopic to an minimum-length embedding of G.

We shall show an enough big class of graphs which have minimum-length embeddings on the torus. The torus is assumed to have a parabolic metric, defined later, and such a torus is called a *flat torus*. The torus admits embeddings of 4-regular graphs which tessellates the torus into quadrilaterals. Such an embedding of a 4-regular graph is called a 4-*regular quadrangulation* on the torus and can be obtained from a rectangular grid of suitable size by identifying each pair of parallel sides. We shall show the following fact on 4-regular quadrangulatoins on the torus:

Theorem 1. *Any* 4*-regular quadrangulation on a flat torus has a minimum-length embedding.*

There have been classified 4-regular quadrangulations on the torus with suitable parameters, up to auto-homeomorphisms over the torus [**6**], as described in Section 4. It should be noticed that an auto-homeomorphism over the torus changes the total length of an embedding in general.

As we shall show later, the minimum-length embedding of a 4-regular quadrangulation on the torus looks like a grid which consists of two sets of cycles placed in parallel and we can slide one of cycles, not changing the total length of its embedding. Thus, if a set of parallel cycles contains two or more cycles, then we can find a sequence of embeddings $f_1, f_2, \ldots$ each of which is isotopic to the original embeddings, but its limit is not. In general, an embedding of a graph on a closed surface is said to be *stable* with respect to minimum length if the limit of any convergent sequence of embeddings of G isotopic to the embedding is an embedding of G.

We shall show the following theorem on the stableness of minimum-length embeddings of 4-regular quadrangulations on the torus. By this theorem, we can conclude that the right embedding of K_5 on the torus is stable.

Theorem 2. *A* 4*-regular quadrangulation* G *on a flat torus is stable with respect to minimum length if and only if* G *can be obtained as an edge-disjoint union of two hamilton cycles which cross each other at each vertex.*

Adding a diagonal to each face of a 4-regular quadrangulation on the torus will yield a 6*-regular triangulation* on the torus. Actually, it has been known that any 6-regular triangulation on the torus can be obtained in such a way [**1, 8**], as described in Section 4. We shall show the following theorem for it:

Theorem 3. *Any* 6*-regular triangulation on a flat torus has a minimum-length embedding which is stable with respect to minimum length.*

In the next section, we shall develop a topological theory for minimum-length embeddings, which guarantees our intuitive arguments later. In Section 3, we shall formulate the torus having parabolic metric, using the universal covering space, which is the Euclidean plane $\mathbb{R}^2$ with Euclidean distance. In Section 4, we shall describe the classifications of 4-regular quadrangulations and 6-regular triangulations on the torus and prove our three main theorems. This paper presents just the first step for this topic and confines ourselves to arguments for embeddings on the torus. In Section 5, we shall show some comments for further studies on this topic under more general situation.

2. Minimum-length embeddings

We have introduced minimum-length embeddings and their stableness in an intuitive way in introduction. Here we shall show the formulation for those in terms of topology. The reader unfamiliar to general topology may skip this section and come back after reading later chapters if necessary.

First we shall discuss a simple necessary condition for an embedding of a graph on a closed surface to have a minimum-length embedding. It is clear that any embedding of any graph on the sphere shrinks to a point continuously and hence it does not have a minimum-length embedding on the sphere. Thus, the surface where a graph has been embedded must be a closed surface other than the sphere. Furthermore, if an embedding of a graph G has a minimum-length embedding on the surface, then “a local planar part” as shown in Figure 2 must be excluded. Shrink this part toward the central point in the figure. Then the edges placed radially toward the outside will be longer, but the total length of the whole embedding will be shorter if there are enough many edges in the inner part. For example, the embedding of K_5 on the torus in the left of Figure 1 has such a local planar part while the right one does not.

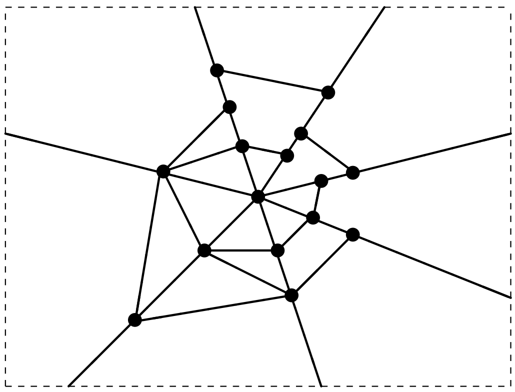

Figure 2. A local planar part in an embedding.

To consider the stableness of embeddings, we need more delicate arguments, as follows. Suppose that a graph G has a path uxv of length 2 passing through a vertex x of degree 2 and that it has been embedded on a closed surface F^2 by a map $f_0\colon G \to F^2$ to be minimum-length. Then we can define a sequence of embeddings $f_t\colon G \to F^2$ $(t \in [0,1))$ so that the image of each embedding f_t coincides with that of f_0 but that $f_t(x)$ approaches to the point $f_0(v)$ $(= f_t(v))$ along the path $f_0(uxv)$ with $d_{F^2}(f_t(x), f_0(v)) = (1-t)\cdot d_{F^2}(f_0(x), f_0(v))$, where $d_{F^2}(\cdot,\cdot)$ denotes the distance between two points on the surface F^2 induced by its metric. Since $\lim_{t\to 1} d_{F^2}(f_t(x), f_t(v)) = 0$, the edge $f_t(xv)$ shrinks to the point $f_0(v)$ in the limit of f_t as $t \to 1$ and the limit is not an embedding of G at all.

A similar phenomenon always happens even if G has no vertex of degree 2. Regard the previous x as just the midpoint of an edge uv in G, not a vertex now, and consider the same sequence of embeddings $f_t\colon G \to F^2$. The images $f_t(uv)$ of the edge uv form the same set of points on F^2 and have the same length, but they are parameterized in different ways. Each of f_t's is a minimum-length embedding of G and its limit also has the same image as f_t's do. However, the limit is not an embedding as a map since one half of the edge uv split at the point x covers the image of $f_0(uv)$ and the other half shrinks to the point $f_0(v)$ in its image.

Thus, if we never set any restriction for embeddings, then any minimum-length embedding could not be stable. To exclude this unexpected phenomenon, we should assume that *each edge must be mapped monotonically in proportion to arc length.* Under this assumption, we can discuss the length of each edge in an embedded graph, not taking how it is mapped along its image into account. We shall assume it implicitly below.

Furthermore, we should define a topology over the set of embeddings to consider "the limit" of a sequence of embeddings. Let $f_0 \colon G \to F^2$ be an embedding of a graph G to a closed surface F^2. Another embedding $f_1 \colon G \to F^2$ is said to be *isotopic* to f_0 on F^2 if there exists a continuous map $H \colon G \times [0,1] \to F^2$ such that $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$ for all $x \in G$ and that $f_t = H(\cdot, t) \colon G \to F^2$ is an embedding of G. Roughly speaking, the image of an embedding f_1 isotopic to f_0 on F^2 can be obtained from that of f_0 by continuous deformation over the surface through f_t's. If we don't force each of f_t's to be an embedding of G in the definition, then f_1 will be said to be *homotopic* to f_0 on F^2.

Let $\mathcal{I}_{f_0}$ be the set of embeddings isotopic to f_0 on F^2, called the *isotopy class* of f_0. That is, $\mathcal{I}_{f_0}$ includes all embeddings of G obtained by deforming f_0 continuously on F^2, as the elements belonging to the set. Then we can define the distance $d(f, f')$ between two embeddings f and f' in $\mathcal{I}_{f_0}$ by:

$$d(f, f') = \sup\{d_{F^2}(f(x), f'(x)) : x \in G\},$$

where x ranges over all points in the 1-dimensional topological space G. This is the standard way to define the distance over a set of functions. It makes $\mathcal{I}_{f_0}$ a metric space and naturally induces its topology.

Suppose that an infinite sequence of embeddings $f_1, f_2, \ldots$ is a *Cauchy sequence* in $\mathcal{I}_{f_0}$, that is, $\lim_{n,m\to\infty} d(f_n, f_m) = 0$. Roughly speaking, embeddings f_n with sufficiently large numbers n are very close to each other and look almost the same. Since $d_{F^2}(f_n(x), f_m(x)) \leq d(f_n, f_m)$, the sequence of points $f_1(x), f_2(x), \ldots$ for any point $x \in G$ must be a Cauchy sequence on the surface F^2. Since F^2 is closed and compact, it must be a complete metric space and the Cauchy sequence of points converges to a point $f_\infty(x)$ on F^2 by a general theory on complete metric spaces. These limit points $\{f_\infty(x) : x \in G\}$ induce a continuous map $f_\infty \colon G \to F^2$. This is the limit of embeddings $f_1, f_2, \ldots$, but it may not be an embedding of G, that is, $f_\infty = \lim_{n\to\infty} f_n \notin \mathcal{I}_{f_0}$ in such a case.

One might wonder if the total length $\|\cdot\| \colon \mathcal{I}_{f_0} \to \mathbb{R}$ would be a continuous function. However, it is not. Let f be an embedding of G which is a point in $\mathcal{I}_{f_0}$, and take any positive real number $\varepsilon > 0$. If $\|\cdot\|$ were continuous, then there would be a positive real number $\delta > 0$ such that if $d(f', f) < \delta$, then $|\,\|f'\| - \|f\|\,| < \varepsilon$. If $f'(x)$ belongs to the δ-open neighborhood of $f(x)$ for all $x \in G$, then we have $d(f', f) < \delta$ according to the definition. For example, replace the image of $f(uv)$ for an edge uv in G with a waving curve of amplitude less than δ joining $f(u)$ and $f(v)$, like the sine curve, and let f' be the embedding f so deformed. Then $d(f', f) < \delta$, but we can make $f'(uv)$ arbitrarily long by waving it as many times as we want, and hence $\|f'\|$ is arbitrarily larger than $\|f\|$, which is contrary to the assumption of $|\,\|f'\| - \|f\|\,| < \varepsilon$. Therefore, the function $\|\cdot\|$ is not continuous.

If we restrict the whole set of embeddings to that of embeddings such that each edge is mapped to a geodesic joining the images of its two ends, then $\|\cdot\|$ will be continuous. As another idea, we may change the topology over $\mathcal{I}_{f_0}$ itself, replacing the distance $d(f, f')$ between two embeddings with

$$\tilde{d}(f, f') = \sqrt{d(f, f')^2 + (\|f\| - \|f'\|)^2}.$$

Then $\|\cdot\|$ will be continuous, too. In particular, if both f and f' are minimum-length embeddings, then we have $\tilde{d}(f,f') = d(f,f)$ since $\|f\| = \|f'\|$. Thus, the stableness of minimum-length embeddings does not change for this new topology. However, these arguments are not so important for our later arguments in this paper and hence we shall omit them.

3. Flat tori

It is well-known that the torus can be obtained from a square or a rectangle by identifying each pair of parallel edges and is also obtained as the quotient space of an action of parallel moves over the plane $\mathbb{R}^2$. We shall introduce a way to give a metric for the torus.

The Euclidean plane $\mathbb{R}^2$ has what is called the *Euclidean metric*, which can be calculated in a usual way by xy-coordinates. A translation over the plane $\mathbb{R}^2$ is called an *isometry* if it preserves the distance between any two points in the plane. A parallel shift induced by one vector is a typical isometry over the plane.

Let $\vec{x}$ and $\vec{y}$ be two independent vectors in $\mathbb{R}^2$ based at the origin $O = (0,0)$ and let Ω be the parallelogram determined by these two vectors $\vec{x}$ and $\vec{y}$. Then the *isometry group* Γ generated by two parallel shifts corresponding to $\vec{x}$ and $\vec{y}$ acts on the plane and the parallel copies of Ω tessellate the whole plane. The original parallelogram Ω is often called a *fundamental region* of this group action.

If two points on the plane can be transfered to each other by the action of Γ, then they are said to be *equivalent* (under the action of Γ). It is easy to see that every point on the plane is equivalent to a point in Ω and that two distinct points inside Ω are not equivalent. These imply that the quotient space of this group action can be obtained from the parallelogram Ω by identifying each pair of its parallel sides and is homeomorphic to a torus.

A *flat torus* is a torus so obtained and its local metric of such a flat torus can be derived as a local copy of the Euclidean metric over $\mathbb{R}^2$. Any two flat tori are homeomorphic to each other, but they have different metric and different area, depending on the group actions over $\mathbb{R}^2$. However, all flat tori have curvature 0 everywhere in the sense of differential geometry.

Let $q\colon \mathbb{R}^2 \to T^2$ be the natural projection from the plane to a flat torus T^2, which maps all mutually equivalent points to the same point on T^2. Both $q = q_\Gamma$ and $T^2 = T^2_\Gamma$ depend on the isometry group Γ, but we omit "Γ" to simplify their notations. The plane $\mathbb{R}^2$ with this projection q is called the *universal covering space* of the torus T^2 in topology. (Refer to [**12**] for the notions in algebraic topology).

Put $p_0 = q(O)$. The sides of Ω corresponding to $\vec{x}$ and $\vec{y}$ are projected to simple closed curves on T^2 based at p_0. These closed curves are often called a *longitude* and a *meridian.* We denote them by L and M and regard them as two generators of the fundamental group $\pi_1(T^2, p_0)$ of T^2. Thus, any closed curve on T^2 based at p_0, or its homotopy class precisely, can be expressed as a linear combination of L and M, say $\lambda \cdot L + \mu \cdot M$ with integers λ and μ since $\pi_1(T^2, p_0) \cong \mathbb{Z} \oplus \mathbb{Z}$ is an abelian group. Such a closed curve γ is lifted to a curve on $\mathbb{R}^2$ starting at the origin $O = (0,0)$ and ending at the point P indicated by $\lambda \cdot \vec{x} + \mu \cdot \vec{y}$. It is clear that we can deform the curve joining two points O and P continuously into the line segment OP, not moving its two ends. This implies that the closed curve γ is homotopic to the closed curve on T^2 obtained as $q(OP)$.

A *geodesic* between two points on a surface F^2, closed or open, having a metric is a curve joining them whose length cannot be shorter by local deformation in general. For example, any geodesic on the Euclidean plane $\mathbb{R}^2$ is nothing but a straight line segment and any geodesic between two points on a flat torus can be lifted to a straight line segment on $\mathbb{R}^2$. A *closed geodesic* also is defined similarly as a locally shortest closed curve on a surface. A geodesic (or a closed geodesic) on a surface F^2 can be regarded as the image of a continuous map $\gamma\colon [0,1] \to F^2$ (or $\gamma\colon S^1 \to F^2$), where S^1 is the unit circle. They are said to be *simple* if these maps γ are injective. Notice that if a geodesic goes over a closed surface globally, it may cross itself and that a closed geodesic may run along a simple closed curve more than one times via the map γ. In these cases, they are not simple.

Proposition 4. *Any closed curve on a flat torus T^2 expressed as $\lambda \cdot L + \mu \cdot M$ in $\pi_1(T^2, p_0)$ is homotopic to a closed geodesic whose length is equal to $|\lambda \cdot \vec{x} + \mu \cdot \vec{y}\,|$. In particular, if λ and μ are relatively prime, then the geodesic is a simple closed curve on T^2.*

Proof. We can slide a given closed curve γ on the flat torus without changing its length so that it contains the base point p_0 afterward and lift it to a curve $\tilde{\gamma}$ between the origin O and the point P indicated by $\lambda \cdot \vec{x} + \mu \cdot \vec{y}$ on the plane $\mathbb{R}^2$. As described in the previous, we can deform the path $\tilde{\gamma}$ into the line segment OP continuously, fixing its end points. Thus γ is homotopic to the closed geodesic $q(OP)$ based at p_0.

The closed geodesic $q(OP)$ on the torus forms one simple closed geodesic as a point set although the projection of OP goes along it many times in general, say d times. If $d \geq 2$, then the closed geodesic $q(OP)$ is not simple and $d-1$ points in $q^{-1}(p_0)$ become intermediate points of the line segment OP placed at equal intervals. Since these points are copies of the origin O, they can be indicated by a linear combination of $\vec{x}$ and $\vec{y}$ with integral coefficients. This implies that d becomes a common divisor of λ and μ. Thus, if λ and μ are relatively prime, then we have $d = 1$ and the projection of OP goes along $q(OP)$ exactly once. Therefore, the closed geodesic $q(OP)$ is simple. □

If a graph G has been embedded on a surface, then a cycle in G can be regarded as a simple closed curve on the surface and it happens that such a cycle becomes a simple closed geodesic on the surface. In this case, we call it a *geodesic cycle.* Each edge in a geodesic cycle must be a geodesic between its ends. If the two geodesics along two such edges meet at a common end, then they make a corner of 180° since they form a part of the closed geodesic and have the same tangent at the common end. The following lemma gives us a criterion for an embedding of a graph to be minimum-length.

Lemma 5. *If an embedding f of a graph G on a flat torus decomposes into an edge-disjoint union of geodesic cycles, then it is a minimum-length embedding.*

Proof. Let $C_1, \dots, C_k$ be mutually edge-disjoint cycles whose union forms the whole of G and let $|f(C_i)|$ denote the summation of lengths of edges $f(e)$ lying along $f(C_i)$ on the flat torus. Then we have:

$$\|f\| = |f(C_1)| + \cdots + |f(C_k)|$$

Since a closed geodesic is the shortest among all closed curves homotopic to it, if each $f(C_i)$ is a geodesic cycle, then we cannot improve the above total length $\|f\|$. Thus, the embedding f must be a minimum-length embedding. □

4. Regular maps on the torus

The 4-regular quadrangulations and the 6-regular triangulations on the torus form two well-classified families of graphs embedded on the torus with suitable parameters as shown below and we can apply Lemma 5 to them. Their detailed descriptions can be found in [**6**] for example. In particular, the 6-regular triangulations on the torus have been classified in [**1**] and [**8**] under different contexts.

Prepare a $(p+1) \times (r+1)$ rectangular grid, which can be obtained as the Cartesian product $P_p \times P_r$ of two paths of lengths p and r. The vertices in this grid can be labeled by $v_{(i,j)}$ so that $v_{(i,0)}v_{(i,1)} \cdots v_{(i,p)}$ forms a vertical path of length p and that $v_{(0,j)}v_{(1,j)} \cdots v_{(r,j)}$ forms a horizontal path of length r. First, identify the pair of parallel sides at the top and the bottom to construct a cylinder $C_p \times P_r$ where $v_{(i,0)} = v_{(i,p)}$ for $i = 0, 1, \dots, r$. Thus, the second component j in the subscript of $v_{(i,j)}$ should be taken modulo p. Next, identify the two ends of this cylinder to make a torus so that $v_{(r,j)}$ coincides with $v_{(0,j+q)}$ (see Figure 3).

The resulting quadrangulation on the torus is denoted by $Q(p,q,r)$. Any 4-regular quadrangulation on the torus can be obtained as $Q(p,q,r)$ with suitable parameters p, q and r, up to auto-homeomorphism over the torus. Add the diagonal $v_{(i,j)}v_{(i+1,j+1)}$ in each quadrilateral face $v_{(i,j)}v_{(i+1,j)}v_{(i+1,j+1)}v_{(i,j+1)}$ of $Q(p,q,r)$. Then we obtain the 6-regular triangulation $T(p,q,r)$ on the torus. Any 6-regular triangulation on the torus is isomorphic to $T(p,q,r)$ for suitable values of p, q and r. Note that each of $Q(p,q,r)$ and $T(p,q,r)$ presents a unique embedding on the torus, up to auto-homeomorphism over the torus, but it has many different embeddings if we take the metric over the torus into account. We shall describe those in our proofs below.

Theorem 6. *Any embedding of a* 4*-regular quadrangulation* $Q(p,q,r)$ *on a flat torus has a minimum-length embedding.*

Proof. The 4-regular quadrangulation $Q(p,q,r)$ decomposes into two sets of disjoint cycles. One consists of r copies of C_p placed in parallel. Thus, each of them can be expressed as a cyclic sequence $v_{(i,0)}v_{(i,1)} \cdots v_{(i,p-1)}$ for $i = 0, 1, \dots, r-1$; we call it "a vertical cycle" here.

The other set consists of cycles obtained by joining "horizontal paths" corresponding to P_r. One goes from $v_{(0,0)}$ to $v_{(r,0)}$ along the bottom horizontal path. After arriving at $v_{(r,0)}$, he jumps to $v_{(0,q)}$ according to the identification of vertical sides and goes along another horizontal path. Continuing this process as far as

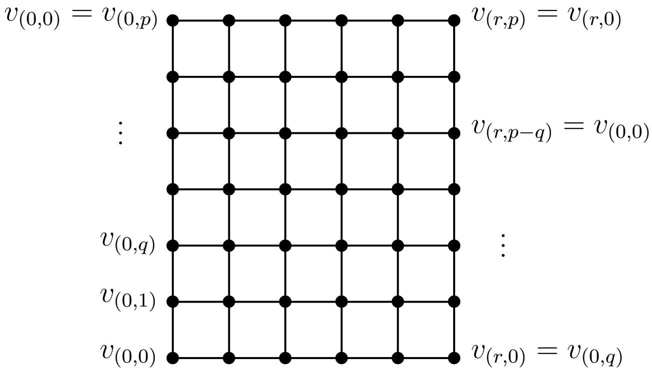

Figure 3. The 4-regular quadrangulation $Q(p, q, r)$ on the torus.

possible, he will come back to $v_{(0,0)}$ finally and get a cycle. This cycle may not cover all horizontal paths in general. In such a case, the cycles obtained similarly are placed in parallel to the first one. It is not so difficult to see that the number of these "horizontal cycles" is equal to $\gcd(p, q)$, which stands for the greatest common divisor of integers p and q.

Now suppose that $Q(p, q, r)$ has been embedded on a flat torus T^2 in one way and consider the universal covering $q\colon \mathbb{R}^2 \to T^2$ which induces the metric over T^2. We may assume that the origin $O = (0, 0)$ projects to $v_{(0,0)}$ placed on T^2. Then the vertical cycle $v_{(0,0)}v_{(0,1)} \cdots v_{(0,p)}(= v_{(0,0)})$ is lifted to a simple curve joining O to another preimage of $v_{(0,0)}$, say P. We can deform such a simple curve into the line segment OP continuously on the plane $\mathbb{R}^2$, fixing its ends. This deformation induces an isotopic deformation for $Q(p, q, r)$ embedded on the flat torus, which moves the vertical cycle to a closed geodesic and arranges the positions of $v_{(0,0)}, v_{(0,1)}, \ldots, v_{(0,p-1)}$ so that they lie along the closed geodesic at equal intervals afterward.

Similarly, we can deform the other vertical cycles into closed geodesics so that they are lifted to a set of parallel lines on the plane. Although each of horizontal paths in $Q(p, q, r)$ may be lifted to a broken line yet, we can modify the positions of $v_{(i,j)}$'s with $i \neq 0$ along the geodesics so that each of horizontal paths also becomes a geodesic and the horizontal cycles form closed geodesics on the flat torus. Then we will find a $(p + 1) \times (r + 1)$ parallelogram grid on the plane $\mathbb{R}^2$ but it may not be a rectangle in general.

Now $Q(p, q, r)$ embedded on the flat torus has been deformed into an embedding which decomposes into two sets of closed geodesics, vertical cycles and horizontal cycles. Therefore, this is a minimum-length embedding by Lemma 5. □

One might wonder if the parallelogram obtained as the lift of the rectangular grid prepared to define $Q(p, q, r)$ would be a fundamental region since its copies tessellate the plane. However, this region may not coincide with the fundamental region Ω used to define the flat torus and the infinite grid obtained as the frame of such a tessellation is not 4-regular if $q \neq 0$. Consider the parallelogram which has the four points corresponding to $v_{(0,0}$, $v_{(0,p)}$, $v_{(r,-q)}$ and $v_{(r,p-q)}$ as its vertices. The parallel shifts of this parallelogram form a tessellation of the plane with a 4-regular grid and it induces the same group action over $\mathbb{R}^2$ as Ω does. However, the parallelogram may be different from Ω, depending on how $Q(p,q,r)$ is embedded on the flat torus.

Notice that the point P appearing in the previous proof depends on the embedding of $Q(p,q,r)$ on a flat torus, or on the homotopy class of its vertical cycle. The length $|OP|$ will be an invariant for minimum-length embeddings of $Q(p,q,r)$ up to isotopy. Thus, two embeddings having different values of $|OP|$ are not isotopic on a fixed flat torus.

Since any 4-regular quadrangulation on a flat torus can be regarded as an embedding of $Q(p,q,r)$ on the torus, Theorem 6 immediately implies Theorem 1. To prove Theorem 2, it suffices to count the number of parallel geodesic cycles in $Q(p,q,r)$ on a flat torus.

Theorem 7. *An embedding of a* 4*-regular quadrangulation* $Q(p,q,r)$ *on a flat torus is stable with respect to minimum length if and only if* $r=1$ *and* $\gcd(p,q)=1$.

Proof. By the previous proof, we have already known that the number of vertical cycles in $Q(p,q,r)$ is equal to r while the number of horizontal cycles in it is equal to $\gcd(p,q)$. If at least one of r and $\gcd(p,q)$ is more than 1, then we find a pair of geodesic cycles in the minimum-length embedding of $Q(p,q,r)$ placed in parallel on the flat torus and can slide one of the geodesic cycles onto the other. This deformation derives a sequence of minimum-length embeddings of $Q(p,q,r)$ whose limit is not an embedding at all. Therefore, the minimum-length embedding of $Q(p,q,r)$ is not stable.

On the other hand, if $r=1$ and $\gcd(p,q)=1$, then $Q(p,q,r)$ has exactly one vertical cycle and one horizontal cycle, say C_V and C_H. Both C_V and C_H must be closed geodesics in any minimum-length embedding of $Q(p,q,r)$ on a flat torus T^2. Then the preimage of C_V by the projection $q\colon \mathbb{R}^2 \to T^2$ consists of parallel lines and these lines contain all vertices which project to the vertices of $Q(p,q,r)$. If the distances between consecutive vertices on these lines were not constant, then the preimage of C_H would be broken lines and hence they would be longer than one consisting only of geodesics. Thus, the edges lying along C_V have the same length and so do the edges along C_H similarly. This implies that any sequence of minimum-length embeddings of $Q(p,q,r)$ shrinks no edge to a point. Therefore, such a minimum-length embedding is stable. □

Using the previous arguments on $Q(p,q,r)$, we can discuss minimum-length embeddings on $T(p,q,r)$ as follows:

Theorem 8. *Any embedding of the* 6*-regular triangulation* $T(p,q,r)$ *on a flat torus has a minimum-length embedding and is stable with respect to minimum length.*

Proof. The 6-regular triangulation $T(p,q,r)$ on the torus contains $Q(p,q,r)$ naturally and can be decomposed into three sets of parallel cycles, vertical cycles, horizontal cycles and "slope 1 cycles" consisting of the diagonals. Suppose that we have an embedding of $T(p,q,r)$ on a flat torus. We can deform it so that $Q(p,q,r)$ becomes minimum-length and that the edges lying on the vertical cycles or on the horizontal cycles have the same length, respectively, as shown in the proof of Theorem 6. It is easy to see that not only the vertical and horizontal cycles contained in $Q(p,q,r)$ but also the slope 1 cycles become closed geodesics in such an embedding on the flat torus since they are covered by straight lines on the plane via the projection $q\colon \mathbb{R}^2 \to T^2$. By Lemma 5, such an embedding of $T(p,q,r)$ is minimum-length. Therefore, any embedding of $T(p,q,r)$ is isotopic to a minimum-length embedding.

Now suppose that we have a minimum-length embedding of $T(p,q,r)$ on a flat torus. Since any closed curve cannot be shorter than a geodesic homotopic to it, the vertical, horizontal and slope 1 cycles in the embedding of $T(p,q,r)$ must be all geodesics. As we have seen in the previous proof, the part corresponding to $Q(p,q,r)$ consists of the set of vertical cycles and the set of horizontal cycles and they are placed in parallel in each set. If they were not placed at equal interval, then the faces of $Q(p,q,r)$ would be parallelograms of different shapes and some of slope 1 cycles would be broken lines and not be geodesic, a contradiction. Thus, each of vertical cycles and horizontal cycles are placed at equal intervals and hence the minimum-length embedding of $T(p,q,r)$ has a unique form. Therefore, the limits of any sequence of minimum-length embeddings of $T(p,q,r)$ becomes an embedding isotopic to the unique form and hence any minimum-length embedding of $T(p,q,r)$ is stable. □

When $r = 1$ and $\gcd(p,q) = 1$, both the unique vertical cycle and the unique horizontal cycle in $Q(p,q,r)$ are hamilton cycles and they cross each other transversely at all vertices. Thus, the condition given in Theorem 2 is equivalent to that in Theorem 7. Therefore, Theorem 2 follows. Also Theorem 3 follows from Theorem 8 on $T(p,q,r)$.

5. Further studies

The basic idea given in this paper will work for other surfaces. For example, we can define a *flat Klein bottle* as the quotient space of an action of an isometry group over the Euclidean plane $\mathbb{R}^2$. However, such an isometry group consists of not only parallel shifts and is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Furthermore, the classification of 4-regular quadrangulations and 6-regular triangulations on the Klein bottle is slightly complicated more than those on the torus, as given in [**4, 6, 9, 10**]; they are not of only one type as well as those on the torus. Also when we consider the stableness of minimum-length embeddings on the Klein bottle, we have to discuss "shrinking a Möbius band". Since we need enough long description for minimum-length embeddings on the Klein bottle, we shall prepare another paper [**5**] for them.

If we try to discuss the case of hyperbolic surfaces of negative constant curvature, we will need knowledge on what is called "hyperbolic geometry", where there are infinitely many lines parallel to a given line and passing through a given point. Furthermore, if the metric over a closed surface induces a non-constant curvature, we cannot use standard arguments like the elementary geometry. So we expect someone to list up a kind of "axioms" of the local differential geometry, that is, basic logics that we can use freely within enough small area, not taking care about the metric of a closed surface into account. Using such axioms, we would like to establish the characterization of graphs embedded on a closed surface that have minimum-length embedding in a combinatorial way, like exclusion of local planar parts described in Section 2 for example.

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