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CERTAIN RESULTS ASSOCIATED WITH THE STRONGLY η-CONVEX FUNCTION WITH MODULUS $\mu \geq 0$

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Abstract. We establish four new results for functions whose second derivatives in absolute value are strongly η-convex. Our theorems generalize some results in the literature. By taking different bifunctions $\eta\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with corresponding values of the modulus $\mu \geq 0$, we obtain more interesting integral inequalities. Some new results are also obtained as drop out of our main results.

1. Introduction

Let $J \subset \mathbb{R}$ be a convex set and J° denote the interior of J. A function $H\colon J \to \mathbb{R}$ is said to be convex on J if the succeeding inequality holds

$$H(\tau x + (1-\tau)y) \leq \tau H(x) + (1-\tau)H(y)$$

for all $x, y \in J$ and $\tau \in [0,1]$.

In 2011, Sarikaya and Aktan [**15**] proved the succeeding result for functions whose second derivative in absolute value is convex.

Theorem 1. *Let $J \subset \mathbb{R}$ be an open interval, $\alpha, \beta \in J$ with $\alpha < \beta$, and $H\colon J \to \mathbb{R}$ be a twice differentiable mapping such that H'' is integrable and $0 \leq \theta \leq 1$. If $|H''|$ is convex on $[\alpha,\beta]$, then the following inequalities hold*

$$\left|(\theta-1)H\Big(\frac{\alpha+\beta}{2}\Big) - \theta\frac{H(\alpha)+H(\beta)}{2} + \frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\,d\tau\right|$$
$$\leq \begin{cases} \frac{(\beta-\alpha)^2}{12}\Big[\big(\theta^4+(1+\theta)(1-\theta)^3+\frac{5\theta-3}{4}\big)\,|H''(\alpha)| \\ +\big(\theta^4+(2-\theta)\theta^3+\frac{1-3\theta}{4}\big)\,|H''(\beta)|\Big] & \text{for } 0\leq\theta\leq\frac{1}{2}, \\ \frac{(\beta-\alpha)^2(3\theta-1)}{48}\,[|H''(\alpha)|+|H''(\beta)|] & \text{for } \frac{1}{2}\leq\theta\leq 1. \end{cases}$$

In the same paper [**15**], the authors also proved

Theorem 2. *Let $J\subset\mathbb{R}$ be an open interval, $\alpha,\beta\in J$ with $\alpha<\beta$, and $H\colon J\to\mathbb{R}$ be a twice differentiable mapping such that H'' is integrable and $0\leq\theta\leq 1$. If*

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$|H''|^q$ *is convex on* $[\alpha,\beta]$, $q \geq 1$, *then the following inequalities hold*

$$\left|(\theta-1)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}+\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\,\mathrm{d}\tau\right|$$

$$\leq \begin{cases} \frac{(\beta-\alpha)^2}{2}\left(\frac{\theta^3}{3}+\frac{1-3\theta}{24}\right)^{1-\frac{1}{q}} \\ \times\left\{\left(\left[\frac{\theta^4}{6}+\frac{3-8\theta}{3\times 2^6}\right]|H''(\alpha)|^q+\left[\frac{(2-\theta)\theta^3}{6}+\frac{5-16\theta}{3\times 2^6}\right]|H''(\beta)|^q\right)^{\frac{1}{q}}\right. \\ \left.+\left(\left[\frac{1+\theta}{6}(1-\theta)^3+\frac{48\theta-27}{3\times 2^6}\right]|H''(\alpha)|^q+\left[\frac{\theta^4}{6}+\frac{3-8\theta}{3\times 2^6}\right]|H''(\beta)|^q\right)^{\frac{1}{q}}\right\} \\ \hfill \text{for } 0\leq\theta\leq\frac{1}{2}, \\ \\ \frac{(\beta-\alpha)^2}{2}\left(\frac{3\theta-1}{24}\right)^{1-\frac{1}{q}}\left\{\left(\frac{8\theta-3}{3\times 2^6}|H''(\alpha)|^q+\frac{16\theta-5}{3\times 2^6}|H''(\beta)|^q\right)^{\frac{1}{q}}\right. \\ \left.+\left(\frac{16\theta-5}{3\times 2^6}|H''(\alpha)|^q+\frac{8\theta-3}{3\times 2^6}|H''(\beta)|^q\right)^{\frac{1}{q}}\right\} \qquad \text{for } \frac{1}{2}\leq\theta\leq 1. \end{cases} \tag{1}$$

Later, the class of s-convex functions was introduced (see [**5**], [**11**] and the references therein). For this class of functions, Özdemir et al. [**14**] obtained, among other things, the following result:

Theorem 3. *Let* $J \subset [0,\infty)$, $\alpha,\beta \in J$ *with* $\alpha < \beta$ *and* $H\colon J \to \mathbb{R}$ *be a twice differentiable on* J° *such that* H'' *is integrable. If* $|H''|$ *is* s*-convex in the second sense on* $[\alpha,\beta]$ *for some fixed* $s \in (0,1]$, *then the following inequalities hold*

$$\begin{aligned} &\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\,\mathrm{d}\tau - H\Big(\frac{\alpha+\beta}{2}\Big)\right| \\ &\leq \frac{(\beta-\alpha)^2}{8(s+1)(s+2)(s+3)}\left[|H''(\alpha)|+(s+1)(s+2)\left|H''\Big(\frac{\alpha+\beta}{2}\Big)\right|\right. \\ &\qquad \left.+|H''(\beta)|\right] \\ &\leq \frac{[1+(s+2)2^{1-s}](\beta-\alpha)^2}{8(s+1)(s+2)(s+3)}\left[|H''(\alpha)|+|H''(\beta)|\right]. \end{aligned} \tag{2}$$

Recently, the notion of convexity was further generalized by Awan et al. [**1**] as follows:

Definition 4. A function $H\colon J \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly η-convex function with respect to $\eta\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and modulus $\mu \geq 0$ if

$$H(\tau x+(1-\tau)y) \leq H(y)+\tau\eta(H(x),H(y))-\mu\tau(1-\tau)(x-y)^2$$

for all $x,y \in J$ and $\tau \in [0,1]$.

Example 5. The function $H(x)=x^2$ is strongly η-convex with respect to the function $\eta(x,y)=2x+y$ and modulus $\mu=1$.

Remark 6. By taking the bifunction $\eta(x,y)=x-y$ and $\mu=0$ in the above definition, we recover the classical definition of convexity. For some results around this new class of functions, we invite the interested reader to see the papers [**1**], [**3**], [**4**], [**8**], and the references therein.

Motivated by the above referenced articles, our purpose in this present paper to achieve the following goals, viz:

1. Generalize Theorem 1 (see Remark 12) and provide a result analogous to Theorem 2 for the strongly η-convex function (see Theorem 13).
2. Establish two new results for this class of functions under consideration (see Theorems 8 and 9).

This paper is organized in the following manner: in Section 2, we formulate and prove our main results. Thereafter, we present some corollaries in Section 3.

2. Main Results

We start by stating the following lemma that will be needed in the proof of the first two theorems.

Lemma 7 ([**10**])**.** *Let $J \subset \mathbb{R}$, $H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha,\beta]$ where $\alpha,\beta \in J$ with $\alpha < \beta$. Then, for any $\theta \in [0,1]$, the following equality holds*

$$\begin{aligned}&\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\,\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\\
&=\frac{(\beta-\alpha)^2}{16}\Big[\int_0^1(\tau^2-2\theta\tau)H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\alpha\Big)\mathrm{d}\tau\\
&\quad+\int_0^1(\tau^2-2\theta\tau)H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\beta\Big)\mathrm{d}\tau\Big].\end{aligned}\tag{3}$$

We now state and prove our first result of the paper.

Theorem 8. *Let $J \subset \mathbb{R}$, $H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha,\beta]$, where $\alpha,\beta \in J$ with $\alpha < \beta$. If $|H''|$ is strongly η-convex (with modulus $\mu \geq 0$), then we have the following inequalities*

$$\begin{aligned}&\Big|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\Big|\\
&\leq\frac{(\beta-\alpha)^2}{16}\Big[\frac{8\theta^3-3\theta+1}{3}\Big(|H''(\alpha)|+|H''(\beta)|\Big)\\
&\quad+\frac{32\theta^4-8\theta+3}{12}\Big(\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\alpha)|\Big)+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\beta)|\Big)\Big)\\
&\quad-\mu\frac{(\beta-\alpha)^2(3-10\theta+160\theta^4-192\theta^5)}{120}\Big]\end{aligned}\tag{4}$$

for $0 \le \theta \le \frac{1}{2}$, *and*

$$
\begin{aligned}
&\Big|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\Big| \\
&\le \frac{(\beta-\alpha)^2}{16}\Big[\Big(\theta-\frac{1}{3}\Big)\Big(|H''(\alpha)|+|H''(\beta)|\Big) \\
&\quad+\frac{8\theta-3}{12}\Big(\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\alpha)|\Big)+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\beta)|\Big)\Big) \\
&\quad-\mu\frac{(\beta-\alpha)^2(10\theta-3)}{120}\Big]
\end{aligned} \tag{5}
$$

for $\frac{1}{2} \le \theta \le 1$.

Proof. From Lemma 7, we have

$$
\begin{aligned}
&\Big|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\Big| \\
&\le \frac{(\beta-\alpha)^2}{16}\Big[\int_0^1|(\tau^2-2\theta\tau)|\Big|H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\alpha\Big)\Big|\mathrm{d}\tau \\
&\quad+\int_0^1|(\tau^2-2\theta\tau)|\Big|H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\beta\Big)\Big|\mathrm{d}\tau\Big] \\
&\le \frac{(\beta-\alpha)^2}{16}\Big[\int_0^1|\tau^2-2\theta\tau| \\
&\quad\times\Big(|H''(\alpha)|+\tau\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\alpha)|\Big)-\mu\tau(1-\tau)\Big(\frac{\beta-\alpha}{2}\Big)^2\Big)\mathrm{d}\tau \\
&\quad+\int_0^1|\tau^2-2\theta\tau|\times\Big(|H''(\beta)|+\tau\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\beta)|\Big) \\
&\quad-\mu\tau(1-\tau)\Big(\frac{\beta-\alpha}{2}\Big)^2\Big)\mathrm{d}\tau\Big] \\
&=\frac{(\beta-\alpha)^2}{16}\Big[|H''(\alpha)|\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau \\
&\quad+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\alpha)|\Big)\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau \\
&\quad-\mu\Big(\frac{\beta-\alpha}{2}\Big)^2\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau+|H''(\beta)|\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau \\
&\quad+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\beta)|\Big)\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau \\
&\quad-\mu\Big(\frac{\beta-\alpha}{2}\Big)^2\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\Big]
\end{aligned}
$$

$$= \frac{(\beta-\alpha)^2}{16}\bigg[\Big(|H''(\alpha)|+|H''(\beta)|\Big)\int_0^1 |\tau^2-2\theta\tau|\mathrm{d}\tau$$
$$+\bigg(\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\alpha)|\Big)+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\beta)|\Big)\bigg)$$
$$\times\int_0^1 \tau|\tau^2-2\theta\tau|\mathrm{d}\tau-2\mu\Big(\frac{\beta-\alpha}{2}\Big)^2\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\bigg].$$

That is,

$$\tag{6}\begin{aligned}&\bigg|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\bigg|\\ &\le\frac{(\beta-\alpha)^2}{16}\bigg[\Big(|H''(\alpha)|+|H''(\beta)|\Big)\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau\\ &\quad+\bigg(\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\alpha)|\Big)+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|,|H''(\beta)|\Big)\bigg)\\ &\quad\times\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau\\ &\quad-\mu\frac{(\beta-\alpha)^2}{2}\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\bigg].\end{aligned}$$

Now, for $0\le\theta\le\frac{1}{2}$, we obtain

$$\tag{7}\begin{aligned}\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau&=\int_0^{2\theta}(2\theta\tau-\tau^2)\mathrm{d}\tau+\int_{2\theta}^1(\tau^2-2\theta\tau)\mathrm{d}\tau\\ &=\frac{4\theta^3}{3}+\frac{4\theta^3-3\theta+1}{3}=\frac{8\theta^3-3\theta+1}{3},\end{aligned}$$

$$\tag{8}\begin{aligned}\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau&=\int_0^{2\theta}(2\theta\tau^2-\tau^3)\mathrm{d}\tau+\int_{2\theta}^1(\tau^3-2\theta\tau^2)\mathrm{d}\tau\\ &=\frac{4\theta^4}{3}+\frac{16\theta^4-8\theta+3}{12}=\frac{32\theta^4-8\theta+3}{12},\end{aligned}$$

and

$$\tag{9}\begin{aligned}\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau&=\int_0^{2\theta}\tau(1-\tau)(2\theta\tau-\tau^2)\mathrm{d}\tau+\int_{2\theta}^1\tau(1-\tau)(\tau^2-2\theta\tau)\mathrm{d}\tau\\ &=\frac{20\theta^4-24\theta^5}{15}+\frac{3-10\theta+80\theta^4-96\theta^5}{60}\\ &=\frac{3-10\theta+160\theta^4-192\theta^5}{60}.\end{aligned}$$

Hence, by using (7)–(9), the inequality in (4) follows from (6).

For $\frac{1}{2}\le\theta\le1$, we also have

$$\tag{10}\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau=\int_0^1(2\theta\tau-\tau^2)\mathrm{d}\tau=\theta-\frac{1}{3},$$

$$\int_0^1 \tau|\tau^2 - 2\theta\tau|\mathrm{d}\tau = \int_0^1 (2\theta\tau^2 - \tau^3)\mathrm{d}\tau = \frac{8\theta - 3}{12}, \tag{11}$$

and

$$\int_0^1 \tau(1-\tau)|\tau^2 - 2\theta\tau|\mathrm{d}\tau = \int_0^1 \tau(1-\tau)(2\theta\tau - \tau^2)\mathrm{d}\tau = \frac{10\theta - 3}{60}. \tag{12}$$

Thus, by using (10)–(12), the desired inequality in (5) follows from (6). □

Theorem 9. *Let $J \subset \mathbb{R}, H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha, \beta]$, where $\alpha, \beta \in J$ with $\alpha < \beta$. If $|H''|^q$ is strongly η-convex for $q > 1$ (with modulus $\mu \geq 0$), then we have the following inequalities*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau - (1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big) - \theta\frac{H(\alpha)+H(\beta)}{2}\right| \\
&\leq \frac{(\beta-\alpha)^2}{16}\Big(\frac{8\theta^3 - 3\theta + 1}{3}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{8\theta^3 - 3\theta + 1}{3}|H''(\alpha)|^q \\
&\quad + \frac{32\theta^4 - 8\theta + 3}{12}\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q, |H''(\alpha)|^q\Big) \\
&\quad - \mu\frac{(\beta-\alpha)^2(3 - 10\theta + 160\theta^4 - 192\theta^5)}{240}\Bigg)^{\frac{1}{q}} \\
&\quad + \Bigg(\frac{8\theta^3 - 3\theta + 1}{3}|H''(\beta)|^q + \frac{32\theta^4 - 8\theta + 3}{12}\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q, |H''(\beta)|^q\Big) \\
&\quad - \mu\frac{(\beta-\alpha)^2(3 - 10\theta + 160\theta^4 - 192\theta^5)}{240}\Bigg)^{\frac{1}{q}}\Bigg]
\end{aligned} \tag{13}$$

for $0 \leq \theta \leq \frac{1}{2}$, and

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau - (1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big) - \theta\frac{H(\alpha)+H(\beta)}{2}\right| \\
&\leq \frac{(\beta-\alpha)^2}{16}\Big(\theta - \frac{1}{3}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\Big(\theta - \frac{1}{3}\Big)|H''(\alpha)|^q \\
&\quad + \frac{8\theta - 3}{12}\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q, |H''(\alpha)|^q\Big) - \mu\frac{(\beta-\alpha)^2(10\theta - 3)}{240}\Bigg)^{\frac{1}{q}} \\
&\quad + \Bigg(\Big(\theta - \frac{1}{3}\Big)|H''(\beta)|^q + \frac{8\theta - 3}{12}\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q, |H''(\beta)|^q\Big) \\
&\quad - \mu\frac{(\beta-\alpha)^2(10\theta - 3)}{240}\Bigg)^{\frac{1}{q}}\Bigg]
\end{aligned} \tag{14}$$

for $\frac{1}{2} \leq \theta \leq 1$.

Proof. Using Lemma 7 and the Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le\frac{(\beta-\alpha)^2}{16}\bigg[\int_0^1|(\tau^2-2\theta\tau)|\Big|H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\alpha\Big)\Big|\mathrm{d}\tau\\
&\qquad+\int_0^1|(\tau^2-2\theta\tau)|\Big|H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\beta\Big)\Big|\mathrm{d}\tau\bigg]\\
&\le\frac{(\beta-\alpha)^2}{16}\bigg[\Big(\int_0^1|(\tau^2-2\theta\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\\
&\qquad\times\Big(\int_0^1|(\tau^2-2\theta\tau)|\Big|H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\alpha\Big)\Big|^q\mathrm{d}\tau\Big)^{\frac{1}{q}}\\
&\qquad+\Big(\int_0^1|(\tau^2-2\theta\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\\
&\qquad\Big(\int_0^1|(\tau^2-2\theta\tau)|\Big|H''\Big(\tau\frac{\alpha+\beta}{2}+(1-\tau)\beta\Big)\Big|^q\mathrm{d}\tau\Big)^{\frac{1}{q}}\bigg]\\
&\le\frac{(\beta-\alpha)^2}{16}\Big(\int_0^1|(\tau^2-2\theta\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\\
&\qquad\times\bigg[\bigg(\int_0^1|\tau^2-2\theta\tau|\Big(|H''(\alpha)|^q+\tau\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q,|H''(\alpha)|^q\Big)\\
&\qquad-\mu\tau(1-\tau)\Big(\frac{\beta-\alpha}{2}\Big)^2\Big)\mathrm{d}\tau\bigg)^{\frac{1}{q}}\\
&\qquad+\bigg(\int_0^1|\tau^2-2\theta\tau|\Big(|H''(\beta)|^q+\tau\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q,|H''(\beta)|^q\Big)\\
&\qquad-\mu\tau(1-\tau)\Big(\frac{\beta-\alpha}{2}\Big)^2\Big)\mathrm{d}\tau\bigg)^{\frac{1}{q}}\bigg]\\
&=\frac{(\beta-\alpha)^2}{16}\Big(\int_0^1|(\tau^2-2\theta\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\bigg[\bigg(|H''(\alpha)|^q\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\qquad+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q,|H''(\alpha)|^q\Big)\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\qquad-\mu\Big(\frac{\beta-\alpha}{2}\Big)^2\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\bigg)^{\frac{1}{q}}+\bigg(|H''(\beta)|^q\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\qquad+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q,|H''(\beta)|^q\Big)\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\qquad-\mu\Big(\frac{\beta-\alpha}{2}\Big)^2\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\bigg)^{\frac{1}{q}}\bigg].
\end{aligned}$$

This implies that

$$
\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le \frac{(\beta-\alpha)^2}{16}\Big(\int_0^1|(\tau^2-2\theta\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\bigg[\Big(|H''(\alpha)|^q\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\quad+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q,|H''(\alpha)|^q\Big)\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\quad-\mu\frac{(\beta-\alpha)^2}{4}\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\Big)^{\frac{1}{q}}\\
&\quad+\Big(|H''(\beta)|^q\int_0^1|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\quad+\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q,|H''(\beta)|^q\Big)\int_0^1\tau|\tau^2-2\theta\tau|\mathrm{d}\tau\\
&\quad-\mu\frac{(\beta-\alpha)^2}{4}\int_0^1\tau(1-\tau)|\tau^2-2\theta\tau|\mathrm{d}\tau\Big)^{\frac{1}{q}}\bigg].
\end{aligned}\tag{15}
$$

Hence, by using (7)–(9) and (10)–(12), respectively, the inequalities in (13)) and (14) follow from (15). ∎

For the next two theorems, we shall be using the succeeding lemma.

Lemma 10 ([**15**]). *Let $J\subset\mathbb{R}$, $H\colon J\to\mathbb{R}$ be a twice differentiable function on J° such that $H''\in L_1[\alpha,\beta]$, where $\alpha,\beta\in J$ with $\alpha<\beta$. Then, for any $\theta\in[0,1]$, the following equality holds*

$$
\begin{aligned}
&\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\\
&=(\beta-\alpha)^2\int_0^1 k(\tau)H''\Big(\tau\alpha+(1-\tau)\beta\Big)\mathrm{d}\tau,
\end{aligned}\tag{16}
$$

where

$$
k(\tau)=\begin{cases}\frac{1}{2}\tau(\tau-\theta), & 0\le\tau\le\frac{1}{2},\\ \frac{1}{2}(1-\tau)(1-\theta-\tau), & \frac{1}{2}\le\tau\le1.\end{cases}
$$

Theorem 11. *Let $J\subset\mathbb{R}, H\colon J\to\mathbb{R}$ be a twice differentiable function on J° such that $H''\in L_1[\alpha,\beta]$, where $\alpha,\beta\in J$ with $\alpha<\beta$. If $|H''|$ is strongly η-convex (with modulus $\mu\ge0$), then we have the following inequalities*

$$
\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le(\beta-\alpha)^2\bigg[\frac{8\theta^3-3\theta+1}{24}|H''(\beta)|+\frac{32\theta^3-12\theta+4}{192}\eta\Big(|H''(\alpha)|,|H''(\beta)|\Big)\\
&\quad-\mu\frac{(\beta-\alpha)^2(9-25\theta+160\theta^4-96\theta^5)}{960}\bigg]
\end{aligned}\tag{17}
$$

for $0 \leq \theta \leq \frac{1}{2}$, *and*

$$\begin{aligned} &\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right| \\ &\leq(\beta-\alpha)^2\bigg[\frac{3\theta-1}{24}|H''(\beta)|+\frac{12\theta-4}{192}\eta\Big(|H''(\alpha)|,|H''(\beta)|\Big) \\ &\quad-\mu\frac{(\beta-\alpha)^2(25\theta-9)}{960}\bigg] \end{aligned} \tag{18}$$

for $\frac{1}{2} \leq \theta \leq 1$.

Proof. Using Lemma 10, we have

$$\begin{aligned} &\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right| \\ &\leq(\beta-\alpha)^2\int_0^1|k(\tau)|\Big|H''\Big(ta+(1-\tau)\beta\Big)\Big|\mathrm{d}\tau \\ &\leq(\beta-\alpha)^2\int_0^1|k(\tau)|\Big(|H''(\beta)|+\tau\eta\Big(|H''(\alpha)|,|H''(\beta)|\Big) \\ &\quad-\mu\tau(1-\tau)(\beta-\alpha)^2\Big)\mathrm{d}\tau \\ &\leq(\beta-\alpha)^2\bigg[|H''(\beta)|\int_0^1|k(\tau)|\mathrm{d}\tau+\eta\Big(|H''(\alpha)|,|H''(\beta)|\Big)\int_0^1\tau|k(\tau)|\mathrm{d}\tau \\ &\quad-\mu(\beta-\alpha)^2\int_0^1\tau(1-\tau)|k(\tau)|\mathrm{d}\tau\bigg]. \end{aligned}$$

That is,

$$\begin{aligned} &\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right| \\ &\leq(\beta-\alpha)^2\bigg[|H''(\beta)|\int_0^1|k(\tau)|\mathrm{d}\tau+\eta\Big(|H''(\alpha)|,|H''(\beta)|\Big)\int_0^1\tau|k(\tau)|\mathrm{d}\tau \\ &\quad-\mu(\beta-\alpha)^2\int_0^1\tau(1-\tau)|k(\tau)|\mathrm{d}\tau\bigg]. \end{aligned} \tag{19}$$

Now, for $0 \leq \theta \leq \frac{1}{2}$, we have

$$\begin{aligned} 2\int_0^1|k(\tau)|\mathrm{d}\tau&=\int_0^{\frac{1}{2}}|\tau(\tau-\theta)|\mathrm{d}\tau+\int_{\frac{1}{2}}^1|(1-\tau)(1-\theta-\tau)|\mathrm{d}\tau \\ &=\int_0^\theta(\theta\tau-\tau^2)\mathrm{d}\tau+\int_\theta^{\frac{1}{2}}(\tau^2-\theta\tau)\mathrm{d}\tau \\ &\quad+\int_{\frac{1}{2}}^{1-\theta}(1-\tau)(1-\theta-\tau)\mathrm{d}\tau+\int_{1-\theta}^1(1-\tau)(\tau-1+\theta)\mathrm{d}\tau \\ &=\frac{\theta^3}{6}+\frac{4\theta^3-3\theta+1}{24}+\frac{4\theta^3-3\theta+1}{24}+\frac{\theta^3}{6}=\frac{16\theta^3-6\theta+2}{24}. \end{aligned}$$

So,

$$\int_0^1 |k(\tau)|\mathrm{d}\tau = \frac{8\theta^3 - 3\theta + 1}{24}. \tag{20}$$

$$\begin{aligned}
2\int_0^1 \tau|k(\tau)|\mathrm{d}\tau &= \int_0^{\frac{1}{2}} \tau^2|(\tau-\theta)|\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \tau(1-\tau)|(1-\theta-\tau)|\mathrm{d}\tau \\
&= \int_0^{\theta} (\theta\tau^2 - \tau^3)\mathrm{d}\tau + \int_{\theta}^{\frac{1}{2}} (\tau^3 - \theta\tau^2)\mathrm{d}\tau \\
&\quad + \int_{\frac{1}{2}}^{1-\theta} \tau(1-\tau)(1-\theta-\tau)\mathrm{d}\tau + \int_{1-\theta}^1 \tau(\tau-1)(1-\theta-\tau)\mathrm{d}\tau \\
&= \frac{\theta^4}{12} + \frac{16\theta^4 - 8\theta + 3}{192} + \frac{5 - 16\theta + 32\theta^3 - 16\theta^4}{192} + \frac{2\theta^3 - \theta^4}{12} \\
&= \frac{16\theta^4 + 16\theta^4 - 8\theta + 3 + 5 - 16\theta + 32\theta^3 - 16\theta^4 + 32\theta^3 - 16\theta^4}{192} \\
&= \frac{64\theta^3 - 24\theta + 8}{192}.
\end{aligned}$$

Therefore,

$$\int_0^1 \tau|k(\tau)|\mathrm{d}\tau = \frac{32\theta^3 - 12\theta + 4}{192}. \tag{21}$$

(22)

$$\begin{aligned}
2\int_0^1 \tau(1-\tau)|k(\tau)|\mathrm{d}\tau &= \int_0^{\frac{1}{2}} \tau^2(1-\tau)|(\tau-\theta)|\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \tau(1-\tau)^2|(1-\theta-\tau)|\mathrm{d}\tau \\
= \int_0^{\theta} \tau^2(1-\tau)(\theta-\tau)\mathrm{d}\tau &+ \int_{\theta}^{\frac{1}{2}} \tau^2(1-\tau)(\tau-\theta)\mathrm{d}\tau \\
&+ \int_{\frac{1}{2}}^{1-\theta} \tau(1-\tau)^2(1-\theta-\tau)\mathrm{d}\tau + \int_{1-\theta}^1 \tau(1-\tau)^2(\tau-1+\theta)\mathrm{d}\tau \\
= \frac{5\theta^4 - 3\theta^5}{60} &+ \frac{9 - 25\theta + 80\theta^4 - 48\theta^5}{960} + \frac{9 - 25\theta + 80\theta^4 - 48\theta^5}{960} \\
&+ \frac{5\theta^4 - 3\theta^5}{60}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\int_0^1 \tau(1-\tau)|k(\tau)|\mathrm{d}\tau &= \frac{5\theta^4 - 3\theta^5}{60} + \frac{9 - 25\theta + 80\theta^4 - 48\theta^5}{960} \\
&= \frac{80\theta^4 - 48\theta^5 + 9 - 25\theta + 80\theta^4 - 48\theta^5}{960} \\
&= \frac{9 - 25\theta + 160\theta^4 - 96\theta^5}{960}.
\end{aligned} \tag{23}$$

Hence, by using (20)–(23), the inequality in (17) follows from (19).

For $\frac{1}{2} \leq \theta \leq 1$, we have

$$2\int_0^1 |k(\tau)|\mathrm{d}\tau = \int_0^{\frac{1}{2}} |\tau(\tau-\theta)|\mathrm{d}\tau + \int_{\frac{1}{2}}^1 |(1-\tau)(1-\theta-\tau)|\mathrm{d}\tau$$
$$= \int_0^{\frac{1}{2}} \theta\tau - \tau^2\mathrm{d}\tau + \int_{\frac{1}{2}}^1 (1-\tau)(\tau-1+\theta)\mathrm{d}\tau$$
$$= \frac{3\theta-1}{24} + \frac{3\theta-1}{24}.$$

Thus,

$$\int_0^1 |k(\tau)|\mathrm{d}\tau = \frac{3\theta-1}{24}. \tag{24}$$

Also,

$$2\int_0^1 \tau|k(\tau)|\mathrm{d}\tau = \int_0^{\frac{1}{2}} \tau^2|(\tau-\theta)|\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \tau(1-\tau)|(1-\theta-\tau)|\mathrm{d}\tau$$
$$= \int_0^{\frac{1}{2}} \theta\tau^2 - \tau^3\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \tau(1-\tau)(\tau-1+\theta)\mathrm{d}\tau$$
$$= \frac{8\theta-3}{192} + \frac{16\theta-5}{192} = \frac{24\theta-8}{192}.$$

This amounts to

$$\int_0^1 \tau|k(\tau)|\mathrm{d}\tau = \frac{12\theta-4}{192}. \tag{25}$$

$$2\int_0^1 \tau(1-\tau)|k(\tau)|\mathrm{d}\tau = \int_0^{\frac{1}{2}} \tau^2(1-\tau)|(\tau-\theta)|\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \tau(1-\tau)^2|(1-\theta-\tau)|\mathrm{d}\tau$$
$$= \int_0^{\frac{1}{2}} \tau^2(1-\tau)(\theta-\tau)\mathrm{d}\tau + \int_{\frac{1}{2}}^1 \tau(1-\tau)^2(\tau-1+\theta)\mathrm{d}\tau$$
$$= \frac{25\theta-9}{960} + \frac{25\theta-9}{960}$$
$$= 2\frac{25\theta-9}{960},$$

which implies that

$$\int_0^1 \tau(1-\tau)|k(\tau)|\mathrm{d}\tau = \frac{25\theta-9}{960}. \tag{26}$$

Hence, by using (19) and (24)–(26), we obtain the inequality in (18). □

Remark 12. By taking $\mu = 0$ and $\eta(x,y) = x - y$ in Theorem 11, we regain Theorem 1.

Theorem 13. *Let $J \subset \mathbb{R}$, $H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha,\beta]$, where $\alpha,\beta \in J$ with $\alpha < \beta$. If $|H''|^q$ is strongly η-convex* *for $q > 1$, then the following inequalities hold*

$$
\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le(\beta-\alpha)^2\Big(\frac{8\theta^3-3\theta+1}{24}\Big)^{1-\frac{1}{q}}\bigg(\frac{8\theta^3-3\theta+1}{24}|H''(\beta)|^q\\
&\qquad+\frac{32\theta^3-12\theta+4}{192}\eta\Big(|H''(\alpha)|^q,|H''(\beta)|^q\Big)\\
&\qquad-\mu\frac{(\beta-\alpha)^2(9-25\theta+160\theta^4-96\theta^5)}{960}\bigg)^{\frac{1}{q}}
\end{aligned}\tag{27}
$$

for $0\le\theta\le\frac{1}{2}$, and

$$
\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le(\beta-\alpha)^2\Big(\frac{3\theta-1}{24}\Big)^{1-\frac{1}{q}}\bigg(\frac{3\theta-1}{24}|H''(\beta)|^q\\
&\qquad+\frac{12\theta-4}{192}\eta\Big(|H''(\alpha)|^q,|H''(\beta)|^q\Big)-\mu\frac{(\beta-\alpha)^2(25\theta-9)}{960}\bigg)^{\frac{1}{q}}
\end{aligned}\tag{28}
$$

for $\frac{1}{2}\le\theta\le 1$.

Proof. Using Lemma 10 and the Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le(\beta-\alpha)^2\Big(\int_0^1|k(\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\bigg(\int_0^1|k(\tau)|\Big|H''\big(\tau\alpha+(1-\tau)\beta\big)\Big|^q\mathrm{d}\tau\bigg)^{\frac{1}{q}}\\
&\le(\beta-\alpha)^2\Big(\int_0^1|k(\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\times\bigg(\int_0^1|k(\tau)|\Big(|H''(\beta)|^q\\
&\qquad+\tau\eta\Big(|H''(\alpha)|^q,|H''(\beta)|^q\Big)-\mu\tau(1-\tau)(\beta-\alpha)^2\Big)\mathrm{d}\tau\bigg)^{\frac{1}{q}}.
\end{aligned}\tag{29}
$$

That is,

$$
\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau-(1-\theta)H\Big(\frac{\alpha+\beta}{2}\Big)-\theta\frac{H(\alpha)+H(\beta)}{2}\right|\\
&\le(\beta-\alpha)^2\Big(\int_0^1|k(\tau)|\mathrm{d}\tau\Big)^{1-\frac{1}{q}}\bigg(|H''(\beta)|^q\int_0^1|k(\tau)|\mathrm{d}\tau\\
&\quad+\eta\Big(|H''(\alpha)|^q,|H''(\beta)|^q\Big)\int_0^1\tau|k(\tau)|\mathrm{d}\tau-\mu(\beta-\alpha)^2\int_0^1\tau(1-\tau)|k(\tau)|\mathrm{d}\tau\bigg)^{\frac{1}{q}}.
\end{aligned}\tag{30}
$$

Hence, by using (20)–(23) and (24)–(26), respectively, the inequalities in (27) and (28) follow from (30). □

3. Some Corollaries

In this section, we give some drop out of our results.

Proposition 14. *Let $J \subset \mathbb{R}$, $H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha, \beta]$, where $\alpha, \beta \in J$ with $\alpha < \beta$. If $|H''|$ is η-convex, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau - H\Big(\frac{\alpha+\beta}{2}\Big)\right| \\
&\leq \frac{(\beta-\alpha)^2}{16}\left[\frac{1}{3}\Big(|H''(\alpha)| + |H''(\beta)|\Big)\right. \\
&\quad \left. + \frac{1}{4}\bigg(\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|, |H''(\alpha)|\Big) + \eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|, |H''(\beta)|\Big)\bigg)\right].
\end{aligned}$$

Proof. The proof follows by setting $\mu = \theta = 0$ in (4) of Theorem 8. □

Proposition 15. *Let $J \subset \mathbb{R}, H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha, \beta]$, where $\alpha, \beta \in J$ with $\alpha < \beta$. If $|H''|^q$ is η-convex for $q > 1$, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau - \frac{H(\alpha)+H(\beta)}{2}\right| \\
&\leq \frac{(\beta-\alpha)^2}{16}\Big(\frac{2}{3}\Big)^{1-\frac{1}{q}}\left[\bigg(\frac{2}{3}|H''(\alpha)|^q + \frac{5}{12}\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q, |H''(\alpha)|^q\Big)\bigg)^{\frac{1}{q}}\right. \\
&\quad \left. + \bigg(\frac{2}{3}|H''(\beta)|^q + \frac{5}{12}\eta\Big(\Big|H''\Big(\frac{\alpha+\beta}{2}\Big)\Big|^q, |H''(\beta)|^q\Big)\bigg)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The desired inequality is achieved by putting $\mu = 0$ and $\theta = 1$ in (14) of Theorem 9. □

Proposition 16. *Let $J \subset \mathbb{R}, H\colon J \to \mathbb{R}$ be a twice differentiable function on J° such that $H'' \in L_1[\alpha, \beta]$, where $\alpha, \beta \in J$ with $\alpha < \beta$. If $|H''|$ is η-convex, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta H(\tau)\mathrm{d}\tau - \frac{1}{2}H\Big(\frac{\alpha+\beta}{2}\Big) - \frac{H(\alpha)+H(\beta)}{4}\right| \\
&\leq (\beta-\alpha)^2\left[\frac{1}{48}|H''(\beta)| + \frac{1}{96}\eta\Big(|H''(\alpha)|, |H''(\beta)|\Big)\right].
\end{aligned}$$

Proof. We get the intended result by substituting $\mu = 0$ and $\theta = 1/2$ in Theorem 11. □

4. Conclusion

Four theorems concerning functions whose second derivative in absolute value is strongly η-convex with modulus $\mu \geq 0$, are hereby established. By taking different bifunctions η with corresponding μ, we get new inequalities that generalize some of the results in the literature. Therefore we hope that results obtained herein will inspire further work in this direction. For more results around this new class of functions, we invite the interested reader to see references [**6**], [**7**], [**12**], [**13**]. Also, see [**2**], [**9**] papers for related results.

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