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# REMARKS ON THE QUALITATIVE BEHAVIOR OF THE UNDAMPED KLEIN-GORDON EQUATION *

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**Abstract.** We present sufficient conditions on the initial data of an undamped Klein-Gordon equation in bounded domains with homogeneous Dirichlet boundary conditions to guarantee the blow up of weak solutions. Our methodology is extended to a class of evolution equations of second order in time. As an example, we consider a generalized Boussinesq equation. Our result is based on a careful analysis of a differential inequality. We compare our results with the ones in the literature.

**Key words.** Klein-Gordon equation, Blow up, High energies, Abstract wave equation, Generalized Boussinesq equation

**AMS subject classifications.** 35L70, 35B35, 35B40

**1. Functional framework and previous results.** For the Cauchy problem associated to any evolution equation on a Banach space, we have the usual questions in terms on the initial data:

- Existence and uniqueness of solutions.
- Non global existence: maximal time of existence $\equiv T_{MAX} < \infty$.
- Global existence: $T_{MAX} = \infty$.
- In the latter case, the behavior of the solution as time approaches infinity.

Here, we present a short overview paper presenting recent advances, published in [1, 2], on the non global existence of solutions corresponding to a non-linear Klein-Gordon equation and to abstract wave equations, in particular to a generalized Boussinesq equation.

We shall first consider the following problem for a Klein-Gordon equation

$$(\mathbf{P}) \begin{cases} u_{tt}(x,t) - \Delta u(x,t) + m^2 u(x,t) = f(u(x,t)), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

where $m \neq 0$ is a real constant, which is assumed to be equal to one without loss of generality, and $\Omega \subset R^n$ is a bounded and open set with sufficiently smooth boundary. We assume that the source term $f$, is locally Lipschitz continuous and satisfies

$$|f(s)| \leq \mu |s|^{r-1}, \;\; sf(s) - rF(s) \geq 0, \; \forall s \in R,$$

where $F(s) \equiv \int_0^s f(t)dt$, and $r > 2, \mu > 0$, are constants. For this problem, Ball [3, 4] proved the following theorem about existence, uniqueness and continuation of weak solutions.

THEOREM 1.1. *Assume that $r \leq 2(n-1)/(n-2)$ if $n \geq 3$. For every initial data $(u_0, v_0) \in H \equiv H_0^1(\Omega) \times L_2(\Omega)$, there exists a unique (local) weak solution*

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$(u_0, v_0) \mapsto (u(t), v(t)),\ v(t) \equiv \frac{d}{dt}u(t)$, *of problem* **(P)**, *that is*

$$\frac{d}{dt}(v(t), w)_2 + (\nabla u(t), \nabla w)_2 + (u(t), w)_2 = (f(u(t)), w)_2,$$

*a. e. in (0,T) and for every* $w \in H_0^1(\Omega)$, *such that* $(u, v) \in C([0, T); H)$. *Here,* $(\cdot, \cdot)_2$ *denotes the inner product in* $L_2(\Omega)$. *Furthermore, the following energy equation holds*

$$E(u(t_0), v(t_0)) = E(u(t), v(t)) \equiv \frac{1}{2}\|v(t)\|_2^2 + J(u(t)), \quad \forall t \geq t_0 \geq 0,$$
$$J(u(t)) \equiv \frac{1}{2}\left(\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2\right) - \int_\Omega F(u(t))dx.$$

*Here,* $\|\cdot\|_q$ *is the norm in* $L_q(\Omega)$. *Finally, if the maximal time of existence* $T_{MAX} < \infty$, *then the solution blows up in finite time. That is,*

$$\lim_{t \nearrow T_{MAX}} \|(u(t), v(t))\|_H^2 \equiv \lim_{t \nearrow T_{MAX}} \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|v(t)\|_2^2 = \infty.$$

*Moreover, by the energy equation,*

$$\lim_{t \nearrow T_{MAX}} \|u(t)\|_r = \infty.$$

REMARK 1. *Problem* **(P)** *is invariant if we reverse the time direction:* $t \mapsto -t$. *The solution backwards* $(u(t), v(t)), t < 0$, *with initial conditions* $(u_0, v_0)$ *corresponds to the solution forwards* $(u(-t), -v(-t)), -t > 0$ *with initial conditions* $(u_0, -v_0)$. *Then, the local existence and uniqueness Theorem 1.1 holds backwards and the results presented in this work for positive times have the corresponding for backwards solutions.*

If the solution $u$ is independent of time, it is called an equilibrium and satisfies

$$(\nabla u, \nabla w)_2 + (u, w)_2 = (f(u), w)_2,$$

for every $w \in H_0^1(\Omega)$. In particular, for $w = u$,

$$I(u) \equiv \|\nabla u\|_2^2 + \|u\|_2^2 - (f(u), u)_2 = 0.$$

We notice that $u = 0$ is an equilibrium. The set of equilibria $u \neq 0$, with minimal energy is called the ground state, and the corresponding value of the energy is positive and denoted by $d$. This number is the mountain pass level of the functional $J$, see [5]. For initial energies $E(u_0, v_0) < d$, a characterization of the qualitative properties of the solutions in terms of the sign of $I(u_0)$ has been proved in [6] by the potential well method. Indeed, if $I(u_0) \geq 0$, respectively $I(u(t_0)) < 0$, the corresponding solution is global and uniformly bounded in $H$, respectively the solution blows up in finite time. For high values of the initial energy the sign of $I(u_0)$ is not sufficient in order to prove qualitative properties of the solution. Certainly, for $E(u_0, v_0) > d$ and for a source term of the form $f(u) \equiv |u|^{r-2}u, r > 2$, in [7] is proved that the solution blows up if $I(u_0) < 0$, $(u_0, v_0)_2 \geq 0$, and $\|u_0\|_2 \geq \sup\{\|u\|_2 : I(u)) = 0, J(u) \leq E(u_0, v_0)\}$. For $E(u_0, v_0) = d$ and $f(u) \equiv |u|^{r-2}u$, in [8] the following is proved: (i) the solution blows up if $I(u_0) < 0$ and $(u_0, v_0)_2 \geq 0$ and (ii) the solution is global and uniformly bounded in $H$ if $I(u_0) > 0$. Recently, several works have proved blow up of solutions with one o several source terms of the form $|u|^{r-2}u$, under sufficient conditions that involve upper bounds on the initial energy as follows.

THEOREM 1.2. *For every solution of problem* **(P)** *with*

$$(u_0, v_0)_2 > 0, \|u_0\|_2 > 0,$$

*the solution blows up in finite time if one of the following holds:*

$$(Wang\ [9]). \qquad E(u_0, v_0) \leq \frac{r-2}{2r}\|u_0\|_2^2, \;\; I(u_0) < 0, \tag{1.1}$$

$$(Korpusov\ [10]). \qquad E(u_0, v_0) < \frac{1}{2}P(u_0, v_0), \tag{1.2}$$

$$(Kutev, et\ al.\ [11]). \qquad E(u_0, v_0) < \frac{r-2}{2r}\|u_0\|_2^2 + \frac{1}{2}P(u_0, v_0), \tag{1.3}$$

$$(Dimova, et\ al.\ [12]). \qquad E(u_0, v_0) \leq \frac{r-2}{2r}\|u_0\|_2^2 + \frac{1}{2}P(u_0, v_0)$$

$$+ \frac{\|u_0\|_2^2}{r}\left[1 - \left\{1 + \frac{P(u_0, v_0)}{\|u_0\|_2^2}\right\}^{-(\frac{r-2}{2})}\right], \tag{1.4}$$

*where* $P(u_0, v_0) \equiv \frac{|(v_0,u_0)_2|^2}{\|u_0\|_2^2}$.

REMARK 2. *For the proof of anyone of the items in this theorem, some differential inequality is employed to prove that the solution only exists up to a finite time:* $T < \infty$. *The estimate of the maximal time of existence by this means is not always optimal, that is,* $T > T_{MAX}$. *See [13, 3, 4] for more discussion. The technique described above belongs to the so called functional method. That is, some functional in terms of a norm of the solution well defined in the sense of Theorem 1.1, satisfies a differential inequality that necessarily implies that such norm blows up in finite time. Consequently, the solution can not be global. This method has been used for many authors to show nonexistence of solutions, see for instance [14] for an early reference where a concavity argument is used.*

REMARK 3. *In [11] is proved that any one of the sufficient conditions (1.1) or (1.2) imply (1.3), and that the contrary is not true. We notice that (1.3) implies (1.4) but the opposite does not occur. In next section we easily show this implication and by this means we propose a new condition to get blow up of the solution in finite time.*

**2. Main result.** In this section we consider solutions with any positive value of the initial energy, in particular with $E(u_0, v_0) \geq d$. The understanding of the complete dynamics in this case is an open question and very much complicated. Here, we limit ourselves to study blow up and give sufficient conditions on $(u_0, v_0) \in H$ and $E(u_0, v_0) > 0$.

We first notice that the right hand-side of (1.3) and (1.4) have the following form

$$\eta_q(u, v) \equiv \frac{1}{2}\Phi(u, v) - \frac{1}{r}\Psi(u)\left(\frac{\Psi(u)}{\Phi(u, v)}\right)^q,$$

where $q \geq 0$ and

$$\Phi(u, v) \equiv \Psi(u) + P(u, v), \qquad \Psi(u) \equiv \|u\|_2^2, \qquad P(u, v) \equiv \frac{|(v, u)_2|^2}{\|u\|_2^2}.$$

The functional $P$ comes from the orthogonal decomposition of the velocity, introduced in [11]. That is,

$$v = \frac{(v, u)_2}{\|u\|_2^2}u + h, \qquad \|v\|_2^2 = \|h\|_2^2 + P(u, v),$$

where $(u,h)_2 = 0$. Indeed, the one in (1.4) is equal to $\eta_{\frac{r-2}{2}}(u_0, v_0)$. We notice that the function $q \mapsto \eta_q(u_0, v_0)$, is strictly increasing for $q \geq 0$, whenever $P(u_0, v_0) > 0$, and that $\eta_0(u_0, v_0)$ is equal to the right-hand side of condition (1.3). Hence, (1.4) is implied by (1.3) but not the contrary. Now, we define a strictly decreasing function $\lambda \mapsto \mu_\lambda(u,v)$, for $0 < \lambda < 1$, by

$$\mu_\lambda(u,v) \equiv \frac{1}{2}\Phi(u,v) - \frac{1}{r}\Psi(u,v)\left(\frac{r-2}{r-2\lambda}\,\frac{\Psi(u)}{\Phi(u,v)}\right)^{\frac{r-2}{2}},$$

with the property that $\mu_\lambda(u,v) \to \eta_{\frac{r-2}{2}}(u,v)$ if $\lambda \to 1$. That is, $\eta_{\frac{r-2}{2}}(u,v) < \mu_\lambda(u,v)$.

Next we present our blow up result whose proof is based on a careful analysis of a differential inequality satisfied by $\Psi(u)$, and where $P(u,v)$ and $\mu_\lambda(u,v)$ are essential to improve the previous results given in Theorem 1.2.

THEOREM 2.1. *(Esquivel-Avila [1]). Consider any solution of problem* **(P)**. *Assume that*

$$\|u_0\|_2 > 0,\ (u_0, v_0)_2 > 0. \tag{2.1}$$

*Hence, $P(u_0, v_0) > 0$, and there exists a nonempty interval*

$$\mathcal{I}_{P(u_0,v_0)} \equiv \left(\alpha_{P(u_0,v_0)}, \beta_{P(u_0,v_0)}\right) \subset \left(0, \frac{1}{2}\Phi(u_0, v_0)\right),$$

*such that if $E(u_0, v_0) \in \mathcal{I}_{P(u_0,v_0)}$, then the solution blows up in finite time. Moreover, for fixed $\Psi(u_0)$,*

$$\lim_{P(u_0,v_0)\to\infty}\left|\beta_{P(u_0,v_0)} - \frac{1}{2}\Phi(u_0, v_0)\right| = 0 = \lim_{P(u_0,v_0)\to\infty}\alpha_{P(u_0,v_0)}.$$

REMARK 4. *We observe that $\beta_{P(u_0,v_0)} = \mu_{\lambda^*}(u_0, v_0)$, where $\lambda^* \in (0,1)$, is uniquely defined by*

$$\lambda^* \equiv \left(\frac{\Psi(u_0)}{\Phi(u_0, v_0)}\right)^{\frac{r}{2}}\left(\frac{r-2}{r-2\lambda^*}\right)^{\frac{r-2}{2}}.$$

*Hence, Theorem 2.1 improves the condition on the upper bound of the initial energy given in Theorem 1.2, (1.1)-(1.4).*

*If $\mu_{\lambda^*}(u_0, v_0) \leq E(u_0, v_0) \leq \mu_\lambda(u_0, v_0)$, for $\lambda \leq \lambda^*$, the qualitative behavior of the solution is unknown. However, given any positive value of the initial energy, if (2.1) holds and $P(u_0, v_0)$ is large enough, then we can always have that $E(u_0, v_0) \in \mathcal{I}_{P(u_0,v_0)}$. Consequently, the corresponding solution blows up in finite time.*

REMARK 5. *For small energies, the result in [6] characterizes blow up of any solution under the condition $I(u_0) < 0$. For high energies, blow up follows from $I(u_0) < 0$ and additional conditions on the initial data, see [7]-[9]. Under the hypotheses of Theorem 2.1, $I(u_0) < 0$ follows if $P(u_0, v_0) > 0$ is sufficiently large, see [1].*

**3. Evolution equations of second order in time.** We extend Theorem 2.1 to the following class of abstract wave equations:

$$(\mathbf{P_M})\begin{cases} Mu_{tt}(t) + Au(t) = \mathcal{F}(u(t)), & t \in (0,T), \\ u(0) = u_0,\ u_t(0) = v_0, \end{cases}$$

where $M : H_{\mathcal{M}} \to H'_{\mathcal{M}}$ and $A : V \to V'$, are linear, positive and symmetric operators, and $V \subset H_{\mathcal{M}} \subset H$ are linear subspaces of the Hilbert space $H$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and $H = H' \subset H'_{\mathcal{M}} \subset V'$ are the dual spaces. Hence, we define the bilinear forms and corresponding inner products and norms

$$\begin{aligned} &\mathcal{M} : H_{\mathcal{M}} \times H_{\mathcal{M}} \to R, \quad \mathcal{M}(u,w) \equiv (Mu,w)_{H_{\mathcal{M}} \times H'_{\mathcal{M}}}, \\ &(u,w)_{\mathcal{M}} \equiv \mathcal{M}(u,w), \quad \|u\|^2_{\mathcal{M}} \equiv (u,u)_{\mathcal{M}}, \quad \forall u,w \in H_{\mathcal{M}} \end{aligned}$$

and

$$\begin{aligned} &\mathcal{A} : V \times V \to R, \quad \mathcal{A}(u,w) \equiv (Au,w)_{V \times V'}, \\ &(u,w)_V \equiv \mathcal{A}(u,w), \quad \|u\|^2_V \equiv (u,w)_V, \quad \forall u,w \in V. \end{aligned}$$

We assume that there exists some constant $c > 0$, such that

$$\|u\|^2_V \geq c\|u\|^2_{\mathcal{M}},\ \forall u \in V. \tag{3.1}$$

Also, we assume that the nonlinear term $\mathcal{F} : V \subset H \to H$, is a potential operator with potential $\mathcal{G} : V \to R$, and

$$\mathcal{F}(0) = 0,\ \ (\mathcal{F}(u),u) - r\mathcal{G}(u) \geq 0,\ \forall u \in V, \tag{3.2}$$

where $r > 2$ is a constant.

We consider solutions in the following functional framework.

For every initial data $(u_0, v_0) \in \mathcal{H} \equiv V \times H_{\mathcal{M}}$, there exists $T > 0$, and a unique local solution $(u_0,v_0) \mapsto (u,v) \in C([0,T);\mathcal{H}),\ v(t) \equiv \frac{d}{dt}u(t)$, of the problem $(\mathbf{P_M})$ in the following sense

$$\frac{d}{dt}\mathcal{M}(v(t),w) + \mathcal{A}(u(t),w) = (\mathcal{F}(u(t)),w),$$

a. e. in $(0,T)$ and for every $w \in V$. Furthermore, the following energy equation holds

$$\begin{aligned} &E(u(t_0),v(t_0)) = E(u(t),v(t)) \equiv \frac{1}{2}\|v(t)\|^2_{\mathcal{M}} + J(u(t)),\ \ t \in [t_0,T),\ t_0 \geq 0, \\ &\qquad J(u(t)) \equiv \frac{1}{2}\|u(t)\|^2_V - \mathcal{G}(u(t)). \end{aligned}$$

We define

$$\Phi(u,v) \equiv c\Psi(u) + P_{\mathcal{M}}(u,v),\ \Psi(u) \equiv \|u\|^2_{\mathcal{M}},\ P_{\mathcal{M}}(u,v) \equiv \frac{|\mathcal{M}(v,u)|^2}{\|u\|^2_{\mathcal{M}}}.$$

Then, we have the following result.

THEOREM 3.1. *(Esquivel-Avila [2]). Consider any solution of problem* $(\mathbf{P_M})$. *Assume that*

$$\|u_0\|_{\mathcal{M}} > 0, \quad \mathcal{M}(u_0,v_0) > 0. \tag{3.3}$$

*Then, there exists a nonempty open interval*

$$\mathcal{I}_{P_{\mathcal{M}}(u_0,v_0)} \equiv \left(\alpha_{P_{\mathcal{M}}(u_0,v_0)}, \beta_{P_{\mathcal{M}}(u_0,v_0)}\right) \subset \left(0, \frac{1}{2}\Phi(u_0,v_0)\right),$$

*such that if* $E(u_0,v_0) \in \mathcal{I}_{P_{\mathcal{M}}(u_0,v_0)}$*, then the solution is not global. Moreover, for fixed* $\Psi(u_0)$*,*

$$\lim_{P_{\mathcal{M}}(u_0,v_0)\to\infty} \left|\beta_{P_{\mathcal{M}}(u_0,v_0)} - \frac{1}{2}\Phi(u_0,v_0)\right| = 0 = \lim_{P_{\mathcal{M}}(u_0,v_0)\to\infty} \alpha_{P_{\mathcal{M}}(u_0,v_0)}.$$

*Here,* $\beta_{P_{\mathcal{M}}(u_0,v_0)} = \mu_{\lambda^*}(u_0,v_0)$*, where* $\lambda^* \in (0,1)$ *is uniquely defined by*

$$\lambda^* \equiv \left(\frac{c\Psi(u_0)}{\Phi(u_0,v_0)}\right)^{\frac{r}{2}} \left(\frac{r-2}{r-2\lambda^*}\right)^{\frac{r-2}{2}},$$

*and*

$$\mu_\lambda(u_0,v_0) \equiv \frac{1}{2}\Phi(u_0,v_0) - \frac{c}{r}\Psi(u_0,v_0)\left(\frac{r-2}{r-2\lambda}\,\frac{c\Psi(u_0)}{\Phi(u_0,v_0)}\right)^{\frac{r-2}{2}}.$$

*Furthermore, given any positive value of the initial energy we can always find initial data satisfying (3.3) with* $P_{\mathcal{M}}(u_0,v_0)$ *sufficiently large so that* $E(u_0,v_0) \in \mathcal{I}_{P_{\mathcal{M}}(u_0,v_0)}$ *and hence the corresponding solution exists only up to a finite time.*

We can apply Theorem 3.1 to several problems, in particular here we present the following Cauchy problem associated to a generalized Boussinesq equation.

$$(\mathbf{P_B})\begin{cases} u_{tt}(x,t) - \beta_1\Delta u(x,t) - \beta_2\Delta u_{tt}(x,t) + \beta_3\Delta^2 u(x,t) & \\ \qquad\qquad +mu(x,t) + \Delta\mathcal{F}(u(x,t)) = 0, & (x,t) \in R^n \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), & x \in R^n, \end{cases}$$

where $\beta_i > 0,\ i = 1,2,3,\ m > 0$ are constants and the source term, that satisfies (3.2), is

$$\mathcal{F}(u) \equiv \mu|u|^{r-2}u,\ \mu > 0, r > 2.$$

Applying $(-\Delta)^{-1}$ to the equation above, we obtain the form of the problem $(\mathbf{P_M})$, where we identify the operators

$$Mu = ((-\Delta)^{-1} + \beta_2 I_d)u, \quad Au = (-\beta_3\Delta + m(-\Delta)^{-1} + \beta_1 I_d)u,$$

and the spaces

$$H = L_2(R^n), \quad H_{\mathcal{M}} = \{u \in L_2(R^n) : (-\Delta)^{-\frac{1}{2}}u \in L_2(R^n)\},$$

and

$$V = \{u \in H^1(R^n) : (-\Delta)^{-\frac{1}{2}}u \in L_2(R^n)\}.$$

If

$$(u,w)_* \equiv ((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}w)_2, \quad \|u\|_*^2 \equiv (u,u)_*,$$

then the bilinear forms, inner products and norms are

$$(u,w)_{\mathcal{M}} \equiv \mathcal{M}(u,w) \equiv (u,w)_* + \beta_2(u,w)_2, \quad \|u\|_{\mathcal{M}}^2 \equiv (u,u)_{\mathcal{M}},$$

and

$$(u,w)_V \equiv \mathcal{A}(u,w) \equiv \beta_3(\nabla u, \nabla w)_2 + m(u,w)_* + \beta_1(u,w)_2, \quad \|u\|_V^2 \equiv (u,u)_V.$$

Hence, (3.1) holds with $c \equiv \min\{m, \frac{\beta_1}{\beta_2}\}$. Fortunately, there exists an existence and uniqueness result in our functional framework and nonexistence of global solutions is due to blow up, see for instance [15, 16]. Then, by Theorem 3.1, if the initial data satisfy

$$\|u_0\|_*^2 + \beta_2\|u_0\|_2^2 > 0, \ (u_0,v_0)_* + \beta_2(u_0,v_0)_2 > 0, \tag{3.4}$$

and the initial energy is such that $E(u_0,v_0) \in \mathcal{I}_{P_{\mathcal{M}}(u_0,v_0)}$, where

$$E(u,v) \equiv \frac{1}{2}\left(\|v\|_*^2 + \beta_2\|v\|_2^2 + \beta_3\|\nabla u\|_2^2 + m\|u\|_*^2 + \beta_1\|u\|_2^2\right) - \frac{\mu}{r}\|u\|_r^r,$$

then the solution blows up in finite time in the norm of $\mathcal{H}$ and, by the energy equation, also in the $L_r(R^n)$ norm. This result improves the ones known in the literature in the following sense. In [17, 18] blow up is proved by means of the analysis of a differential inequality and by the construction of invariant sets, if (3.4) holds and the initial energy is such that

$$E(u_0,v_0) \le \eta_0(u_0,v_0) \equiv \frac{r-2}{2r}c\left(\|u_0\|_*^2 + \beta_2\|u_0\|_2^2\right) + \frac{1}{2}\frac{|(u_0,v_0)_* + \beta_2(u_0,v_0)_2|^2}{\|u_0\|_*^2 + \beta_2\|u_0\|_2^2}.$$

We notice that $\eta_0(u_0,v_0) = \frac{1}{2}\Phi(u_0,v_0) - \frac{c}{r}\Psi(u_0,v_0) \in \mathcal{I}_{P_{\mathcal{M}}(u_0,v_0)}$. Then, Theorem 3.1 agrees with the result in [17, 18] and states that blow up occur even for larger initial energies, that is, if

$$\eta_0(u_0,v_0) < E(u_0,v_0) < \mu_{\lambda^*}(u_0,v_0).$$

Moreover, given any positive value of the initial energy there exist initial data satisfying (3.4) and with

$$P_{\mathcal{M}}(u_0,v_0) \equiv \frac{|(u_0,v_0)_* + \beta_2(u_0,v_0)_2|^2}{\|u_0\|_*^2 + \beta_2\|u_0\|_2^2},$$

sufficiently large, so that $E(u_0,v_0) \in \mathcal{I}_{P_{\mathcal{M}}(u_0,v_0)}$ holds and consequently the corresponding solution blows up in finite time.

REMARK 6. *For each concrete example of* $(\mathbf{P_M})$, *if the potential well method is applicable as it is in* $(\mathbf{P})$, *then there are conditions to get blow up when* $E(u_0,v_0) < d$. *Theorem 3.1 gives sufficient conditions for* $\alpha_{P_{\mathcal{M}}(u_0,v_0)} < E(u_0,v_0) < \beta_{P_{\mathcal{M}}(u_0,v_0)}$. *In case that* $E(u_0,v_0) \le \alpha_{P_{\mathcal{M}}(u_0,v_0)}$ *the blow up problem is resolved as follows. (i) If* $E(u_0,v_0) < \min\{\alpha_{P_{\mathcal{M}}(u_0,v_0)}, d\}$, *by the potential well method. (ii) If* $d \le E(u_0,v_0) \le \alpha_{P_{\mathcal{M}}(u_0,v_0)}$, *by the techniques in [17, 18].*

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