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BOUNDEDNESS IN A FULLY PARABOLIC CHEMOTAXIS SYSTEM WITH SIGNAL-DEPENDENT SENSITIVITY AND LOGISTIC TERM*

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Abstract. This paper deals with the chemotaxis system with signal-dependent sensitivity and logistic term

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \mu u(1-u), \\ v_t &= \Delta v + u - v \end{aligned}$$

in $\Omega \times (0,\infty)$, where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary, $\mu > 0$ is a constant and χ is a function generalizing

$$\chi(s) = \frac{K}{(1+s)^2} \quad (K > 0,\ s > 0).$$

In the case that $\mu = 0$ global existence and boundedness were established under some conditions ([14]); however, conditions for global existence and boundedness in the above system have not been studied. The purpose of this paper is to construct conditions for global existence and boundedness in the above system.

Key words. chemotaxis; signal-dependent sensitivity; logistic term; global existence.

AMS subject classifications. Primary: 35K51; Secondary: 35A01, 92C17.

1. Introduction.

Chemotaxis is the property such that species move towards higher concentration of a chemical substance when they plunge into hunger. The following problem which describes the movement of species with chemotaxis

$$u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \mu u(1-u), \quad v_t = \Delta v + u - v,$$

where χ is a function and $\mu \geq 0$ is a constant, is called a *Keller–Segel system* or a *chemotaxis system*, and is studied intensively. The function χ appearing in the above problem is called *signal-dependent sensitivity*, and examples of this function χ are as follows: $\chi(s) = K$ (constant), $\chi(s) = \frac{K}{s}$ (singular), $\chi(s) = \frac{K}{(1+s)^2}$ (regular) for $s > 0$ with some constant $K > 0$. Previous works which deal with the constant sensitivity can be found in [2, 7, 8, 15, 18, 19]; the singular sensitivity is treated in [3, 5, 6, 9, 10]; we can find works related to the regular sensitivity in [5, 6, 11, 13, 14, 16, 17, 20]; variation of chemotaxis systems are in [1]. Here we focus on the case that χ is a function generalizing the regular sensitivity:

$$\chi(s) \leq \frac{K}{(a+s)^k} \quad (s > 0) \tag{1.1}$$

with some constants $a \geq 0$, $k > 1$ and $K > 0$. In a mathematical view, one of difficulties caused by the sensitivity function χ is to deal with the additional term $u\chi'(v)|\nabla v|^2$ which does not appear in the case that χ is a constant. In the case that

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$\mu = 0$, by using an energy estimate to overcome the difficulties of the sensitivity function, under the condition that χ fulfils (1.1) with some constants $a \geq 0$, $k > 1$ and $K > 0$ satisfying

$$K < k(a+\eta)^{k-1}\sqrt{\frac{2}{n}}, \tag{1.2}$$

where η is a constant defined as

$$\eta := \sup_{\tau>0}\left(\min\left\{e^{-2\tau}\min_{x\in\overline{\Omega}} v_0(x),\ c_0\|u_0\|_{L^1(\Omega)}(1-e^{-\tau})\right\}\right) \geq 0$$

(see [4, 14]), global existence and boundedness were established ([14]). Recently, Fujie–Senba [5, 6] established conditions for global existence and boundedness in a problem generalizing the chemotaxis system with $\mu = 0$. More related works which deal with a two-species chemotaxis system with competitive kinetics can be found in [11, 12, 13, 16, 17, 20]; global existence and boundedness are in [11, 13, 16, 17, 20]; asymptotic behavior is shown in [11, 12].

In summary, the conditions (1.1)–(1.2) lead to global existence and boundedness in the chemotaxis system with $\mu = 0$. However, the case that $\mu > 0$ has not been studied. The purpose of this work is to derive conditions for global existence and boundedness in the chemotaxis system.

In this paper we consider the chemotaxis system with signal-dependent sensitivity and logistic term

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\chi(v)\nabla v) + \mu u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v + u - v, & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$; $\mu > 0$ is a constant; the initial data u_0 and v_0 are assumed to be nonnegative functions. The unknown function $u(x,t)$ represents the population density of species and $v(x,t)$ shows the concentration of the substance at place x and time t. As to the sensitivity function χ, we are interested in functions generalizing

$$\chi(s) = \frac{K}{(1+s)^2} \quad (s>0),$$

where $K > 0$ is a constant.

In order to achieve our purpose we shall suppose that χ satisfies that

$$\chi \in C^{1+\lambda}((0,\infty)) \quad \text{and} \quad 0 \leq \chi(s) \leq \frac{K}{(a+s)^k} \quad (s>0) \tag{1.4}$$

with some $\lambda > 0$, $k > 1$, $a > 0$ and $K > 0$ fulfiling

$$K < ka^{k-1}\sqrt{\frac{2}{n}}. \tag{1.5}$$

Now the main result reads as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with smooth boundary and let $\mu > 0$. Assume that χ satisfies* (1.4) *with some $\lambda > 0$, $k > 1$, $a > 0$, $K > 0$ fulfiling* (1.5)*. Then for any u_0, v_0 satisfying*

$$0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\} \quad \text{and} \quad 0 \leq v_0 \in W^{1,q}(\Omega) \setminus \{0\} \tag{1.6}$$

with some $q > n$, there exists an exactly one pair (u, v) of positive functions

$$u,\ v \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$$

which solves (1.3)*. Moreover, the solution (u, v) is uniformly bounded, i.e., there exists a constant $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C$$

for all $t > 0$.

Here we give one remark: The condition (1.5) is more restricted condition than (1.2) except the case that $\eta = 0$ (which is the case that $\min_{x\in\Omega} v_0(x) = 0$). The reason is that it is difficult to see the uniform-in-time lower estimate for v because of lacking information about the lower estimate for u. Moreover, the condition (1.5) is independent of $\mu > 0$: The question "can the logistic term relax conditions for global existence and boundedness?" is still open problem in (1.3).

The strategy for the proof of Theorem 1.1 is to construct the L^p-estimate for u with some $p > \frac{n}{2}$. One of keys for this strategy is to derive the inequality

$$\frac{d}{dt} \int_\Omega u^p \varphi(v) \leq c \int_\Omega u^p \varphi(v) - \mu p \int_\Omega u^{p+1} \varphi(v)$$

for some constant $c > 0$, where

$$\varphi(s) := \exp\left\{ -r \int_0^s \frac{1}{(a + \tau)^k} \, d\tau \right\} \quad (s \geq 0)$$

with some $r > 0$. Thanks to this strategy, we obtain

$$\int_\Omega u^p \varphi(v) \leq C$$

with some $C > 0$, which together with the lower estimate for φ implies the L^p-estimate for u. Thus in light of the well-known semigroup estimates, we can attain the L^∞-estimate for u.

2. Proof of the main result. In this section we will prove Theorem 1.1. We first recall the well-known result about local existence of solutions to (1.3) (see e.g., [1, Lemma 3.1]).

LEMMA 2.1. *Assume that χ satisfies (1.4) with some $\lambda > 0$, $k > 1$, $a > 0$, $K > 0$ and the initial data u_0, v_0 fulfil (1.6) for some $q > n$. Then there exist $T_{\max} \in (0, \infty]$ and exactly one pair (u, v) of positive functions*

$$u \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$
$$v \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap L^{\infty}_{\mathrm{loc}}([0, T_{\max}); W^{1,q}(\Omega))$$

which solves (1.3) in the classical sense. Moreover, if $T_{\max} < \infty$, then

$$\lim_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty.$$

In the following, we let (u, v) be the solution of (1.3) on $[0, T_{\max})$ as in Lemma 2.1. For the proof of Theorem 1.1 we will recall a useful fact to derive the L^∞-estimate for u.

LEMMA 2.2. *Assume that the solution (u, v) of (1.3) satisfies*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \le C(p) \tag{2.1}$$

for all $t \in (0, T_{\max})$ with some $p > \frac{n}{2}$ and $C(p) > 0$. Then there exists a constant $C' > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} \le C'$$

for all $t \in (0, T_{\max})$.

Proof. The same argument as in the proof of [1, Lemma 3.2] yields this result. □

Thanks to Lemmas 2.1 and 2.2 we will only make sure that the L^p-estimate for u holds with some $p > \frac{n}{2}$ to show global existence and boundedness of solutions to (1.3). To establish (2.1) we introduce the functions g and φ by

$$g(s) := -r \int_0^s \frac{1}{(a+\tau)^k}\, d\tau, \quad \varphi(s) := \exp\{g(s)\} \quad (s \ge 0), \tag{2.2}$$

where $r > 0$ is a constant fixed later. Here we note from straightforward calculations that

$$\varphi(s) = C_\varphi \exp\left\{ \frac{r}{(k-1)(a+s)^{k-1}} \right\}$$

with $C_\varphi = \exp\{-r(k-1)^{-1} a^{-k+1}\} > 0$. Now we shall prove the following inequality by using the test function $\varphi(v)$.

LEMMA 2.3. *Assume that χ satisfies (1.4) with some $\lambda > 0$, $k > 1$, $a > 0$, $K > 0$. Then there exists $c > 0$ such that*

$$\frac{d}{dt} \int_\Omega u^p \varphi(v) \le \int_\Omega u^p H_r(v) \varphi(v) |\nabla v|^2 + c \int_\Omega u^p \varphi(v) - \mu p \int_\Omega u^{p+1} \varphi(v), \tag{2.3}$$

where H_r is the function defined by

$$H_r(s) := -\frac{kr}{(a+s)^{k+1}} + \left(\frac{p(p-1)K^2}{4} + \frac{r^2}{p-1} \right) \frac{1}{(a+s)^{2k}} \tag{2.4}$$

for $s \geq 0$.

Proof. Let $p \geq 1$. From (1.3) we have

$$\frac{d}{dt}\int_\Omega u^p\varphi(v) = p\int_\Omega u^{p-1}\varphi(v)\nabla\cdot(\nabla u - u\chi(v)\nabla v) + \mu p\int_\Omega u^p\varphi(v)(1-u) + \int_\Omega u^p\varphi'(v)(\Delta v - v + u). \tag{2.5}$$

Then integration by parts derives

$$\begin{aligned} &p\int_\Omega u^{p-1}\varphi(v)\nabla\cdot(\nabla u - u\chi(v)\nabla v) + \int_\Omega u^p\varphi'(v)\Delta v \\ &= -p\int_\Omega \nabla(u^{p-1}\varphi(v))\cdot(\nabla u - u\chi(v)\nabla v) - \int_\Omega \nabla(u^p\varphi'(v))\cdot\nabla v \\ &= -p(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 + \int_\Omega u^{p-1}\left(p(p-1)\varphi(v)\chi(v) - 2p\varphi'(v)\right)\nabla u\cdot\nabla v \\ &\quad + \int_\Omega u^p(-\varphi''(v) + p\varphi'(v)\chi(v))|\nabla v|^2. \end{aligned} \tag{2.6}$$

Due to the Young inequality, we infer that

$$\begin{aligned} &\int_\Omega u^{p-1}\left(p(p-1)\varphi(v)\chi(v) - 2p\varphi'(v)\right)\nabla u\cdot\nabla v \\ &\leq p(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2 + \int_\Omega u^p\frac{(p(p-1)\varphi(v)\chi(v) - 2p\varphi'(v))^2}{4p(p-1)\varphi(v)}|\nabla v|^2. \end{aligned} \tag{2.7}$$

Thus a combination of (2.5), (2.6) and (2.7) yields that

$$\frac{d}{dt}\int_\Omega u^p\varphi(v) \leq \int_\Omega u^pF_\varphi(v)|\nabla v|^2 + \mu p\int_\Omega u^p\varphi(v)(1-u) + \int_\Omega u^p\varphi'(v)(-v+u), \tag{2.8}$$

where

$$F_\varphi(s) := -\varphi''(s) + \frac{p(p-1)}{4}\chi(s)^2\varphi(s) + \frac{p\varphi'(s)^2}{(p-1)\varphi(s)} \quad (s \geq 0).$$

Noting that

$$\varphi'(s) = g'(s)\varphi(s) \quad \text{and} \quad \varphi''(s) = g''(s)\varphi(s) + g'(s)^2\varphi(s) \quad (s \geq 0),$$

we can rewrite the function $F_\varphi(s)$ as

$$F_\varphi(s) = \left(-g''(s) + \frac{p(p-1)}{4}\chi(s)^2 + \frac{g'(s)^2}{p-1}\right)\varphi(s) \quad (s \geq 0).$$

Recalling by (2.2) that

$$g'(s) = \frac{-r}{(a+s)^k} \quad \text{and} \quad g''(s) = \frac{rk}{(a+s)^{k+1}} \quad (s \geq 0),$$

we obtain from (1.4) that

$$F_\varphi(s) \leq H_r(s)\varphi(s) \quad \text{for all } s \geq 0, \tag{2.9}$$

where H_r is defined as (2.4). Therefore we see from (2.8) together with (2.9) that

$$\frac{d}{dt}\int_\Omega u^p\varphi(v) \le \int_\Omega u^p H_r(v)\varphi(v)|\nabla v|^2 + \mu p\int_\Omega u^p\varphi(v)(1-u) - r\int_\Omega u^p\varphi(v)\frac{(-v+u)}{(a+v)^k}.$$

We finally verify from the boundedness of the function $s \mapsto \frac{s}{(a+s)^k}$ on $[0,\infty)$ $(k>1)$ and the positivity of u, v and φ that there is a constant $c_1 > 0$ satisfying

$$-r\int_\Omega u^p\varphi(v)\frac{(-v+u)}{(a+v)^k} \le c_1\int_\Omega u^p\varphi(v),$$

and thus we obtain (2.3). □

Now we shall confirm the following inequality which enables us to see the L^p-boundedness of u.

Lemma 2.4. *Assume that* (1.4) *and* (1.5) *are satisfied with some* $\lambda > 0$, $k > 1$, $a > 0$ *and* $K > 0$. *Then there exist* $p > \frac{n}{2}$ *and* $r > 0$ *such that*

$$H_r(s) \le 0 \quad \textit{for all } s \ge 0, \tag{2.10}$$

where H_r *is defined as* (2.4), *which implies that*

$$\frac{d}{dt}\int_\Omega u^p\varphi(v) \le c\int_\Omega u^p\varphi(v) - \mu p\int_\Omega u^{p+1}\varphi(v) \tag{2.11}$$

holds.

Proof. The same argument as in the proof of [14, Lemma 4.1] with $\varepsilon = 0$ leads to (2.10). Moreover, from a combination of Lemma 2.3 and (2.10) we obtain (2.11). □

Now we are ready to show the L^p-estimate for u. By using an argument similar to that in the proof of [13, Lemma 3.2] we can verify the following lemma.

Lemma 2.5. *Assume that* (1.4) *and* (1.5) *are satisfied with some* $\lambda > 0$, $k > 1$, $a > 0$ *and* $K > 0$. *Then there exist* $p > \frac{n}{2}$ *and* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C$$

for all $t \in (0, T_{\max})$.

Proof. From Lemma 2.4 we obtain (2.11) with some $p > \frac{n}{2}$ and $r > 0$. We shall show the L^p-estimate for u by using (2.11). We first note from the definition of φ (see (2.2)) that

$$C_\varphi \le \varphi(s) \le 1 \quad (s \ge 0). \tag{2.12}$$

Noticing from the Hölder inequality and (2.12) that

$$\int_\Omega u^p\varphi(v) \le \left(\int_\Omega \varphi(v)\right)^{\frac{1}{p+1}}\left(\int_\Omega u^{p+1}\varphi(v)\right)^{\frac{p}{p+1}} \le |\Omega|^{\frac{1}{p+1}}\left(\int_\Omega u^{p+1}\varphi(v)\right)^{\frac{p}{p+1}},$$

we infer from (2.11) that

$$\frac{d}{dt}\int_\Omega u^p\varphi(v) \le c\int_\Omega u^p\varphi(v) - \mu p|\Omega|^{-\frac{1}{p+1}}\left(\int_\Omega u^p\varphi(v)\right)^{\frac{p+1}{p}},$$

which implies that there exists $C > 0$ satisfying

$$\int_\Omega u^p \varphi(v) \le C.$$

Therefore we obtain from (2.12) that

$$\int_\Omega u^p \le C C_\varphi^{-1},$$

which entails this lemma. □

Proof of Theorem 1.1. Lemmas 2.2 and 2.5 directly lead to the conclusion of Theorem 1.1. □

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