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NONLINEAR DIFFUSION EQUATIONS WITH PERTURBATION TERMS ON UNBOUNDED DOMAINS

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Abstract. This paper considers the initial-boundary value problem for the nonlinear diffusion equation with the perturbation term

$$u_t + (-\Delta + 1)\beta(u) + G(u) = g \quad \text{in } \Omega \times (0, T)$$

in an unbounded domain $\Omega \subset \mathbb{R}^N$ with smooth bounded boundary, where $N \in \mathbb{N}$, $T > 0$, β is a single-valued maximal monotone function on $\mathbb{R}$, e.g.,

$$\beta(r) = |r|^{q-1}r \ (q > 0, q \neq 1)$$

and G is a function on $\mathbb{R}$ which can be regarded as a Lipschitz continuous operator from $(H^1(\Omega))^*$ to $(H^1(\Omega))^*$. The present work establishes existence and estimates for the above problem.

Key words. porous media equations, fast diffusion equations, subdifferential operators

AMS subject classifications. 35K59, 35K35, 47H05

1. Introduction. In this paper we consider the initial-boundary value problem for the nonlinear diffusion equation with the perturbation term

$$\begin{cases} u_t + (-\Delta + 1)\beta(u) + G(u) = g & \text{in } \Omega \times (0, T), \\ \partial_\nu \beta(u) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is an *unbounded* domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$ with smooth bounded boundary $\partial\Omega$ (e.g., $\Omega = \mathbb{R}^N \setminus \overline{B(0, R)}$, where $B(0, R)$ is the open ball with center 0 and radius $R > 0$) or $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$, $T > 0$, and ∂_ν denotes the derivative with respect to the outward normal of $\partial\Omega$. Though the precise conditions for β, G, g and u_0 will be given in (A1)-(A4) stated later, we roughly explain that β is a single-valued maximal monotone function, e.g.,

$$\beta(r) = |r|^{q-1}r,$$

where the problem is the porous media equation in the case that $q > 1$ (see e.g., [1, 13, 17, 18]) and is the fast diffusion equation in the case that $0 < q < 1$ (see e.g., [5, 15, 17]); G can be regarded as a Lipschitz continuous operator from $(H^1(\Omega))^*$ to $(H^1(\Omega))^*$; g and u_0 are known functions.

Nonlinear diffusion equations on unbounded domains are not so substantially studied from a viewpoint of the operator theory, whereas in the case that $\Omega = \mathbb{R}^N$ the equations are studied by the method of real analysis (see e.g., [11]). The case of unbounded domains would be important in both mathematics and physics. Also, since compact methods do not work directly on unbounded domains, it would be worth studying the case of unbounded domains mathematically. Also, the perturbation term $G(u)$ makes proving existence for (P) without growth conditions for β be difficult (see

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Remark in the end of this section). Although we give an example of G only as $G(u) = u$ in this paper, if we can weaken the condition for G and we can take $G(u) = -\beta(u)$, then we can possibly deal with the "pure" diffusion equation as $u_t - \Delta\beta(u) = g$.

In the case that $G \equiv 0$, in [12] and [9], existence of weak solutions to (P) and their estimates were shown by monotonicity methods.

The new point of this paper is that the perturbation term "$G(u)$" is added to the left-hand side of the equation $u_t + (-\Delta + 1)\beta(u) = g$ studied in [12] and [9]. The purpose of this paper is to show existence of weak solutions to (P) and to obtain their estimates. In particular, we prove existence for (P) by using Brézis's theory which is a monotonicity method for an abstract evolution equation including a subdifferential operator and a perturbation term.

We first give assumptions, notations and definitions used in this paper before introducing main results.

Assume that β, G, g and u_0 satisfy the following conditions:

(A1) The following (A1a), (A1b) and (A1c) hold:

(A1a) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function and

$$\beta(r) = \hat{\beta}\,'(r) = \partial\hat{\beta}(r),$$

where $\hat{\beta}\,'$ and $\partial\hat{\beta}$ respectively denote the differential and subdifferential of a proper differentiable (lower semicontinuous) convex function $\hat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\hat{\beta}(0) = 0$. This entails $\beta(0) = 0$.

(A1b) There exist constants $m > 1$ and $c_0, c_0' > 0$ such that for all $r \in \mathbb{R}$,

$$\hat{\beta}(r) \geq c_0|r|^m$$

and

$$|\beta(r)| \leq c_0'|r|^{m-1}$$

hold.

(A1c) For all $z \in H^1(\Omega)$, if $\hat{\beta}(z) \in L^1(\Omega)$, then $\beta(z) \in L^1_{\mathrm{loc}}(\Omega)$. For all $z \in H^1(\Omega)$ and all $\psi \in C_c^\infty(\Omega)$, if $\hat{\beta}(z) \in L^1(\Omega)$, then $\hat{\beta}(z + \psi) \in L^1(\Omega)$.

(A2) $G : (H^1(\Omega))^* \to (H^1(\Omega))^*$ is a Lipschitz continuous operator.

(A3) $g \in L^2\big(0, T; L^2(\Omega)\big)$.

(A4) $u_0 \in L^2(\Omega)$ and $\hat{\beta}(u_0) \in L^1(\Omega)$.

From (A3) we can fix a solution $f \in L^2\big(0, T; H^2(\Omega)\big)$ of

$$\begin{cases} (-\Delta + 1)f(t) = g(t) & \text{a.e. in } \Omega, \\ \partial_\nu f(t) = 0 & \text{in the sense of traces on } \partial\Omega \end{cases}$$

for a.a. $t \in (0, T)$, that is,

$$\int_\Omega \nabla f(t) \cdot \nabla z + \int_\Omega f(t)z = \int_\Omega g(t)z \quad \text{for all } z \in H^1(\Omega).$$

An example of (A2) is given as $G(v^*) = v^*$ for all $v^* \in (H^1(\Omega))^*$.

We define the Hilbert spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega)$$

with inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$, respectively. Moreover we put

$$W := \{z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega\}.$$

The notation V^* denotes the dual space of V with duality pairing $\langle\cdot,\cdot\rangle_{V^*,V}$. Moreover the Riesz representation theorem ensures the existence of a bijective mapping $F : V \to V^*$ satisfying

$$\langle Fv_1, v_2\rangle_{V^*,V} := (v_1, v_2)_V \quad \text{for all } v_1, v_2 \in V$$

and we define the inner product in V^* as

$$(v_1^*, v_2^*)_{V^*} := \langle v_1^*, F^{-1}v_2^*\rangle_{V^*,V} \quad \text{for all } v_1^*, v_2^* \in V^*.$$

We remark that (A3) implies

$$Ff(t) = g(t) \quad \text{for a.a. } t \in (0,T). \tag{1.1}$$

We give the definition of weak solutions to (P).

Definition 1.1. *A pair* (u,μ) *with*

$$u \in H^1(0,T;V^*), \quad \mu \in L^2(0,T;V)$$

is called a weak solution of (P) *if* (u,μ) *satisfies*

$$\langle u'(t) + G(u(t)), z\rangle_{V^*,V} + (\mu(t), z)_V = 0 \quad \textit{for all } z \in V \textit{ and a.a. } t \in (0,T), \tag{1.2}$$

$$\mu(t) = \beta(u(t)) - f(t) \quad \textit{in } V \quad \textit{for a.a. } t \in (0,T), \tag{1.3}$$

$$u(0) = u_0 \quad \textit{a.e. on } \Omega. \tag{1.4}$$

We next state the main result which asserts existence and estimates for (P).

Theorem 1.2. *Assume* (A1)-(A4). *Then there exists a unique weak solution* (u,μ) *of* (P) *satisfying* $u \in H^1(0,T;V^*), \mu \in L^2(0,T;V)$. *Moreover, if it holds that* $G(v^*) = av^*$ *for* $v^* \in V^*$, *where* $a \in \mathbb{R}$, *then there exists a constant* $M > 0$ *such that for a.a.* $t \in (0,T)$, $u(t) \in H$ *and*

$$|u(t)|_H^2 \le M, \tag{1.5}$$

$$\int_0^t |u'(s)|_{V^*}^2\, ds + a|u(t)|_{V^*}^2 \le M, \tag{1.6}$$

$$\int_0^t |\mu(s)|_V^2\, ds \le M, \tag{1.7}$$

$$\int_0^t |\beta(u(s))|_V^2\, ds \le M. \tag{1.8}$$

In the case that $G \equiv 0$, in [12], existence of weak solutions to (P) was proved by rewriting (P) to

$$u'(t) + \partial\phi(u(t)) = g(t) \quad \text{in } V^*,$$

where ϕ is a proper lower semicontinuous convex function on V^* and $\partial\phi$ is the subdifferential of ϕ, and by applying Brézis's theory ([3, Theorem 3.6]). Also, the m-growth condition for β was assumed to derive the lower semicontinuity of $\phi : V^* \to \overline{\mathbb{R}}$. The examples are the porous media equation and the fast diffusion equation. Recently, in [9], the approximation

$$u_\varepsilon'(t) + (-\Delta + 1)(\varepsilon(-\Delta + 1)u_\varepsilon(t) + \beta(u_\varepsilon(t)) + \pi_\varepsilon(u_\varepsilon(t))) = g \qquad \text{(P)}_\varepsilon$$

was considered and existence of weak solutions to (P)_ε with their estimates was shown; moreover, existence of weak solutions to (P) with their estimates was obtained without growth conditions for β, and existence of weak solutions to (P), their estimates were obtained without growth conditions for β by passing to the limit in (P)_ε as $\varepsilon \searrow 0$. In addition to the porous media equation and fast diffusion equation, the examples of (P) include the Stefan problem (see e.g., [2, 4, 6, 7, 8, 10]) which is described by (P) with

$$\beta(r) = \begin{cases} k_s r & \text{if } r < 0, \\ 0 & \text{if } 0 \le r \le L, \\ k_\ell(r - L) & \text{if } r > L \end{cases}$$

for all $r \in \mathbb{R}$, where k_s, k_ℓ, L are positive constants.

The strategy for the proof of Theorem 1.2 is to prove existence for (P) under the m-growth condition for β by setting some proper lower semicontinuous convex function $\phi : V^* \to \overline{\mathbb{R}}$ appropriately as in [12, Section 3], by rewriting (P) to

$$u'(t) + \partial\phi(u(t)) + G(u(t)) = g \quad \text{in } V^*$$

and by applying Brézis's theory to the above abstract evolution equation with the perturbation term.

Remark. At the moment, we do not know whether existence of weak solutions to (P) can be proved in a similar way to [9] or not. Since existence of weak solutions to the approximation of (P)

$$u_\varepsilon'(t) + (-\Delta + 1)\big(\varepsilon(-\Delta + 1)u_\varepsilon(t) + \beta(u_\varepsilon(t)) + \pi_\varepsilon(u_\varepsilon(t))\big) + G(u_\varepsilon(t)) = g \qquad (1.9)$$

can be proved in a similar way to the above strategy for (P), we can prove existence of weak solutions to (P) by passing the limit in (1.9) if we can obtain estimates for (1.9). In this paper we will directly prove existence of weak solutions to (P) under the m-growth condition for β without approximation (1.9). We hope that we can avoid growth conditions for β in a future work.

The plan of this paper is as follows. In Section 2 we prove existence of weak solutions to (P). Section 3 obtains estimates for (P). In Section 4 we present the porous media equation and the fast diffusion equation as examples of (P).

2. Proof of Theorem 1.2 (existence). In this section we will prove existence of a unique weak solution to (P). The following lemma is known in the Brézis's theory for a nonlinear evolution equation with a perturbation term including a subdifferential operator (see e.g., [3, Proposition 3.12]) and plays an important role in this section.

LEMMA 2.1. *Let X be a real Hilbert space, let $\psi : X \to \overline{\mathbb{R}}$ be a proper l.s.c. convex function and let $G : X \to X$ be a Lipschitz continuous operator. If $u_0 \in D(\psi)$ and $\tilde{f} \in L^2(0,T;X)$, then there exists a unique function u such that $u \in H^1(0,T;X)$, $u(t) \in D(\partial\psi)$ for a.a. $t \in (0,T)$ and u solves the following initial value problem:*

$$\begin{cases} u'(t) + \partial\psi(u(t)) + G(u(t)) \ni \tilde{f}(t) \quad \text{in } X \quad \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in } X. \end{cases}$$

Proof of Theorem 1.2 (*existence*). Defining $\phi : V^* \to \overline{\mathbb{R}}$ as

$$\phi(z) = \begin{cases} \displaystyle\int_\Omega \hat{\beta}(z(x))\,dx & \text{if } z \in D(\phi) := \{z \in V^* \cap L^m(\Omega) \mid \hat{\beta}(z) \in L^1(\Omega)\}, \\ +\infty & \text{otherwise,} \end{cases}$$

we deduce from [12, Section 3] that this ϕ is proper lower semicontinuous convex on V^* and

$$\beta(z) \in V, \quad \partial\phi(z) = F\beta(z) \tag{2.1}$$

hold for all $z \in D(\partial\phi)$. Hence, from (1.1) and (2.1) we can rewrite (1.2)-(1.4) in Definition 1.1 to

$$\begin{cases} u'(t) + \partial\phi(u(t)) + G(u(t)) = g(t) \quad \text{in } V^* \quad \text{for a.a. } t \in [0,T], \\ u(0) = u_0 \quad \text{in } V^*. \end{cases} \tag{2.2}$$

Invoking Lemma 2.1, we can find a unique solution $u \in H^1(0,T;V^*)$ of (2.2) and $u(t) \in D(\partial\phi)$ for a.a. $t \in (0,T)$. Hence there exists a unique weak solution of (P). □

3. Proof of Theorem 1.2 (estimates). We will obtain the estimates for weak solutions of (P) in this section.

Proof of Theorem 1.2 (*estimates*). In addition to (A2) we assume further that

$$G(v^*) = av^*$$

for all $v^* \in V^*$, where $a \in \mathbb{R}$. For $\lambda > 0$ we put

$$A := -\Delta : D(A) := W \subset H \to H,$$
$$J_\lambda := (I + \lambda A)^{-1} : H \to H,$$
$$A_\lambda := \lambda^{-1}(I - J_\lambda) : H \to H,$$

and

$$\tilde{A} := F - I : V \to V^*,$$
$$\tilde{J}_\lambda := \left(I + \lambda\tilde{A}\right)^{-1} : V^* \to V.$$

Let $u \in H^1(0,T;V^*)$ be a unique solution of (2.2). We first show (1.5). Noting that $\tilde{J}_\lambda^{1/2} : V^* \to H$ is defined as a bounded operator (see e.g., [14, Lemma 3.3]) and putting

$$u_\lambda(t) := \tilde{J}_\lambda^{1/2} u(t) \quad \text{for all } t \in (0,T),$$

we derive from [12, Lemma 3.3] that

$$u_\lambda \in H^1(0,T;H)$$

and

$$u_\lambda'(t) + \tilde{J}_\lambda^{1/2} F\beta(u(t)) + \tilde{J}_\lambda^{1/2} G(u(t)) = J_\lambda^{1/2} g(t).$$

Then we obtain

$$\frac{1}{2}\frac{d}{ds}|u_\lambda(s)|_H^2 \le \frac{1}{2}|g(s)|_H^2 + \frac{1}{2}|u_\lambda(s)|_H^2 + \bigl(\tilde{J}_\lambda^{1/2}G(u(s)), u_\lambda(s)\bigr)_H \tag{3.1}$$

in a similar way to [12, Section 3]. Here we have

$$\begin{aligned}\bigl(\tilde{J}_\lambda^{1/2}G(u(s)), u_\lambda(s)\bigr)_H &= a\bigl(\tilde{J}_\lambda^{1/2}u(s), u_\lambda(s)\bigr)_H \\ &= a|u_\lambda(s)|_H^2. \end{aligned}\tag{3.2}$$

Thus combining (3.1) and (3.2) yields

$$\frac{1}{2}\frac{d}{ds}|u_\lambda(s)|_H^2 \le \frac{1}{2}|g(s)|_H^2 + \Bigl(a + \frac{1}{2}\Bigr)|u_\lambda(s)|_H^2,$$

and hence the inequality

$$|u_\lambda(t)|_H^2 \le e^{|2a+1|T}|u_0|_H^2 + e^{|2a+1|T}|g|_{L^2(0,T;H)}^2$$

holds for all $t \in (0,T)$. Therefore it follows from a similar way to [12, Section 3, Proof of Theorem 1.1 (continued)] that for a.a. $t \in (0,T)$,

$$u(t) \in H$$

and there exists a positive constant C such that

$$\|u\|_{L^\infty(0,T;H)} \le C,$$

which means that the estimate (1.5) holds.

Next we verify (1.6). The equation in (2.2) yields that

$$\begin{aligned}\bigl|u'(s)\bigr|_{V^*}^2 &= -\bigl(u'(s), \partial\phi(u(s))\bigr)_{V^*} + \bigl(u'(s), Ff(s)\bigr)_{V^*} - a\bigl(u'(s), u(s)\bigr)_{V^*} \\ &= -\bigl(u'(s), \partial\phi(u(s))\bigr)_{V^*} + \bigl(u'(s), Ff(s)\bigr)_{V^*} - \frac{a}{2}\frac{d}{ds}|u(s)|_{V^*}^2. \end{aligned}\tag{3.3}$$

Here we have

$$\bigl(u'(s), \partial\phi(u(s))\bigr)_{V^*} = \frac{d}{ds}\phi(u(s))$$

(see e.g., Showalter [16, Lemma IV.4.3]) and it follows from the definition of $(\cdot,\cdot)_{V^*}$ and Young's inequality that

$$\begin{aligned}\bigl(u'(s), Ff(s)\bigr)_{V^*} &= \bigl\langle u'(s), f(s)\bigr\rangle_{V^*,V} \\ &\le \frac{1}{2}\bigl|u'(s)\bigr|_{V^*}^2 + \frac{1}{2}|f(s)|_V^2. \end{aligned}$$

Integrating (3.3) combined with these facts leads to the inequality

$$\frac{1}{2}\int_0^t \bigl|u'(s)\bigr|_{V^*}^2\,ds \le -\phi(u(t)) + \phi(u_0) + \frac{1}{2}|f|_{L^2(0,T;V)}^2 - \frac{a}{2}|u(t)|_{V^*}^2 + \frac{a}{2}|u_0|_{V^*}^2,$$

i.e.,

$$\frac{1}{2}\int_0^t \bigl|u'(s)\bigr|_{V^*}^2\,ds + \int_\Omega \hat{\beta}(u(t)) + \frac{a}{2}|u(t)|_{V^*}^2 \le \int_\Omega \hat{\beta}(u_0) + \frac{1}{2}|f|_{L^2(0,T;V)}^2 + \frac{a}{2}|u_0|_{V^*}^2.$$

Since (A1a) implies

$$\int_\Omega \hat{\beta}(u(t)) \geq 0,$$

it holds that

$$\int_0^t \big|u'(s)\big|_{V^*}^2 \, ds + a|u(t)|_{V^*}^2 \leq 2\int_\Omega \hat{\beta}(u_0) + |f|_{L^2(0,T;V)}^2 + a|u_0|_{V^*}^2.$$

Next we show (1.7). The fact that $\mu(s) = -F^{-1}\big(u'(s)\big)$ implies

$$\begin{aligned}\int_0^t |\mu(s)|_V^2 \, ds &= \int_0^t \big|F^{-1}\big(u'(s)\big)\big|_V^2 \, ds \\ &= \int_0^t \big|u'(s)\big|_{V^*}^2 \, ds.\end{aligned}$$

Hence (1.7) can be obtained from (1.6).

Next we prove (1.8). We see from (1.3) and the definition of $\mu(\cdot)$ that

$$\begin{aligned}|\beta(u(s))|_V^2 &= \big(-F^{-1}\big(u'(s)\big) - au(s) + f(s), \beta(u(s))\big)_V \\ &\leq \big(\big|F^{-1}\big(u'(s)\big)\big|_V + |a||u(s)|_{V^*} + |f(s)|_V\big)|\beta(u(s))|_V \\ &\leq \big|F^{-1}\big(u'(s)\big)\big|_V^2 + a^2|u(s)|_{V^*}^2 + |f(s)|_V^2 + \frac{3}{4}|\beta(u(s))|_V^2 \\ &= \big|u'(s)\big|_{V^*}^2 + a^2|u(s)|_{V^*}^2 + |f(s)|_V^2 + \frac{3}{4}|\beta(u(s))|_V^2.\end{aligned}$$

Integrating this inequality, we have

$$\int_0^t |\beta(u(s))|_V^2 \, ds \leq 4\int_0^t \big|u'(s)\big|_{V^*}^2 \, ds + 4a^2\int_0^t |u(s)|_{V^*}^2 \, ds + 4|f|_{L^2(0,T;V)}^2.$$

Therefore there exists a constant $M > 0$ satisfying (1.5), (1.6), (1.7) and (1.8). Moreover, (1.5) means that $u \in L^\infty(0,T;H)$. □

4. Examples. An example of $G : (H^1(\Omega))^* \to (H^1(\Omega))^*$ is given by $G(v^*) = v^*$ for all $v^* \in (H^1(\Omega))^*$. As to β, we give the following two examples.

The porous media equation. We consider

$$\beta(r) = |r|^{q-1}r \quad (q > 1).$$

This β is the function in the porous media equation (see e.g., [1, 13, 17, 18]).

The fast diffusion equation. Consider

$$\beta(r) = |r|^{q-1}r \quad (0 < q < 1).$$

This β is the function in the fast diffusion equation (see e.g., [5, 15, 17]).

In both examples we can show that β satisfies (A1), (A4) (see [12, Section 6]).

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