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PROPAGATION OF ERRORS IN DYNAMIC ITERATIVE SCHEMES

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Abstract. We consider iterative schemes applied to systems of linear ordinary differential equations and investigate their convergence in terms of magnitudes of the coefficients given in the systems. We address the question of whether the reordering of equations in a given system improves the convergence of an iterative scheme.

Key words. Dynamic iterations, waveform relaxation, Gauss-Seidel schemes, convergence, error bounds

AMS subject classifications. 65L04, 65L20, 65L70

1. Introduction. We investigate convergence of dynamic iteration schemes, see e.g. [2], [4], whose successive iterates are vector functions of the time variable t rather than just a collection of scalars (as in static iterations). The schemes are also called waveform relaxation techniques and their advantages are described e.g. in [3]. The references [3], [2], [4] provide a broad overview on the dynamic iteration schemes (designed for time-dependent initial value problems) versus static iteration schemes (designed for linear algebraic systems). Convergence analyses for dynamic iteration schemes are provided in [3], [2], [4] and the references therein. However, the comparison of different choices of dynamic iteration schemes obtained through a change in the order of the differential equations in a given system is not considered in these references.

In this paper, we show that the choice of the components to be computed using previous iterates and the components to be computed using present iterates affects the efficiency of resulting iterative schemes. To illustrate this, we consider dynamic iterative schemes for the following system

$$\begin{cases} \dfrac{d}{dt}x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + g_1(t), \\ \dfrac{d}{dt}x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + g_2(t), \quad t > 0. \end{cases} \tag{1.1}$$

supplemented by the initial conditions

$$x_1(0) = x_{1,0}, \quad x_2(0) = x_{2,0}, \tag{1.2}$$

where $a_{11} \le 0$, $a_{22} \le 0$, a_{12}, a_{21}, $x_{1,0}$, $x_{2,0}$ are given real numbers and $g_i(t)$ are given real valued functions.

For (1.1)–(1.2), we consider the following alternative iterative schemes

$$\begin{cases} \dfrac{d}{dt}x_1^{k+1}(t) = a_{11}x_1^{k+1}(t) + a_{12}x_2^{k}(t) + g_1(t), \\ \dfrac{d}{dt}x_2^{k+1}(t) = a_{21}x_1^{k+1}(t) + a_{22}x_2^{k+1}(t) + g_2(t), \quad t > 0. \end{cases} \tag{1.3}$$

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and

$$\begin{cases} \dfrac{d}{dt} y_2^{k+1}(t) = a_{22} y_2^{k+1}(t) + a_{21} y_1^k(t) + g_2(t), \\ \dfrac{d}{dt} y_1^{k+1}(t) = a_{12} y_2^{k+1}(t) + a_{11} y_1^{k+1}(t) + g_1(t), \quad t > 0. \end{cases} \tag{1.4}$$

supplemented by the initial conditions

$$x_1^k(0) = y_1^k(0) = x_{1,0}, \quad x_2^k(0) = y_2^k(0) = x_{2,0}. \tag{1.5}$$

Scheme (1.3) is initiated from an arbitrary function $x_2^0(t)$ and (1.4) is initiated from another arbitrary function $y_1^0(t)$. Schemes (1.3) and (1.4) are called Gauss-Seidel waveform relaxation schemes see, e.g., [2], [4].

Note that (1.4) is obtained from (1.1) by switching the equations in (1.1). Moreover, schemes (1.3) and (1.4) differ through the fact that scheme (1.3) is slowed down by the previous iterate $x_2^k(t)$ that is multiplied by the coefficient a_{12} while scheme (1.4) is slowed down by the previous iterate $y_1^k(t)$ multiplied by a_{21}.

Suppose that both k^{th} iterates $x_2^k(t)$ and $y_1^k(t)$ give rise to the same error

$$\mathcal{E}^k(t) = x_2^k(t) - x_2(t) = y_1^k(t) - y_1(t).$$

Then, in scheme (1.3), the error $\mathcal{E}^k(t)$ is multiplied by the coefficient a_{12} while, in scheme (1.4), $\mathcal{E}^k(t)$ is multiplied by a_{21}. Let us additionally suppose that a_{12} is much greater than a_{21}, for example, $a_{12} = 10^6$ and $a_{21} = 10^{-6}$. Then, in scheme (1.3), the error $\mathcal{E}^k(t)$ is multiplied by 10^6 (that is, it is significantly enlarged) while, in scheme (1.4), the error $\mathcal{E}^k(t)$ is multiplied by 10^{-6} (so, it is significantly reduced). Therefore, a natural question arises. Which of the schemes (1.3) or (1.4) is faster? In other words, which of the sequences

$$\Big\{ \big(x_1^k(t), x_2^k(t) \big) \Big\}_{k=0}^{\infty}, \qquad \Big\{ \big(y_1^k(t), y_2^k(t) \big) \Big\}_{k=0}^{\infty} \tag{1.6}$$

converges to $\big(x_1(t), x_2(t) \big)$ faster?

This brings about further questions. Is it better to reorder the differential equations in system (1.1) before the Gauss-Seidel waveform relaxation scheme is applied to get faster convergence of the resulting successive iterates? The goal of the paper is to address the above questions.

2. Convergence analysis involving the spectral radius of a linear integral operator. In this section, we follow [3] and define the linear integral operator

$$\mathcal{K}x(t) = \int_0^t \exp\big((t-s)A \big) Bx(s) ds,$$

where A and B are complex square matrices of the same size. Then system (1.3) is written in the form

$$x^{k+1}(t) = \mathcal{K}x^k(t) + \int_0^t \exp\big((t-s)A \big) g(s) ds + \exp\big((t-s)A \big) x_0$$

with

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}, \quad g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$$

and the spectral radius of $\mathcal{K}$ is written in the form

$$\rho(\mathcal{K}) = \left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|,$$

see [3]. If

$$\tilde{A} = \begin{bmatrix} a_{22} & 0 \\ a_{12} & a_{11} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & a_{21} \\ 0 & 0 \end{bmatrix}, \quad \tilde{g}(t) = \begin{bmatrix} g_2(t) \\ g_1(t) \end{bmatrix}, \quad \tilde{x}_0 = \begin{bmatrix} x_{2,0} \\ x_{1,0} \end{bmatrix}$$

then (1.4) is written in the form

$$y^{k+1}(t) = \tilde{\mathcal{K}} y^k(t) + \int_0^t \exp\big((t-s)\tilde{A}\big)\tilde{g}(s)ds + \exp\big((t-s)\tilde{A}\big)\tilde{x}_0,$$

where

$$\tilde{\mathcal{K}} x(t) = \int_0^t \exp\big((t-s)\tilde{A}\big)\tilde{B}x(s)ds,$$

and

$$\rho(\tilde{\mathcal{K}}) = \left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|.$$

Note that the spectral radius for (1.4) is the same as for (1.3). Therefore, the spectral radius does not give rise to any answer to the question of which of the schemes (1.3) or (1.4) converge faster, though numerical experiments presented in Section 5 illustrate that both schemes converge at different rates, showing that one is more efficient than the other.

3. Explicit formulas for errors and conclusions for improving convergence of iterative schemes. The roles of the parameters in the propagation of errors can be traced more precisely from exact formulas of the errors than from error bounds. In this section, we investigate the roles of the parameters a_{11}, a_{12}, a_{21}, a_{22} in the propagation of errors arising during computations of the sequences of vector functions (1.6) from the alternative numerical schemes (1.3) or (1.4) and address the question of which of the schemes converges faster.

To realize this goal, we investigate exact formulas for the errors

$$e_i^k(t) = x_i(t) - x_i^k(t), \quad i = 1, 2, \ k = 0, 1, \dots \tag{3.1}$$

and

$$E_i^k(t) = x_i(t) - y_i^k(t), \quad i = 1, 2, \ k = 0, 1, \dots, \tag{3.2}$$

which are provided through the following theorem.

THEOREM 3.1. *Let*

$$w(\xi) = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} \sum_{i=0}^{k-1} a_{11}^{k-1-i} a_{22}^i. \tag{3.3}$$

Then the errors (3.1) are given by the formulas

$$e_1^k(t_{k+1}) = a_{12}^k a_{21}^{k-1} \int_0^{t_{k+1}} \int_0^{t_k} \dots \int_0^{t_2} e^{a_{11}(t_{k+1}-t_k)} \prod_{j=1}^{k-1} w(t_{j+1}-t_j) \quad (3.4)$$
$$e_2^0(t_1) dt_1 dt_2 \dots dt_k,$$

$$e_2^k(t_{k+1}) = a_{12}^k a_{21}^k \int_0^{t_{k+1}} \int_0^{t_k} \dots \int_0^{t_2} \prod_{j=1}^{k} w(t_{j+1}-t_j) \quad (3.5)$$
$$e_2^0(t_1) dt_1 dt_2 \dots dt_k,$$

where $0 < t_1 < t_2 < \dots < t_{k+1}$ *and* $k = 1, 2, \dots$

Proof. From (1.1)–(3.1), we have

$$\begin{cases} \frac{d}{dt} e_1^{k+1}(t) = a_{11} e_1^{k+1}(t) + a_{12} e_2^k(t), \\ \frac{d}{dt} e_2^{k+1}(t) = a_{21} e_1^{k+1}(t) + a_{22} e_2^{k+1}(t), \end{cases} \quad (3.6)$$

and

$$e_1^k(0) = e_2^k(0) = 0.$$

Therefore, the error $e^k(t) = (e_1^k(t), e_2^k(t))^T$ is given recursively by

$$e^{k+1}(t) = \int_0^t \exp\left((t-s) \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \right) \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} e^k(s) ds, \quad (3.7)$$

for $k = 0, 1, 2 \dots$. It can be proved by induction that

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}^k = \begin{bmatrix} a_{11}^k & 0 \\ a_{21} \sum_{j=0}^{k-1} a_{11}^{k-1-j} a_{22}^j & a_{22}^k \end{bmatrix},$$

for $k = 1, 2, \dots$ This leads to

$$\exp\left((t-s) \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{(t-s)^1}{1!} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} + \dots +$$
$$\frac{(t-s)^i}{i!} \begin{bmatrix} a_{11}^i & 0 \\ a_{21} \sum_{j=0}^{i-1} a_{11}^{i-1-j} a_{22}^j & a_{22}^i \end{bmatrix} + \dots$$
$$= \begin{bmatrix} \sum_{i=0}^{\infty} \frac{a_{11}^i (t-s)^i}{i!} & 0 \\ a_{21} \sum_{i=1}^{\infty} \frac{(t-s)^i}{i!} \sum_{j=0}^{i-1} a_{11}^{i-1-j} a_{22}^j & \sum_{i=0}^{\infty} \frac{a_{22}^i (t-s)^i}{i!} \end{bmatrix}$$
$$= \begin{bmatrix} e^{a_{11}(t-s)} & 0 \\ a_{21} w(t-s) & e^{a_{22}(t-s)} \end{bmatrix},$$

which gives

$$\exp\left((t-s)\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}\right)\begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{12}e^{a_{11}(t-s)} \\ 0 & a_{12}a_{21}w(t-s) \end{bmatrix}.$$

From this and from (3.7) we have

$$e_1^{k+1}(t) = a_{12}\int_0^t \exp\Big(a_{11}(t-s)\Big)e_2^k(s)ds, \tag{3.8}$$

$$e_2^{k+1}(t) = a_{12}a_{21}\int_0^t w(t-s)e_2^k(s)ds, \tag{3.9}$$

for $k = 0, 1, \ldots$. We now use (3.9) to prove (3.5). It is easy to check that (3.9) for $k = 0$ implies (3.5) for $k = 1$, (here, $t_2 = t$ and $t_1 = s$). Assuming (3.5) holds for some k, we will prove it for $k+1$. From (3.9) we have

$$\begin{aligned} &e_2^{k+1}(t_{k+2}) = a_{12}a_{21}\int_0^{t_{k+2}} w(t_{k+2}-t_{k+1})e_2^k(t_{k+1})dt_{k+1} \\ &= a_{12}^{k+1}a_{21}^{k+1}\int_0^{t_{k+2}} w(t_{k+2}-t_{k+1})\int_0^{t_{k+1}}\int_0^{t_k}\ldots\int_0^{t_2}\prod_{j=1}^{k} w(t_{j+1}-t_j)\times \\ &\quad e_2^0(t_1)dt_1dt_2\ldots dt_kdt_{k+1} \\ &= a_{12}^{k+1}a_{21}^{k+1}\int_0^{t_{k+2}}\int_0^{t_{k+1}}\int_0^{t_k}\ldots\int_0^{t_2}\prod_{j=1}^{k+1} w(t_{j+1}-t_j)e_2^0(t_1)dt_1dt_2\ldots dt_kdt_{k+1}, \end{aligned}$$

which proves (3.5). We now use (3.5) and (3.8) to prove (3.4). From (3.5) and (3.8) we have

$$\begin{aligned} &e_1^{k+1}(t_{k+2}) = a_{12}\int_0^{t_{k+2}} \exp\Big(a_{11}(t_{k+2}-t_{k+1})\Big)e_2^k(t_{k+1})dt_{k+1} \\ &= a_{12}^{k+1}a_{21}^{k}\int_0^{t_{k+2}} \exp\Big(a_{11}(t_{k+2}-t_{k+1})\Big)\int_0^{t_{k+1}}\int_0^{t_k}\ldots\int_0^{t_2}\prod_{j=1}^{k} w(t_{j+1}-t_j)\times \\ &e_2^0(t_1)dt_1dt_2\ldots dt_kdt_{k+1}, \end{aligned}$$

which finishes the proof of the theorem. □

We now apply Theorem 3.1 to (1.4) and compare the errors arising in both schemes, (1.3) and (1.4). Since

$$\sum_{i=0}^{k-1} a_{11}^{k-1-i}a_{22}^i = \sum_{i=0}^{k-1} a_{22}^{k-1-i}a_{11}^i,$$

from (3.4) and (3.5), we have

$$E_2^k(t_{k+1}) = a_{21}^k a_{12}^{k-1} \int_0^{t_{k+1}} \int_0^{t_k} \dots \int_0^{t_2} e^{a_{22}(t_{k+1}-t_k)} \times \prod_{j=1}^{k-1} w(t_{j+1}-t_j) E_1^0(t_1) dt_1 dt_2 \dots dt_k, \tag{3.10}$$

$$E_1^k(t_{k+1}) = a_{21}^k a_{12}^k \int_0^{t_{k+1}} \int_0^{t_k} \dots \int_0^{t_2} \prod_{j=1}^{k} w(t_{j+1}-t_j) \times E_1^0(t_1) dt_1 dt_2 \dots dt_k, \tag{3.11}$$

for $k = 1, 2, \dots$ and $t_{k+1} > 0$.

REMARK. *Note that the starting function $x_1^0(t)$ has no influence on the convergence of the scheme (1.3) and the starting function $y_2^0(t)$ has no influence on the convergence of the scheme (1.4).*

The formulas (3.4)–(3.5), for the scheme (1.3), and the formulas (3.10)–(3.11), for the scheme (1.4), show how the starting error $e_2^{(0)} = x_2 - x_2^{(0)}$ propagates in (1.3) and how the starting error $e_1^{(0)} = x_1 - y_1^{(0)}$ propagates in (1.4).

To choose the faster scheme, we compare (3.4)–(3.5) with (3.10)–(3.11) in the following Corollary.

COROLLARY 3.2. *If*

$$e_2^0 \equiv E_1^0. \tag{3.12}$$

and

$$a_{11} < a_{22} \quad \textit{and} \quad |a_{12}| < |a_{21}|, \tag{3.13}$$

then scheme (1.3) converges faster than scheme (1.4). If (3.12) holds and the inequalities in (3.13) are reversed then scheme (1.4) converges faster than scheme (1.3).

Corollary 3.2 shows that if (3.13) holds, then even though (1.3) and (1.4) are initiated with the same error, it propagates differently in both schemes.

Results for higher-dimensional systems are developed in [5].

4. Using parameters in the derivation of error bounds. Applying the variation of constants formula it is easy to obtain the following classical error bound

$$\|e^k(t)\| \le \frac{1}{k!}\Big(\exp(t\|L+D\|)\|U\|\Big)^k \max\{\|e^0(s)\| : 0 \le s \le t\},$$

see [2]. However, sharper error estimation can be obtained by using the exact formulas (3.4) and (3.5).

THEOREM 4.1. *Let*

$$S_k = \frac{1}{k!}\Big(\frac{|a_{12}a_{21}|}{|a_{11}|+|a_{22}|}\Big)^k \int_0^t s^k \exp\Big(s(|a_{11}|+|a_{22}|)\Big) ds \max_{s\in[0,t]} |e_2^0(s)|, \tag{4.1}$$

for $k = 0, 1, \dots$. Then

$$|e_1^k(t)| < |a_{12}| S_{k-1}, \tag{4.2}$$

$$|e_2^k(t)| < \frac{|a_{12}a_{21}|}{|a_{11}|+|a_{22}|} S_{k-1}, \tag{4.3}$$

for $k = 1, 2, \dots$ *Moreover*

$$\lim_{k\to\infty} S_k = 0. \tag{4.4}$$

Proof. Let w be defined as in Theorem 3.1 and $\alpha = |a_{11}| + |a_{22}|$. Since

$$0 < t_1 < t_2 < \cdots < t_k < t_{k+1}$$

in (3.4) and (3.5), then from the definition (3.3) we have

$$\begin{aligned}\Big|w(t_{j+1}-t_j)\Big| &\le \sum_{k=1}^{\infty} \frac{(t_{j+1}-t_j)^k}{k!} \sum_{i=0}^{k-1} |a_{11}|^{k-1-i}|a_{22}|^i \\ &\le \sum_{k=1}^{\infty} \frac{(t_{j+1}-t_j)^k}{k!} \sum_{i=0}^{k-1} \binom{k-1}{i} |a_{11}|^{k-1-i}|a_{22}|^i \\ &= \sum_{k=1}^{\infty} \frac{(t_{j+1}-t_j)^k}{k!} \alpha^{k-1} < \frac{1}{\alpha} \exp\Big(\alpha(t_{j+1}-t_j)\Big),\end{aligned}$$

and

$$\Big|\prod_{j=1}^{k-1} w(t_{j+1}-t_j)\Big| = \prod_{j=1}^{k-1} \Big|w(t_{j+1}-t_j)\Big| < \prod_{j=1}^{k-1} \frac{1}{\alpha} \exp\Big(\alpha(t_{j+1}-t_j)\Big) = \frac{1}{\alpha^{k-1}} \exp\Big(\alpha(t_k - t_1)\Big).$$

This, together with (3.4), implies that

$$\begin{aligned}|e_1^k(t_{k+1})| \le |a_{12}|^k|a_{21}|^{k-1} &\int_0^{t_{k+1}}\int_0^{t_k}\dots\int_0^{t_2} \exp\Big(a_{11}(t_{k+1}-t_k)\Big) \\ &\frac{1}{\alpha^{k-1}} \exp\Big(\alpha(t_k-t_1)\Big)|e_2^0(t_1)|dt_1dt_2\dots dt_k \\ \le |a_{12}|^k|a_{21}|^{k-1}\alpha^{1-k} &\max_{0\le\tau\le t_{k+1}} |e_2^0(\tau)| \int_0^{t_{k+1}}\int_0^{t_k}\dots\int_0^{t_2} \exp\Big(\alpha(t_{k+1}-t_1)\Big)dt_1dt_2\dots dt_k.\end{aligned}$$

We now show that

$$\int_0^{t_{k+1}}\int_0^{t_k}\dots\int_0^{t_2} \exp\Big(\alpha(t_{k+1}-t_1)\Big)dt_1dt_2\dots dt_k = \frac{1}{(k-1)!}\int_0^{t_{k+1}} s^{k-1}e^{\alpha s}ds. \tag{4.5}$$

Since

$$\frac{1}{k-1}\int_0^t s^{k-1}e^{\alpha s}ds = \int_0^t e^{\alpha(t-z)}\int_0^z s^{k-2}e^{\alpha s}dsdz,$$

the right-hand side of (4.5) is

$$\frac{1}{(k-1)!}\int_0^{t_{k+1}} t_k^{k-1}e^{\alpha t_k}dt_k = \frac{1}{(k-2)!}\int_0^{t_{k+1}} e^{\alpha(t_{k+1}-t_k)}\int_0^{t_k} t_{k-1}^{k-2}e^{\alpha t_{k-1}}dt_{k-1}dt_k =$$

$$\frac{1}{(k-3)!}\int_0^{t_{k+1}} e^{\alpha(t_{k+1}-t_k)}\int_0^{t_k} e^{\alpha(t_k-t_{k-1})}\int_0^{t_{k-1}} t_{k-2}^{k-3}e^{\alpha t_{k-2}}dt_{k-2}dt_{k-1}dt_k = \dots$$

$$\frac{1}{1!}\int_0^{t_{k+1}} e^{\alpha(t_{k+1}-t_k)}\int_0^{t_k} e^{\alpha(t_k-t_{k-1})}\int_0^{t_{k-1}} e^{\alpha(t_{k-1}-t_{k-2})}\dots\int_0^{t_3} t_2e^{\alpha t_2}dt_2\dots dt_{k-2}dt_{k-1}dt_k =$$

$$\int_0^{t_{k+1}}\int_0^{t_k}\int_0^{t_{k-1}}\dots\int_0^{t_3} t_2e^{\alpha(t_{k+1}-t_3+t_2)}dt_2\dots dt_{k-2}dt_{k-1}dt_k =$$

$$\int_0^{t_{k+1}}\int_0^{t_k}\int_0^{t_{k-1}}\dots\int_0^{t_3} (t_3-t_2)e^{\alpha(t_{k+1}-t_2)}dt_2\dots dt_{k-2}dt_{k-1}dt_k.$$

This, together with

$$\int_0^{t_3} (t_3 - t_2) e^{\alpha(t_{k+1} - t_2)} dt_2 = \int_0^{t_3} (t_3 - t_2) \Big(\frac{d}{dt_2} \int_0^{t_2} e^{\alpha(t_{k+1} - t_1)} dt_1 \Big) dt_2 =$$
$$\Big[(t_3 - t_2) \int_0^{t_2} e^{\alpha(t_{k+1} - t_1)} dt_1 \Big]_{t_2=0}^{t_2=t_3} + \int_0^{t_3} \int_0^{t_2} e^{\alpha(t_{k+1} - t_1)} dt_1 dt_2,$$

implies (4.5) and the proof of (4.2) is finished. The proof of (4.3) is similar. We now show (4.4). Since

$$0 \le \frac{S_k}{S_{k-1}} \le \frac{t}{k} \frac{|a_{12}a_{21}|}{|a_{11}| + |a_{22}|},$$

it follows that

$$\lim_{k \to \infty} \frac{S_k}{S_{k-1}} = 0,$$

which proves (4.4) and finishes the proof of the theorem. □

5. Numerical experiments. In this section, we present results of numerical experiments for (1.1). We apply the alternative schemes (1.3) and (1.4) to (1.1) and compare their corresponding errors. To integrate (1.3) and (1.4) in time, we apply BDF3 with the step size $h = 10^{-3}$. Time integration gives rise to the numerical approximations

$$x_{1,n}^k \approx x_1(t_n), \quad x_{2,n}^k \approx x_2(t_n),$$

for (1.3) and

$$y_{1,n}^k \approx x_1(t_n), \quad y_{2,n}^k \approx x_2(t_n),$$

for (1.4), at the grid-points $t_n = nh$, $n = 0, 1, \ldots$. We measure the errors

$$\max_{i=1,2} \big\{ |x_i(t_n) - x_{i,n}^k| \big\}, \tag{5.1}$$

$$\max_{i=1,2} \big\{ |x_i(t_n) - y_{i,n}^k| \big\}, \tag{5.2}$$

and observe the convergence of the schemes (1.3) and (1.4) by plotting (5.1) and (5.2) as functions of $k = 0, 1, \ldots$ for a fixed n.

The errors (5.1) and (5.2) resulting from the different schemes ((5.1) corresponds to (1.3) and (5.2) corresponds to (1.4)) are plotted in Figures 5.1 and 5.2 for $n = 1000$. In both figures, the dotted line presents the error (5.1) and the solid line presents the error (5.2).

Figure 5.1 presents the errors for problem (1.1)–(1.2) with $g_1 \equiv g_2 \equiv 0$ and the initial values $x_{1,0} = 0$ and $x_{2,0} = 0$. Figure 5.2 presents the errors for problem (1.1)–(1.2) with the initial values $x_{1,0} = 1$ and $x_{2,0} = 0$ and the inhomogeneous functions $g_1(t)$ and $g_2(t)$ defined in such a way that the exact solution to this problem is $x_1(t) = \cos t$, $x_2(t) = \sin t$, cp. [1, Sec. 203].

Figures 5.1 and 5.2 illustrate that scheme (1.3) converges faster than scheme (1.4). Note that condition (3.13) is satisfied by the scheme whose error is presented by the

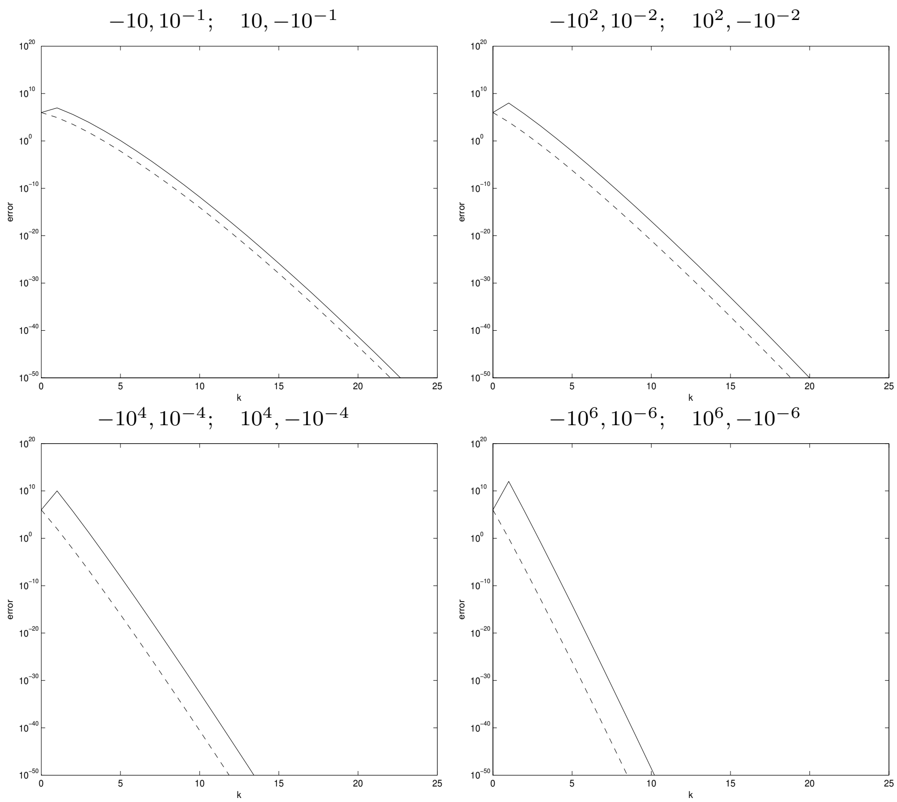

FIG. 5.1. *Numerical errors (5.1) using (1.3) (dotted) and numerical errors (5.2) using (1.4) (solid) for (1.1)–(1.2) in the homogeneous case with* $g_1 \equiv g_2 \equiv 0$.

dotted line and is not satisfied by the scheme whose error is presented by the solid line. This illustrates the conclusion derived in Corollary 3.2 in both homogeneous and non-homogeneous cases.

The errors presented in Figures 5.1 and 5.2 were obtained by running numerical experiments with different coefficients, which we list above each subfigure in the order a_{11}, a_{12}, a_{21}, a_{22}. Note that all these lists of coefficients satisfy condition (3.13) and, therefore, Corollary 3.2 implies that for all these problems (each problem with a different list of a_{ij}) scheme (1.3) convergerges faster than scheme (1.4).

Note that the error (5.1) (presented by the dotted lines), that is,

$$\Big(x_i(t_n) - x_i^k(t_n)\Big) + \Big(x_i^k(t_n) - x_{i,n}^k\Big),$$

is composed of two components: the error $x_i(t_n) - x_i^k(t_n)$ of the iteration and the error $x_i^k(t_n) - x_{i,n}^k$ of the ODE solver. Since integration in t is exact for the problem considered in Figure 5.1, the only non-zero component of (5.1) is the error $e_i^k(t_n)$ of the iteration presented in Figure 5.1. The same conclusion can be derived for the error (1.4) presented by the solid lines.

In Figure 5.2, the error (5.1) (dotted lines) has two non-zero components: the iteration error $e_i^k(t_n)$, which tends to zero as $k \to \infty$, and the time integration error $x_i^k(t_n) - x_{i,n}^k$ which is illustrated by the persistent horizontal lines in Figure 5.2. The same conclusion can be derived for the error (1.4) (solid lines).

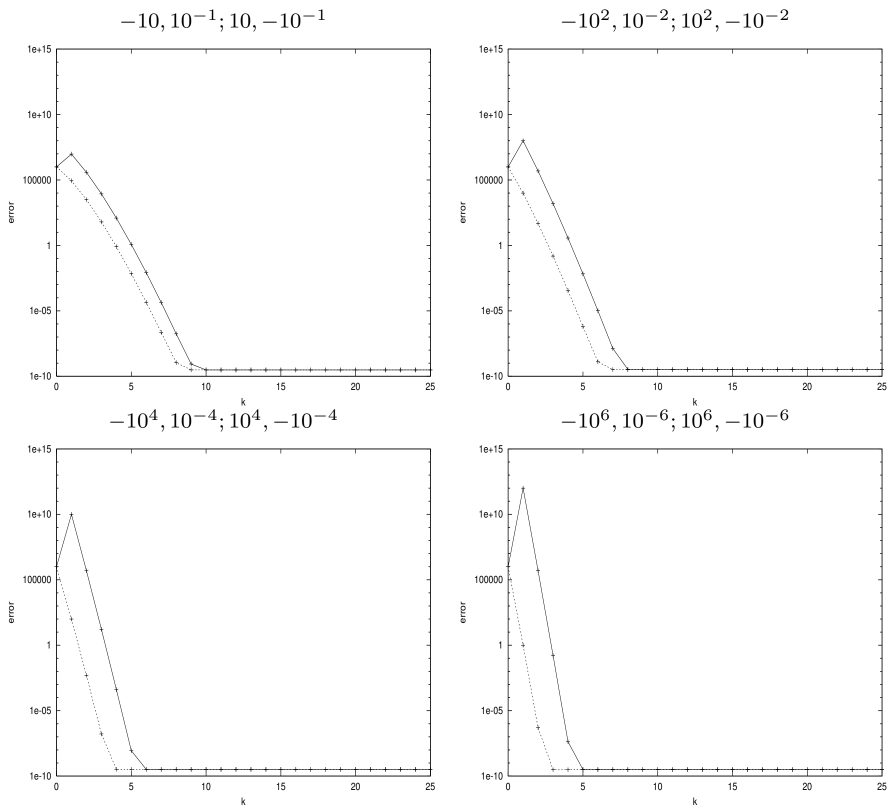

FIG. 5.2. *Numerical errors (5.1) using (1.3) (dotted) and numerical errors (5.2) using (1.4) (solid) for (1.1)–(1.2) in the non-homogeneous case with non-zero source functions $g_1(t)$ and $g_2(t)$.*

6. Concluding remarks and future work. In this paper, we addressed the question of whether the convergence of dynamic iterations depends on the magnitudes of the coefficients multiplied by present and previous iterates. From Sections 3, 4, and 5, we conclude that the order of the differential equations given in a larger dimensional system may slow down or speed up the convergence of the dynamic iterations applied to it. Therefore, we conclude that the order of the equations should be thoughtfully optimized before dynamic iterations are used. The conclusions derived from Sections 3, 4, and 5 give suggestions for choices of present and previous iterates in larger dimensional systems. Our future work [5] addresses the questions raised in this paper in the case of higher-dimensional systems.

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