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ON BEHAVIOR OF SOLUTIONS TO A CHEMOTAXIS SYSTEM WITH A NONLINEAR SENSITIVITY FUNCTION*

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Abstract. In this paper, we consider solutions to the following chemotaxis system with general sensitivity

$$\begin{cases} \tau u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) & \text{in } \Omega \times (0,\infty), \\ \eta v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

Here, τ and η are positive constants, χ is a smooth function on $(0,\infty)$ satisfying $\chi'(\cdot) > 0$ and Ω is a bounded domain of $\mathbf{R}^n$ $(n \geq 2)$.

It is well known that the chemotaxis system with direct sensitivity ($\chi(v) = \chi_0 v$, $\chi_0 > 0$) has blowup solutions in the case where $n \geq 2$. On the other hand, in the case where $\chi(v) = \chi_0 \log v$ with $0 < \chi_0 \ll 1$, any solution to the system exists globally in time and is bounded.

We present a sufficient condition for the boundedness of solutions to the system and some related systems.

Key words. Chemotaxis system, nonlinear sensitivity, time-global existence

AMS subject classifications. 35B45, 35K45, 35Q92, 92C17

1. Introduction.

We treat this system,

$$(PP) \begin{cases} \tau u_t = \nabla \cdot (\nabla u - u \nabla \chi(v)) & \text{in } \Omega \times (0,T), \\ \eta v_t = \Delta v - v + u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}$$

Here, η and τ (time constants) are positive constants, $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, χ is smooth on $(0,\infty)$ satisfying $\chi'(v) > 0$ $(v > 0)$, $\nu = \nu(x)$ is the outer normal unite vector at $x \in \partial\Omega$ and initial conditions u_0 and v_0 are positive in $\overline{\Omega}$.

This system (PP) is introduced to describe the aggregation of cellular slime molds. Normally the living things move around as individual amoebas, performing a simple random walk. But when the environmental situation worsens, they suddenly change their behavior and aggregate to a single milt-cellular body. During this aggregation process, a chemical signal is secreted by cells to guide the collective movements. Unknown functions u and v in (PP) represent the density of the living things and the chemical concentration, respectively.

The maximal principle guarantees that

$$u > 0 \quad \text{and} \quad v > 0 \quad \text{in } \Omega \times (0, T_{max}).$$

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Here, T_{max} is the maximal existence time of the classical solution (u, v). It follows from the boundary condition that

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for } t \in [0, T_{max}). \tag{1.1}$$

The function $\chi(v)$ represents the relation between the movement of cells and the chemical concentration. The term $u\chi'(v)\nabla v$ represents the flow due to the stimulus of the chemical substance. This property is so called chemotaxis. Then, the positivity of χ' means that the chemical substance is an attractant. When $\chi(v) = av$ and a is positive constant, we refer to this function as linear sensitivity function. The following functions are used in biological models frequently.

$$\chi(v) = av,\ a\log v,\ \frac{av}{b+v} \quad (a > 0,\ b > 0).$$

Except the linear sensitivity function, they satisfy that

$$\lim_{v\to\infty} \chi'(v) = 0. \tag{1.2}$$

This property represents saturation of the stimulus.

The following are our problem and our landmark.

Our problem

(i) Find a condition of sensitivity functions for the boundedness of solutions.

(ii) Find a condition of sensitivity functions for the existence of blowup solutions.

Our conjecture

(i) All solutions exist globally in time and are bounded, if one of the following two conditions holds:

· $\lim_{v\to\infty} \chi'(v) = 0$ and $n = 2$, or

· $\limsup_{v\to\infty} v\chi'(v) < \frac{n}{n-2}$ and $n \geq 3$.

(ii) There exist blowup solutions, if $\limsup_{v\to\infty} v\chi'(v) > \frac{n}{n-2}$ and $n \geq 3$.

Here, we say that a solution (u, v) to (PP) blows up at a time T, if

$$\limsup_{t\to T} \left(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}\right) = \infty.$$

2. Known results. In this section, we describe known results.

Firstly, we describe those in the case where $\chi(v)$ is a linear function.

THEOREM 2.1. *Suppose that $\chi(v) = \chi_1 v$, $\chi_1 > 0$, $\eta > 0$ and $\tau > 0$ and that Ω is a bounded domain of $\mathbf{R}^n$ $(n \geq 2)$ with smooth boundary. Then, the following hold:*

(i) *Suppose $n = 2$. Then, solutions exist globally in time and are bounded, if one of the following two conditions holds ([10]):*
  · *$\|u_0\|_{L^1(\Omega)} < 4\pi/\chi_1$, or*
  · *Ω is a bounded disk and u_0 is a radial function satisfying $\|u_0\|_{L^1(\Omega)} < 8\pi/\chi_1$.*

(ii) *If Ω is a bounded disk of $\mathbf{R}^2$ and u_0 is a radial function satisfying $\|u_0\|_{L^1(\Omega)} > 8\pi/\chi_1$, there are blowup solutions ([7]).*

(iii) *If Ω is a bounded ball of $\mathbf{R}^n$ $(n \geq 3)$, there are blowup solutions ([16]).*

Then, in the linear sensitivity case, the behavior of solutions depends on the constant χ_1 and the L^1 norm of the solution u if $n = 2$, and there exist blowup solutions for any positive constant χ_1 if $n \geq 3$.

When χ is a nonlinear function satisfying (1.2), classical solutions to (PP) satisfy the following properties.

THEOREM 2.2. *Suppose that Ω is a bounded domain of $\mathbf{R}^n$ $(n \geq 2)$ with smooth boundary. Then, the following hold:*

(i) *If $\chi'(v) \le a/(b+v)^p$, $a > 0$ and $p > 1$, solutions to (PP) exist globally in time and are bounded ([14, 5]).*

(ii) *If $\chi(v) = a\log v$ and $a < \sqrt{2/n}$, solutions to (PP) exist globally in time and are bounded ([15, 1]).*

The above sensitivity functions $\chi(v)$ satisfy that $\limsup_{v\to\infty} v\chi'(v) < \sqrt{2/n}$. Then, those conditions for global existence of classical solutions are not critical in the sense of our conjecture.

3. limiting systems. When the sensitivity function is a linear function, the condition for global existence of classical solutions is critical. The condition comes from a Lyapunov function and the Trudinger-Moser inequality ([10, 7, 16]). On the other hand, when the sensitivity function is not linear, it seems that conditions presented at the moment are not critical. In this case, we do not have any tools such as the Lyapunov function. Then, we consider the limiting system of (PP) as τ or $\eta = 0$. Because, those systems are simpler than (PP).

First, we consider the limiting system of (PP) as $\tau = 0$. For simplicity, we assume $\eta = 1$.

$$
(PE)\begin{cases} u_t = \nabla\cdot(\nabla u - u\nabla\chi(v)) & \text{in } \Omega\times(0,T),\\ 0 = \Delta v - v + u & \text{in } \Omega\times(0,T),\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}
$$

Classical solutions to this system satisfy the following properties.

THEOREM 3.1. *Suppose that Ω is a bounded domain of $\mathbf{R}^n$ ($n \ge 2$) with smooth boundary and that $\lim_{v\to\infty}\chi'(v) = 0$. Then, the following hold:*

(i) *If $n = 2$, solutions to (PE) exist globally in time and are bounded ([2]).*

(ii) *If $n \ge 3$, Ω is a bounded ball, u_0 is radial, $\chi(v) = a\log v$ and $a < 2/(n-2)$, then solutions to (PE) exist globally in time and are bounded ([11]).*

(iii) *If $n \ge 3$, Ω is a bounded ball, u_0 is radial, $\chi(v) = a\log v$ and $a > 2n/(n-2)$, there are blowup solutions to (PE) ([11]).*

We think that the assumption (1.2) is almost necessary condition in two dimensional case. Because, if Ω is a bounded disk of $\mathbf{R}^2$ and $\inf_{v>0}\chi'(v) > 0$, we can find blowup solutions to (PE) by using an argument similar to the one in [9].

In the case of $n \ge 3$, the conditions for global existence of solutions and existence of blowup solutions are not critical. Because, in our conjecture, the critical number is $n/(n-2)$.

Next, we consider the limiting system of (PP) as $\tau = 0$. For simplicity, we assume $\eta = 1$.

$$
(EP)\begin{cases} 0 = \nabla\cdot(\nabla u - u\nabla\chi(v)) & \text{in } \Omega\times(0,T),\\ v_t = \Delta v - v + u & \text{in } \Omega\times(0,T),\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ v(\cdot,0) = v_0 & \text{in } \Omega,\\ \displaystyle\int_\Omega u(x,t)dx = \lambda & \text{in } (0,T), \end{cases}
$$

where λ is a given positive constant. The last condition means the conservation of mass. Since solutions to the original system (PP) satisfy (1.1), then we impose this property also for solutions to (EP).

This system (EP) can be transformed into a non-local parabolic equation. In fact, the first equation and the boundary condition of (EP) guarantee that

$$\log u = \chi(v) + C,$$

where C is a constant. This and the last condition of (EP) ensure that

$$u = \frac{\lambda \exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx}.$$

Therefore, the system (EP) is equivalent to the following system,

$$(NLP) \begin{cases} v_t = \Delta v - v + \dfrac{\lambda \exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx} & \text{in } \Omega \times (0,T), \\ u = \dfrac{\lambda \exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}$$

Here, λ is a given positive constant.

Classical solutions to (NLP) satisfy the following properties.

THEOREM 3.2 ([12]). *Suppose that Ω is a bounded domain of $\mathbf{R}^n$ $(n \geq 2)$ with smooth boundary and that χ satisfies (1.2). Then, the following hold:*

(i) *If $n = 2$, solutions to (NLP) exist globally in time and are bounded.*

(ii) *If $n \geq 3$ and $\limsup_{v\to\infty} v\chi'(v) < n/(n-2)$, solutions to (NLP) exist globally in time and are bounded.*

(iii) *If $n \geq 3$, Ω is a bounded ball of $\mathbf{R}^n$, $\chi(v) = a \log v$ and $a > n/(n-2)$, there are blowup solutions to (NLP).*

In two dimensional case, (1.2) is the sufficient condition for the global existence of solutions to (PE) and (NLP). We expect that (1.2) is also the sufficient condition for (PP). In the case of $n \geq 3$, the threshold number $n/(n-2)$ in Theorem 3.2 is same as the one in our conjecture. Then, we think that this result is an evidence for our conjecture.

4. Our results. Considering results on the limiting systems mentioned in the previous section, we consider also almost limiting systems which are the systems (PP) in the case where τ or η is sufficient small.

In two dimensional case, classical solutions to those almost systems satisfy the following properties.

THEOREM 4.1 ([3, 4]). *Suppose that Ω is a bounded domain of $\mathbf{R}^2$ with smooth boundary and that $\lim_{v\to\infty} \chi'(v) = 0$. Then, the following hold:*

(i) *If Ω is a bounded disk, (u_0, v_0) is radial and η is sufficiently small, then solutions to (PP) exist globally in time and are bounded.*

(ii) *If Ω is convex and τ is sufficiently small, then solutions to (PP) exist globally in time and are bounded.*

REMARK 4.2. *If our conjecture is correct, the smallness of constants η and τ and the symmetry of (u_0, v_0) are not necessary in two dimensional case.*

In high dimensional case, classical solutions to the almost limiting system satisfy the following property.

THEOREM 4.3 ([4]). *If $n \geq 3$, Ω is a bounded and convex domain of $\mathbf{R}^n$, τ is sufficiently small and $\limsup_{v\to\infty} v\chi'(v) < n/(n-2)$, then solutions to (PP) exist globally in time and are bounded.*

REMARK 4.4. *If our conjecture is correct, we expect that the smallness of τ and the convexity of Ω are not necessary. Moreover, the research on blowup solutions is necessary.*

5. Idea of proof of Theorem 4.3. In this section, we describe the idea of the proof of Theorem 4.3. For simplicity, we assume $\eta = 1$.

LEMMA 5.1. *There exist positive constants $T_{min} > 0$ and $L > 0$ satisfying*

$$\|(u,v)\|_{C([0,T_{min}]\times\overline{\Omega})} \le L \quad \text{for } \tau \in (0,1].$$

LEMMA 5.2. *There exists a positive constant v_{min} satisfying*

$$v \ge v_{min} \quad \text{in } \Omega \times [0,T_{max}) \quad \text{for } \tau \in (0,1].$$

Lemma 5.1 comes from the standard energy argument and Lemma 5.2 comes from $\min_{\overline{\Omega}} v_0 > 0$ and $\|u\|_{L^1(\Omega)} > 0$.

Let $z = \dfrac{\exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx}$ and $w = \dfrac{u}{z}$. Those functions satisfy the following system,

$$(TPP)\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - v + w\dfrac{\exp(\chi(v))}{\int_\Omega \exp(\chi(v))\,dx} & \text{in } \Omega\times(0,T),\\ \tau\dfrac{\partial w}{\partial t} = \dfrac{1}{z}\nabla\cdot(z\nabla w) - \dfrac{\tau}{z}\dfrac{\partial z}{\partial t}w & \text{in } \Omega\times(0,T),\\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ v(\cdot,0) = v_0, \quad w(\cdot,0) = \displaystyle\int_\Omega \exp(\chi(v_0))dx\frac{u_0}{\exp(\chi(v_0))} & \text{in } \Omega. \end{cases}$$

Let $H = 2\max(\|u_0\|_{L^1(\Omega)}, \|w(0)\|_{L^\infty(\Omega)}, L)$ and let

$$S(\tau) = \sup\{T>0; \sup_{0<t<T}\|w(t)\|_{L^\infty(\Omega)} \le H\},$$

where L is the constant in Lemma 5.1.

LEMMA 5.3. *There exists a constant $\theta \in (0,1)$ such that*

$$\|v\|_{C^{2+\theta,(2+\theta)/2}(\Omega\times[0,S(\tau)])} < C(H),$$

where here and henceforth we will denote by $C(H)$ a positive generic constant (possibly changing from line to line) depending on H.

Proof. For $q > n/2$, $n/q < 2\beta < 2$, the semi-group property of the Laplacian guarantees that

$$\begin{aligned} \|v(t)\|_{L^\infty(\Omega)} &\le \|v_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}w(s)z(s)\|_{L^\infty(\Omega)}ds \\ &\le \|v_0\|_{L^\infty(\Omega)} + C\int_0^t \frac{e^{s-t}}{(t-s)^\beta}\|w(s)\|_{L^\infty(\Omega)}\|z(s)\|_{L^q(\Omega)}ds. \end{aligned}$$

Here and henceforth, we will denote by C a positive generic constant (possibly changing from line to line). We see that

$$\|z(t)\|_{L^q(\Omega)} = \frac{\|\exp(\chi(v))\|_{L^q(\Omega)}}{\|\exp(\chi(v))\|_{L^1(\Omega)}}$$

$$\leq \frac{\|\exp(\chi(v))\|_{L^1(\Omega)}^{1/q}\|\exp(\chi(v))\|_{L^\infty(\Omega)}^{(q-1)/q}}{\|\exp(\chi(v))\|_{L^1(\Omega)}}$$
$$\leq C(H)\frac{\|(v+1)^\mu\|_{L^\infty(\Omega)}^{(q-1)/q}}{(|\Omega|\exp(\chi(v_{min})))^{(q-1)/q}}.$$

Since $\limsup_{v\to\infty} v\chi'(v) < \mu < n/(n-2)$, we can take q and β such that

$$q > n/2, \quad n/q < 2\beta < 1 \quad \text{and} \quad \mu\frac{q-1}{q} < 1.$$

Then, we have that $\|v\|_{L^\infty(\Omega\times[0,S(\tau)])} \leq C(H)$. We obtain this lemma from this estimate and the parabolic regularity argument. □

By those and the parabolic regularity argument, we get a unique classical solution (v,w) to (TPP) in $\Omega\times[0,S(\tau)]$.

We will show that $S(\tau)=\infty$ if τ is sufficiently small. Assume to the contrary that $S(\tau)<\infty$ for $\tau\in(0,1]$. For an integer $J\geq 2$ and $j=0,1,2,\cdots,J$, put $T=S(\tau)/J$ and $z_j=z(jT)$. Then, for $j=0,1,2,\cdots,J-1$ and $t\in(jT,(j+1)T]$ we have

$$\tau w_t = \frac{1}{z_j}\nabla\cdot z_j\nabla w + \nabla\log\frac{z}{z_j}\cdot\nabla w - \tau\frac{z_t}{z}w \quad \text{in } \Omega.$$

For $j=0,1,2,\cdots,J-1$, put $\zeta=(t-jT)/\tau$, $W(x,\zeta)=w(x,t)$, $Z(x,\zeta)=z(x,t)$, $Z_0(x)=z_j(x)$ and $Q(x,\zeta)=z_t(x,t)/z(x,t)$. Then, those functions satisfy that

$$\frac{\partial W}{\partial\zeta} = \frac{1}{Z_0}\nabla\cdot Z_0\nabla W + \nabla\log\frac{Z}{Z_0}\cdot\nabla W - \tau QW \quad \text{in } \Omega\times(0,T/\tau).$$

Put $\mathcal{A}=Z_0^{-1}\nabla\cdot Z_0\nabla$ in Ω with $\partial\cdot/\partial\nu=0$ on $\partial\Omega$. The function W satisfies that

$$W(\zeta)=e^{\zeta\mathcal{A}}W(0)+\int_0^\zeta e^{(\zeta-\xi)\mathcal{A}}F(\xi)d\xi \quad \text{for } \zeta\in(0,T/\tau),$$

where

$$F=\nabla\log\frac{Z}{Z_0}\cdot\nabla W-\tau QW.$$

There exists a positive constant Λ depending on $\inf_\Omega Z_0$, $\|Z_0\|_\infty$ and Ω such that

$$\|\nabla e^{\zeta\mathcal{A}}W(0)\|_{L^q(\Omega)} \leq Ce^{-\zeta\Lambda}\|\nabla W(0)\|_{L^q(\Omega)} \quad \text{for } \zeta\in(0,T/\tau),$$
$$\int_0^\zeta \|\nabla e^{(\zeta-\xi)\mathcal{A}}\nabla\log\frac{Z(\xi)}{Z_0}\cdot\nabla W(\xi)\|_{L^q(\Omega)}d\xi \leq C(H)T^{\theta/2}e^{-\zeta\Lambda}\sup_{\xi\in[0,\zeta]}e^{\xi\Lambda}\|\nabla W(\xi)\|_{L^q(\Omega)}$$
$$\text{for } \zeta\in(0,T/\tau),$$
$$\int_0^\zeta \|\nabla e^{(\zeta-\xi)\mathcal{A}}\tau Q(\xi)W(\xi)\|_{L^q(\Omega)}d\xi \leq C(H)\tau^{(q-1)/q} \quad \text{for } \zeta\in(0,T/\tau)$$

and that

$$\sup_{\xi\in[0,T/\tau]}e^{\xi\Lambda}\|\nabla W(\xi)\|_{L^q(\Omega)}$$
$$\leq C\|\nabla W(0)\|_{L^q(\Omega)}+C(H)T^{\theta/2}\sup_{\xi\in[0,T/\tau]}e^{\xi\Lambda}\|\nabla W(\xi)\|_{L^q(\Omega)}+C(H)\tau^{(q-1)/q}e^{T\Lambda/\tau}.$$

Here, θ is the constant in Lemma 5.3. Taking $0 < \tau \ll T \ll 1$, we have that

$$\|\nabla w((j+1)T)\|_{L^q(\Omega)} \le Ce^{-T\Lambda/\tau}\|\nabla w(jT)\|_{L^q(\Omega)} + C(H)\tau^{(q-1)/q}$$
$$\text{for } j = 0, 1, 2, \cdots, J-1.$$

and that

$$\|\nabla w(jT)\|_{L^q(\Omega)} \le Ce^{-jT\Lambda/\tau}\|\nabla w(jT)\|_{L^q(\Omega)} + C(H)\tau^{(q-1)/q} \qquad \text{for } j = 1, 2, 3, \cdots, J.$$

Those estimates guarantee that

$$\|\nabla w(t-jT)\|_{L^q(\Omega)} \le Ce^{-(t-jT)\Lambda/\tau}\|\nabla w(jT)\|_{L^q(\Omega)} + C(H)\tau^{(q-1)/q}$$

for $t \in [jT, (j+1)T]$. Take $x(t) \in \overline{\Omega}$ such that $w(x(t), t) = \|w(t)\|_{L^\infty(\Omega)}$. We have that

$$\begin{aligned}\lambda &= \int_\Omega u(t)dx = \int_\Omega w(t)z(t)dx \\ &\ge \int_\Omega w(x(t), t)z(t)dx - \mathrm{diam}(\Omega)\int_\Omega \frac{|w(x,t) - w(x(t), t)|}{|x - x(t)|}z(t)dx,\end{aligned}$$

where $\mathrm{diam}(\Omega) = \sup\{|x-y|; x, y \in \Omega\}$. Then, we obtain that

$$\|w(t)\|_{L^\infty(\Omega)} \le \lambda + C(\Omega, H, q)\|\nabla w(t)\|_{L^q(\Omega)} < H \qquad \text{for } t \in [0, S(\tau)],$$

if τ is sufficiently small. This means that $S(\tau) = \infty$, a contradiction. Then, we have that $S(\tau) = \infty$ if τ is sufficiently small. Therefore, we get Theorem 4.3.

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