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AN ELEMENTARY PROOF OF ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $U'' = VU$ *

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Abstract. We provide an elementary proof of the asymptotic behavior of solutions of second order differential equations without successive approximation argument.

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1. Introduction. The asymptotic behavior of the solutions of the ordinary differential equation

$$u''(x) = V(x)u(x), \qquad x \in (0, \infty) \tag{1.1}$$

is an important tool in various fields of mathematics and mathematical physics, in particular when special functions are involved. It can be found in [3, Section 6.2] and partially in [1, Chapter 10] and in [2, Chapter IV] that if $V(x) = f(x) + g(x)$, that is,

$$u''(x) = \big(f(x) + g(x)\big)u(x), \qquad x \in (0, \infty) \tag{1.2}$$

and

$$\psi_{f,g} := |f|^{-\frac{1}{4}} \left(-\frac{d^2}{dx^2} + g\right) |f|^{-\frac{1}{4}} \text{ is absolutely integrable in } (0, \infty), \tag{1.3}$$

then two solutions of (1.2) behave like

$$u(x) \approx |f|^{-1/4} e^{\pm \int_0^x |f(s)|^{1/2}\, ds}, \quad u(x) \approx |f|^{-1/4} e^{\pm i \int_0^x |f(s)|^{1/2}\, ds}.$$

The proof is usually done treating first the cases $f = \pm 1$ and then reducing to them the general case, by the Liouville transformation. We follow the same approach but simplify the cases $f = \pm 1$ by using Gronwall's Lemma, instead of successive approximations. In order to keep the exposition at an elementary level, we avoid also Lebesgue integration and dominated convergence (which could shorten some proofs); note that we only use the notation $f \in L^1(I)$ when f is absolutely integrable in I. We consider both the behavior at infinity and near isolated singularities and apply the results to Bessel functions. We also recall that the general case

$$u''(x) + g(x)u'(x) = V(x)u(x)$$

can be reduced to the form (1.1) (with another V) by writing $u = \frac{1}{2}(\exp \int g)v$.

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This kind of analysis can be applied to the spectral analysis for Schrödinger operator with singular potentials (for example $S = -\Delta + V(|x|)$ with $V(r) \sim r^{-\delta}$ near the origin). Actually, the essential selfadjointness of the Schrödinger operator S can be treated by using the limit-point and limit-circle criteria (see e.g., Reed–Simon [4]) which require the behavior of two solutions to $u - u'' + \frac{N-1}{r}u + Vu = 0$. The behavior of two solutions above leads also to resolvent estimates for S. From this view-piont, the elemental consideration in the present paper helps in understanding various spectral phenomena for second-order differential operators.

2. Behavior near infinity in the simplest cases. First we consider the cases $f \equiv 1$ and $f \equiv -1$ and we prove the following results to which the general case reduces.

PROPOSITION 2.1. *If $f = 1$, $g \in L^1(0,\infty)$, then there exist two solutions u_1 and u_2 of (1.2) such that, as $x \to \infty$,*

$$e^{-x}u_1(x) \to 1, \qquad e^{-x}u_1'(x) \to 1, \tag{2.1}$$

$$e^{x}u_2(x) \to 1, \qquad e^{x}u_2'(x) \to -1. \tag{2.2}$$

PROPOSITION 2.2. *If $f = -1$, $g \in L^1(0,\infty)$, then there exist two solutions v_1 and v_2 of (1.2) such that, as $x \to \infty$,*

$$e^{-ix}u_1(x) \to 1, \qquad e^{-ix}u_1'(x) \to i, \tag{2.3}$$

$$e^{ix}u_2(x) \to 1, \qquad e^{ix}u_2'(x) \to -i. \tag{2.4}$$

By variation of parameters, every solution of (1.2) can be written as

$$u(x) = c_1e^{\zeta x} + c_2e^{-\zeta x} + \frac{1}{2\zeta}\int_a^x (e^{\zeta(x-s)} - e^{-\zeta(x-s)})g(s)u(s)\,ds, \quad x \in [a,\infty), \tag{2.5}$$

with $c_1, c_2 \in \mathbb{C}$, $\zeta = 1, i, -i$ and $a > 0$. In the following Lemma we choose $c_1 = 1, c_2 = 0$ to construct a solution which behaves like $e^{\zeta x}$ as $x \to \infty$, $\zeta = 1, i, -i$.

LEMMA 2.3. *Let $\zeta \in \{1, i, -i\}$, $a > 0$ and $g \in L^1(a,\infty)$. If $u \in C^2([a,\infty))$ satisfies*

$$u(x) = e^{\zeta x} + \frac{1}{2\zeta}\int_a^x (e^{\zeta(x-s)} - e^{-\zeta(x-s)})g(s)u(s)\,ds, \qquad x \in [a,\infty),$$

then $z(x) := e^{-\zeta x}u(x)$ satisfies

$$|z(x)| \le e^{\int_a^x |g(r)|\,dr}, \qquad x \in [a,\infty) \tag{2.6}$$

$$\|zg\|_{L^1(a,\infty)} \le e^{\|g\|_{L^1(a,\infty)}} - 1. \tag{2.7}$$

Proof. Note that

$$z(x) = 1 + \frac{1}{2\zeta}\int_a^x (1 - e^{-2\zeta(x-s)})g(s)z(s)\,ds, \quad x \in [a,\infty).$$

Since $|1 - e^{-2\zeta(x-s)}| \le 2$ for $s \le x$, we see that for $x \ge a$,

$$|z(x)| \le 1 + \left|\frac{1}{2\zeta}\int_a^x (1 - e^{-2\zeta(x-s)})g(s)z(s)\,ds\right| \le 1 + \int_a^x |g(s)|\,|z(s)|\,ds.$$

Thus Gronwall's lemma implies (2.6), in particular z is bounded on $[a,\infty)$ and then $zg \in L^1(a,\infty)$. Moreover we have

$$\|zg\|_{L^1(a,\infty)} \le \int_a^\infty |g(s)|\, e^{\int_a^s |g(r)|\,dr}\, ds = e^{\|g\|_{L^1(a,\infty)}} - 1.$$

□

Proof of Proposition 2.1. Let $a > 0$ such that $\|g\|_{L^1(a,\infty)} < \log 2$ and let u be in Lemma 2.3 with $\zeta = 1$. Then u is one solution of (1.2) with $f = 1$. Set $z(x) = e^{-x}u(x)$. Then noting that as $x \to \infty$,

$$\begin{aligned}\left|\int_a^x e^{-2(x-s)}g(s)z(s)\,ds\right| &\le \int_a^{\frac{a+x}{2}} e^{-2(x-s)}|g(s)z(s)|\,ds + \int_{\frac{a+x}{2}}^x |g(s)z(s)|\,ds \\ &\le e^{-x+a}\|gz\|_{L^1(a,\infty)} + \|gz\|_{L^1(\frac{a+x}{2},\infty)} \to 0,\end{aligned}$$

we see that z satisfies

$$\begin{aligned} z(x) \to z_\infty &:= 1 + \int_a^\infty g(s)z(s)\,ds \quad \text{as } x\to\infty, \\ z'(x) &= \int_a^x e^{-2(x-s)}g(s)z(s)\,ds \to 0 \quad \text{as } x \to \infty.\end{aligned}$$

By (2.7), we deduce that $\|zg\|_{L^1(a,\infty)} < 1$. Therefore $|z_\infty - 1| \le \|zg\|_{L^1(a,\infty)} < 1$ and hence $z_\infty \ne 0$. The function $u_1(x) := z_\infty^{-1}e^x z(x)$ satisfies (2.1). Moreover, since u_1^{-2} is integrable near ∞, another solution of (1.2) is given by

$$u_2(x) = 2u_1(x)\int_x^\infty \frac{1}{u_1(s)^2}\,ds. \tag{2.8}$$

Integrating by parts we deduce that, as $x \to \infty$,

$$\begin{aligned} e^x u_2(x) &= 2z_\infty e^{2x} z(x)\int_x^\infty \frac{1}{e^{2s}[z(s)]^2}\,ds \\ &= z_\infty e^{2x}z(x)\left(-\left[\frac{1}{e^{2s}[z(s)]^2}\right]_{s=x}^{s=\infty} - 2\int_x^\infty \frac{z'(s)}{e^{2s}[z(s)]^3}\,ds\right) \to 1\end{aligned}$$

and

$$[e^x u_2(x)]' = 2z_\infty e^{2x}z'(x)\int_x^\infty \frac{1}{e^{2s}[z(s)]^2}\,ds + 2e^x u_2(x) - \frac{2z_\infty}{z(x)} \to 0.$$

□

Proof of Proposition 2.2. Let $a > 0$ such that $\|g\|_{L^1(a,\infty)} < \log 2$ and let $\tilde u_1$ and $\tilde u_2$ be as in Lemma 2.3 with $\zeta = i$ and with $\zeta = -i$, respectively. Noting that both $\tilde u_1$ and $\tilde u_2$ satisfy (1.2) with $f = -1$, and setting $z_1(x) = e^{-ix}\tilde u_1(x)$ and $z_2(x) = e^{ix}\tilde u_2(x)$, we have as $x \to \infty$

$$\begin{aligned} e^{2ix}\left(z_1(x) - 1 - \frac{1}{2i}\int_a^\infty g(s)z_1(s)\,ds\right) &\to \frac{1}{2i}\int_a^\infty e^{2is}g(s)z_1(s)\,ds, \\ e^{-2ix}\left(z_2(x) - 1 + \frac{1}{2i}\int_a^\infty g(s)z_2(s)\,ds\right) &\to -\frac{1}{2i}\int_a^\infty e^{-2is}g(s)z_2(s)\,ds\end{aligned}$$

and

$$e^{2ix}z_1'(x) \to \int_a^\infty e^{2is}g(s)z_1(s)\,ds, \qquad e^{-2ix}z_2'(x) \to \int_a^\infty e^{-2is}g(s)z_2(s)\,ds.$$

It follows that $\tilde{u}_1 \approx \xi_1 e^{ix} + \xi_2 e^{-ix}$, $\tilde{u}_1' \approx i\xi_1 e^{ix} - i\xi_2 e^{-ix}$ and $\tilde{u}_2 \approx \eta_1 e^{ix} + \eta_2 e^{-ix}$, $\tilde{u}_2' \approx i\eta_1 e^{ix} - i\eta_2 e^{-ix}$ as $x \to \infty$ where

$$\xi_1 = 1 + \frac{1}{2i}\int_a^\infty g(s)z_1(s)\,ds, \qquad \xi_2 = -\frac{1}{2i}\int_a^\infty e^{2is}g(s)z_1(s)\,ds,$$

and similarly for η_1, η_2. From (2.7) we see that $|\xi_1| > 1/2$, $|\xi_2| < 1/2$, $|\eta_1| < 1/2$ and $|\eta_2| > 1/2$ and hence $|\xi_1\eta_2 - \xi_2\eta_1| > 0$ and $\tilde{u}_1$ and $\tilde{u}_2$ are linearly independent. Therefore we can construct solutions u_1 and u_2 which satisfy (2.3) and (2.4), respectively. □

We consider now the case $f = 0$, assuming extra conditions on g.

Proposition 2.4. *Assume that $xg \in L^1(0,\infty)$. Then there exist two solutions u_1 and u_2 of*

$$u''(x) = g(x)u(x) \tag{2.9}$$

such that

$$x^{-1}u_1(x) \to 1, \qquad u_1'(x) \to 1,$$
$$u_2(x) \to 1, \qquad xu_2'(x) \to 0$$

as $x \to \infty$, respectively.

Proof. Set $u(x) := xz(x)$. Then $z'' + (2/x)z' = gz$ and, assuming $z'(a) = 0$ we obtain

$$z'(x) = x^{-2}\int_a^x s^2 g(s)z(s)\,ds. \tag{2.10}$$

Then assuming $z(a) = 1$

$$\begin{aligned} |z(x) - 1| &\le \int_b^x t^{-2}\left(\int_a^t s^2|g(s)z(s)|\,ds\right)dt \\ &= \int_a^x \left(\int_s^x t^{-2}\,dt\right)s^2|g(s)z(s)|\,ds \le \int_a^x s|g(s)z(s)|\,ds. \end{aligned} \tag{2.11}$$

Gronwall's lemma yields

$$|z(x)| \le e^{\int_a^x s|g(s)|\,ds}$$

hence z is bounded and $z' \in L^1(a,\infty)$ by (2.10). As in the proof of Proposition 2.1, $z(x) \to z_\infty \neq 0$ if a is sufficiently large. Moreover, since as $x \to \infty$,

$$|xz'(x)| \le \sqrt{\frac{a}{x}}\int_a^{\sqrt{ax}} s|g(s)z(s)|\,ds + \int_{\sqrt{ax}}^x s|g(s)z(s)|\,ds \to 0,$$

$u_1(x) := z_\infty^{-1}xz(x)$ satisfies the statement. Another solution u_2 of (1.2) is given by

$$u_2(x) := u_1(x)\int_x^\infty \frac{1}{u_1(s)^2}\,ds.$$

As in the proof of Proposition 3.1 we can verify that u_2 satisfies $u_2(x) \to 1$ and $xu_2'(x) \to 0$ as $x \to \infty$. □

Observe the integrability condition for xg near ∞ is necessary. In fact, if $g(x) = cx^{-2}$ the above equation has solutions x^α if $\alpha^2 - \alpha = c$.

3. Behavior near infinity in the general case. We recall that the function $\psi_{f,g}$ is defined in (1.3) and set $v_j(x) = |f|^{1/4}u_j(x)$, $j = 1,2$ if u_1, u_2 are solutions of (1.2). The hypothesis $|f|^{1/2}$ not summable near ∞ guarantees that the Liouville transformation Φ of Lemma 3.3 maps (a,∞) onto $(0,\infty)$, so that the results of the previous section apply. When it is not satisfied Φ maps (a,∞) onto a bounded interval $(0,b)$ and the behavior of the solutions of (3.5) near b is more elementary (in some cases one can use Proposition 2.4).

PROPOSITION 3.1. *Assume that $f(x) > 0$ in (a,∞), $|f|^{1/2} \notin L^1(a,\infty)$ and $\psi_{f,g} \in L^1(a,\infty)$. Then there exist two solutions u_1 and u_2 of (1.2) such that as $x \to \infty$*

$$e^{-\int_a^x |f(r)|^{1/2}dr} v_1(x) \to 1, \qquad |f(x)|^{-1/2} e^{-\int_a^x |f(r)|^{1/2}dr} v_1'(x) \to 1, \tag{3.1}$$
$$e^{\int_a^x |f(r)|^{1/2}dr} v_2(x) \to 1, \qquad |f(x)|^{-1/2} e^{\int_a^x |f(r)|^{1/2}dr} v_2'(x) \to -1. \tag{3.2}$$

PROPOSITION 3.2. *Assume that $f(x) < 0$ in (a,∞), $|f|^{1/2} \notin L^1(a,\infty)$ and $\psi_{f,g} \in L^1(a,\infty)$. Then there exists two solutions u_1 and u_2 of (1.2) such that asx $\to \infty$*

$$e^{-i\int_a^x |f(r)|^{1/2}dr} v_1(x) \to 1, \qquad |f(x)|^{-1/2} e^{-i\int_a^x |f(r)|^{1/2}dr} v_1'(x) \to i, \tag{3.3}$$
$$e^{i\int_a^x |f(r)|^{1/2}dr} v_2(x) \to 1, \qquad |f(x)|^{-1/2} e^{i\int_a^x |f(r)|^{1/2}dr} v_2'(x) \to -i. \tag{3.4}$$

The proof is based on the well-known Liouville transformation that we recall below.

LEMMA 3.3. *Let $a > 0$ and assume that $f \in C^2([a,\infty))$ satisfies $|f(x)| > 0$, $|f|^{1/2} \notin L^1(a,\infty)$. Define $\Phi \in C^2([a,\infty))$ by*

$$\Phi(x) := \int_a^x |f(r)|^{1/2}\,dr, \quad x \in [a,\infty).$$

Then $\Phi^{-1} : [0,\infty) \to [a,\infty)$ and if u satisfies (1.2) the function

$$w(y) := |f(\Phi^{-1}(y))|^{1/4} u(\Phi^{-1}(y)), \quad y \in [0,\infty)$$

satisfies

$$w''(y) = \left(\frac{f(\Phi^{-1}(y))}{|f(\Phi^{-1}(y))|} + \frac{\psi_{f,g}(\Phi^{-1}(y))}{|f(\Phi^{-1}(y))|^{1/2}} \right) w(y). \tag{3.5}$$

Proof. Note that $\Phi'(x) = |f(x)|^{1/2}$ and $\frac{d(\Phi^{-1})}{dy}(y) = |f(\Phi^{-1}(y))|^{-1/2}$. Setting

$w(y) = |f(\Phi^{-1}(y))|^{1/4}u(\Phi^{-1}(y))$ (and using $\xi = \Phi^{-1}(y)$ for simplicity), we have

$$\begin{aligned} w'(y) &= \frac{d}{dx}\left[|f|^{1/4}u\right](\xi)\frac{d(\Phi^{-1})}{dy}(y) \\ &= |f(\xi)|^{-1/4}u'(\xi) + \left[|f|^{-1/2}\frac{d}{dx}|f|^{1/4}\right](\xi)u(\xi) \\ &= \left[|f|^{-1/4}u' - \frac{d}{dx}(|f|^{-1/4})u\right](\xi), \\ w''(y) &= \frac{d}{dx}\left[|f|^{-1/4}u' - \frac{d}{dx}(|f|^{-1/4})u\right](\xi)\frac{d(\Phi^{-1})}{dy}(y) \\ &= |f(\xi)|^{-3/4}u''(\xi) - \left[|f|^{-1/2}\frac{d^2}{dx^2}|f|^{-1/4}\right](\xi)u(\xi) \\ &= |f(\xi)|^{-1}(f(\xi)+g(\xi))w(y) - \left[|f|^{-3/4}\frac{d^2}{dx^2}|f|^{-1/4}\right](\xi)w(y). \end{aligned}$$

Thus we obtain (3.5). □

Proof. [Proof of Propositions 3.1 and 3.2] It suffices to apply Propositions 2.1 and 2.2 to the respective cases $f > 0$ and $f < 0$. Set $h(y) = \psi_{f,g}(\Phi^{-1}(y))|f(\Phi^{-1}(y))|^{-1/2}$. Then

$$\int_0^b |h(y)|\,dy = \int_a^\infty |\psi_{f,g}(x)|\,dx.$$

Therefore Propositions 2.1 and 2.2 are applicable to $w'' = \pm w + hw$, respectively. Finally, using Lemma 3.3 and taking $u(x) = |f(x)|^{-1/4}w(\Phi(x))$, we obtain the respective assertions in Propositions 3.1 and 3.2. □

4. Behavior near interior singularities. If f and g have local singularities at x_0, then the behavior of solutions near x_0 is also considerable. For simplicity, we take $x_0 = 0$. The following propositions are meaningful when $|f|^{1/2}$ is not integrable near 0, in particular when $|f|^{1/2} = cx^{-1}$. We recall that $v_j(x) = |f(x)|^{1/4}u_j(x)$, $j = 1, 2$.

PROPOSITION 4.1. *Assume that $f(x) > 0$ in $(0,\infty)$ and $\psi_{f,g} \in L^1(0,\infty)$. Then there exist two solutions u_1 and u_2 of* (1.2) *such that as $x \downarrow 0$*

$$\begin{aligned} e^{-\int_x^1 |f(r)|^{1/2}dr}v_1(x) &\to 1, \qquad |f(x)|^{-1/2}e^{-\int_x^1 |f(r)|^{1/2}dr}v_1'(x) \to -1, \\ e^{\int_x^1 |f(r)|^{1/2}dr}v_2(x) &\to 1, \qquad |f(x)|^{-1/2}e^{\int_x^1 |f(r)|^{1/2}dr}v_2'(x) \to 1. \end{aligned}$$

PROPOSITION 4.2. *Assume that $f(x) < 0$ in $(0,\infty)$ and $\psi_{f,g} \in L^1(0,\infty)$. Then there exist two solutions u_1 and u_2 of* (1.2) *such that as $x \downarrow 0$*

$$\begin{aligned} e^{-\int_x^1 |f(r)|^{1/2}dr}v_1(x) &\to 1, \qquad |f(x)|^{-1/2}e^{-\int_x^1 |f(r)|^{1/2}dr}v_1'(x) \to -i, \\ e^{\int_x^1 |f(r)|^{1/2}dr}v_2(x) &\to 1, \qquad |f(x)|^{-1/2}e^{\int_x^1 |f(r)|^{1/2}dr}v_2'(x) \to i. \end{aligned}$$

Proof of Propositions 4.1 and 4.2. Setting $w(s) := su(s^{-1})$ we see that

$$\begin{aligned} w''(s) &= s^{-3}u''(s^{-1}) \\ &= s^{-3}(f(s^{-1}) + g(s^{-1}))u(s^{-1}) = s^{-4}(f(s^{-1}) + g(s^{-1}))w(s). \end{aligned}$$

Let $\tilde{f}(s) := s^{-4}f(s^{-1})$ and $\tilde{g}(s) := s^{-4}g(s^{-1})$. Noting that

$$\begin{aligned}\psi_{\tilde{f},\tilde{g}}(s) &= s|f(s^{-1})|^{-1/4}\left(-\frac{d^2}{ds^2} + s^{-4}g(s^{-1})\right)\left(s|f(s^{-1})|^{-1/4}\right)\\ &= s^{-2}|f(s^{-1})|^{-1/4}\left(-\frac{d^2}{dx^2}|f|^{-1/4} + g|f|^{-1/4}\right)(s^{-1})\\ &= s^{-2}\psi_{f,g}(s^{-1}),\end{aligned}$$

we have $\psi_{\tilde{f},\tilde{g}} \in L^1((0,\infty))$, and hence Propositions 3.1 and 3.2 can be applied. Since

$$\int_1^s |\tilde{f}(r)|^{1/2}dr = \int_{1/s}^1 |f(t)|^{1/2}dt,$$

we obtain the respective assertions in Propositions 4.1 and 4.2. □

5. Examples from special functions. Some examples illustrate the application of the results of the previous sections.

EXAMPLE 1 (Modified Bessel functions). *We consider the modified Bessel equation of order* ν

$$u'' + \frac{u'}{r} - \left(1 + \frac{\nu^2}{r^2}\right)u = 0, \tag{5.1}$$

All solutions of (5.1) *can be written through the modified Bessel functions* I_ν *and* K_ν. *Both* I_ν *and* K_ν *are positive,* I_ν *is monotone increasing and* K_ν *is monotone decreasing (see e.g., [3, Theorem 7.8.1]). Proposition 2.1 and Proposition 4.1 give the precise behavior of* I_ν *and* K_ν *near* ∞ *and near* 0, *respectively. In fact,* (5.1) *can be written as*

$$(\sqrt{r}u)'' = \left(1 + \frac{4\nu^2 - 1}{4r^2}\right)(\sqrt{r}u). \tag{5.2}$$

Since $1/r^2$ *is integrable near* ∞, *choosing* $f = 1$ *and* $g = \frac{4\nu^2-1}{4r^2}$, *we see from Proposition 2.1 that*

$$\sqrt{r}e^{-r}I_\nu(r) \to c_1 \neq 0 \quad \textit{and} \quad \sqrt{r}e^{r}K_\nu(r) \to c_2 \neq 0 \qquad \textit{as } r \to \infty.$$

Moreover, if $\nu \neq 0$, *then choosing* $f(r) = \frac{\nu^2}{r^2}$ *and* $g(r) = 1 - \frac{1}{4r^2}$, *that is,* $\psi_{f,g}(r) = r/\nu$, *from Proposition 4.1 we have*

$$r^{-\nu}I_\nu(r) \to c_3 \neq 0 \quad \text{and} \quad r^{\nu}K_\nu(r) \to c_4 \neq 0 \qquad \textit{as } r \downarrow 0.$$

If $\nu = 0$, *then putting* $w(s) = u(e^{-s})$ *we obtain*

$$w''(s) = e^{-2s}w(s), \qquad s \in \mathbb{R}.$$

Therefore using Proposition 2.4 with $\tilde{g}(s) = e^{-2s}$ *and taking* $u(x) = w(-\log x)$, *we have*

$$I_0(r) \to c_5 \neq 0 \quad \textit{and} \quad |\log r|^{-1}K_0(r) \to c_6 \neq 0 \qquad \textit{as } r \downarrow 0.$$

EXAMPLE 2 (Fundamental solution of $\lambda-\Delta$). *For $n \geq 3$, $\lambda \geq 0$ the fundamental solution v_λ of $\lambda - \Delta$ can be computed by integrating the heat kernel:*

$$v_\lambda(r) = \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\lambda t - \frac{r^2}{4t}}\, dt,$$

where $r = |x|$. Clearly $v_\lambda(r) \leq v_0(r) = cr^{2-n}$, $v_\lambda(r) \to 0$ as $r \to \infty$. The function $v = v_\lambda$ satisfies

$$v'' + \frac{n-1}{r} v' = \lambda v$$

or, setting $v = r^{(1-n)/2} w$,

$$w'' = \left(\lambda + \frac{n^2-1}{4r^2}\right) w.$$

Proceeding as in the example above we see that $r^{2-n} v(r) \to c_1 \neq 0$ as $r \to 0$ and $r^{(n-1)/2} e^{\sqrt{\lambda} r} v(r) \to c_2 \neq 0$ as $r \to \infty$.

EXAMPLE 3 (Bessel functions). *Next we consider the Bessel equation of order ν*

$$u'' + \frac{u'}{r} + \left(1 - \frac{\nu^2}{r^2}\right) u = 0, \tag{5.3}$$

or equivalently,

$$(\sqrt{r} u)'' = \left(-1 + \frac{4\nu^2 - 1}{4r^2}\right)(\sqrt{r} u).$$

All solutions of (5.3) can be written through the Bessel functions J_ν and Y_ν. As in Example 1, from Propositions 4.1 (for $\nu > 0$) and 2.4 (for $\nu = 0$) we obtain the behavior of J_ν and Y_ν near 0

$$r^{-\nu} J_\nu(r) \to c_1 \neq 0, \quad \text{and} \quad r^\nu Y_\nu(r) \to c_2 \neq 0 \qquad \textit{as } r \downarrow 0$$

and if $\nu = 0$,

$$|\log r| J_0(r) \to c_3 \neq 0, \quad \text{and} \quad Y_0(r) \to c_4 \neq 0 \qquad \textit{as } r \downarrow 0.$$

In view of Proposition 2.2 the behavior of J_ν and Y_μ near ∞ is given by

$$|\sqrt{r} J_\nu(r) - c_5 \cos(r + \theta_1)| \to 0, \quad \text{and} \quad |\sqrt{r} Y_\nu(r) - c_6 \cos(r + \theta_2)| \to 0,$$

as $r \to \infty$, where $c_5 \neq 0$, $c_6 \neq 0$ and $\theta_1, \theta_2 \in [0, \pi)$ satisfy $\theta_1 \neq \theta_2$.

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