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MULTIPLE POSITIVE SOLUTIONS FOR A p-LAPLACE CRITICAL PROBLEM (p > 1), VIA MORSE THEORY

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Abstract. We consider the quasilinear elliptic problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is bounded in $\mathbf{R}^N$, $N \geq p^2$, $1 < p \leq q < p^*$, $p^* = \frac{Np}{N-p}$, $\lambda > 0$ is a parameter.
Denoting by $\mathcal{P}_1(\Omega)$ the Poincaré polynomial of Ω, we state that, for any $p > 1$, there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, either (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, (P_λ) can be approached by a sequence of problems $(P_n)_{n\in\mathbf{N}}$, each having at least $\mathcal{P}_1(\Omega)$ distinct solutions. These results have been proved in [12] only as regards the case $p \geq 2$, while they will be completely proved in the forthcoming work [13] in the case $p \in (1, 2)$.
Note that, when $p \in (1, 2)$, the Euler functional associated to (P_λ) is never C^2, so the approach already used for $p \geq 2$ fails. This problem will be faced exploiting recent results given in [7] and [8].

Key words. Morse theory in Banach spaces, p-laplace equations, critical exponent, critical groups, multiplicity, perturbation results, functionals with lack of smoothness, generalized Morse index

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1. Introduction. Let us consider the quasilinear elliptic problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p\, u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary, $N \geq p^2$, $1 < p \leq q < p^*$, $p^* = Np/(N-p)$, $\lambda > 0$ is a parameter.

This problem was introduced by Brezis and Nirenberg in the famous paper [3], in the semilinear case in which $p = q = 2$. Their result was later extended to the quasilinear case $p = q \neq 2$ by Azorero and Peral [2], and Guedda and Veron [14]. Alves and Ding in [1] achieved a multiplicity result for the quasilinear problem (P_λ), under the hypothesis $p \geq 2$. More precisely, they proved that, if $N \geq p^2$ and $2 \leq p \leq q < p^*$, then (P_λ) has at least $cat(\Omega)$ solutions, where $cat(\Omega)$ denotes the Ljusternik-Schnirelmann category of Ω in itself.

Our goal is to exploit Morse theory in order to improve the previous result and extend it to the case $p > 1$.

Solutions to (P_λ) are critical points of the energy functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{q}\int_\Omega (u^+)^q\, dx - \frac{1}{p^*}\int_\Omega (u^+)^{p^*}\, dx.$$

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When $p \neq 2$, $W_0^{1,p}(\Omega)$ is a Banach space, but not a Hilbert one, and this brings a lot of problems when trying to apply Morse theory.

Furthermore, when $p \in (1,2)$, I_λ is just a C^1 functional, not C^2.

2. Morse theory: recalls and considerations.

We need to recall some notions about this topic (cf. [5, 6]).
In the sequel, let $\mathbf{K}$ be a field.

Definition 2.1. *For any $B \subset A \subset \mathbf{R}^n$, we denote $\mathcal{P}_t(A,B)$ the Poincaré polynomial of the topological pair (A,B), defined by*

$$\mathcal{P}_t(A,B) = \sum_{k=0}^{+\infty} \dim H^k(A,B)\, t^k.$$

where $H^k(A,B)$ stands for the k-th Alexander-Spanier relative cohomology group of (A,B), with coefficient in $\mathbf{K}$; we also set $H^k(A) = H^k(A,\emptyset)$ so that

$$\mathcal{P}_t(A) = \mathcal{P}_t(A,\emptyset) \tag{2.1}$$

is the Poincaré polynomial of A.

Definition 2.2. *Let Y be a Banach space and J a C^1 functional on Y. Let C be a closed subset of Y. A sequence (u_n) in C is a Palais-Smale sequence for J if $\|J(u_n)\| \leq M$ uniformly in n, while $J'(u_n) \to 0$ as $n \to +\infty$.*
We say that J satisfies (P.S.) on C, if any Palais-Smale sequence in C has a strongly convergent subsequence.
Let $c \in \mathbf{R}$. We say that J satisfies $(P.S.)_c$ if any sequence (u_n) in Y, such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to +\infty$, has a strongly convergent subsequence.

Definition 2.3. *Let Y be a Banach space, $J \in C^2(Y,\mathbf{R})$ and z a critical point of J. The Morse index of J in z is the supremum of the dimensions of the subspaces of Y on which $J''(z)$ is negative definite. It is denoted by $m(J,z)$. The large Morse index of J in z is the supremum of the dimensions of the subspaces of Y on which $J''(z)$ is negative semidefinite. It is denoted by $m^*(J,z)$.*

Definition 2.4. *Let Y be a Banach space, $J \in C^1(Y,\mathbf{R})$ and z a critical point of J. We call*

$$C_q(J,z) = H^q(J^c, J^c \setminus \{z\})$$

the q-th critical group of J at z, where $c = J(z)$, $q \in \mathbf{N}$ and $H^q(A,B)$ stands for the q-th Alexander-Spanier cohomology group of the pair (A,B) with coefficients in $\mathbf{K}$.
We call multiplicity of z the number

$$\sum_{q=0}^{+\infty} \dim C_q(J,z). \tag{2.2}$$

In the context of a C^1 functional defined on a Banach space, a topological version of Morse relation holds.

THEOREM 2.5. *Let Y be a Banach space, $J \in C^1(Y, \mathbf{R})$ and z a critical point of J. Let $a, b \in \mathbf{R}$ be two regular values for J, with $a < b$. If J satisfies $(P.S.)_c$ condition for any $c \in (a,b)$, and $z_1, \dots, z_l$ are the critical points of J in $J^{-1}(a,b)$, then*

$$\sum_{q=0}^{+\infty} \left(\sum_{j=1}^{l} \dim C_q(J, z_j) \right) t^q = \mathcal{P}_t(J^b, J^a) + (1+t)Q(t)$$

where $Q(t)$ is a formal series with coefficients in $\mathbf{N} \cup \{+\infty\}$.

By the previous theorem, building a suitable barycenter map, in [12] we proved that there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, (P_λ) has at least $\mathcal{P}_1(\Omega)$ solutions, possibly counted with their multiplicities, (see (2.1) and (2.2)).

This is an improvement on the result previously obtained in [1] via Ljusternik-Schnirelmann theory, at least when Ω is a topologically rich domain. In fact, for example, if Ω is obtained by cutting off k holes from an open contractible domain, then $\mathcal{P}_1(\Omega) = k+1$, while $cat(\Omega) = 2$.

At the same time, we do not know what is the minimum number of *distinct* solutions to (P_λ), as we have no information about the multiplicity of each solution. The situation would have been different if the energy functional had been defined on a Hilbert space.

In fact, if H is a Hilbert space, $J \in C^2(H, \mathbf{R})$, and z is a nondegenerate critical point of J, namely if $J''(z) : H \to H^*$ is invertible, then, using Morse splitting Lemma, a crucial relation between differential and topological information about z holds.

THEOREM 2.6. *If z is a nondegenerate critical point of J, then*

$$C_q(J,z) \cong \mathbf{K} \quad \text{if } q = m(J,z),$$
$$C_q(J,z) = \{0\} \quad \text{if } q \neq m(J,z)$$

being $m(J,z)$ the Morse index of J in z.

Consequently, in a Hilbert space, the multiplicity of any nondegenerate critical point is 1.

Moreover, nondegeneracy assumption holds quite often, as proved by the remarkable result [15] due to Marino and Prodi, in which it is showed that, if the second derivative is a Fredholm operator, an isolated degenerate critical point can be "solved" in a finite number of nondegenerate critical points by a small local perturbation of J.

When we pass to consider a functional $J \in C^2(Y, \mathbf{R})$ defined on a Banach space (not Hilbert) Y, a lot of difficulties arise, in fact:

- it is not clear what can be a reasonable definition of nondegenerate critical point, as it makes no sense to require that the second derivative of J in a critical point is invertible, since a Banach space, in general (and $W_0^{1,p}(\Omega)$, in particular), is not isomorphic to its dual space;
- moreover in [16] it has been proved that the existence of a nondegenerate critical point having finite Morse index, implies the existence of an equivalent Hilbert structure on Y;
- Morse Lemma does not hold;

• $J''(z)$ is not a even Fredholm operator from Y to Y^*, because if $J''(z)$ is a Fredholm operator, then Y is isomorphic to its dual space;

• extensions of Morse lemma of Gromoll-Meyer type can not be applied, and no perturbation argument of Marino-Prodi type can be applied.

In this context, is it possible to relate critical groups to differential features of critical points?

Various experts addressed the issue. We just quote, among them, Uhlenbeck [18], Tromba [17] and Chang [4].

First of all, they tried to give a suitable definition of nondegenerate critical points in Banach spaces, but these definitions were quite involved and not easy to verify. For example, let us see the following one given in [4].

DEFINITION 2.7. *Let X be a Banach space and $f : X \to \mathbf{R}$ a C^2 function. A critical point x_0 of f is said to be s-nondegenerate if*

- *x_0 is isolated;*
- *there exists a neighborhood U of x_0 and an hyperbolic operator $L_{x_0} \equiv L : X \to X$ such that*
  $\langle f''(x_0)Lx, y\rangle = \langle f''(x_0)x, Ly\rangle \quad \forall x, y \in X;$
  $\langle f''(x_0)Lx, x\rangle > 0 \quad \forall\, x \in X \setminus \{0\};$
  $\langle f'(x), L(x - x_0)\rangle > 0 \quad \forall\, x \in f^c \cap (U \setminus \{x_0\})$, *where* $c = f(x_0)$.

Our approach is different.

3. A new approach: I) case $\mathbf{p \geq 2}$.

Let us return to consider problem (P_λ). In the case $p \geq 2$, the energy functional I_λ is of class C^2 on $W_0^{1,p}(\Omega)$, which is not a Hilbert space when $p > 2$. In [11] we consider a class of functionals including

$$J_{\alpha,f}(u) = \frac{1}{p}\int_\Omega \left(\left(\alpha^2 + |\nabla u|^2\right)^{\frac{p}{2}}\right) dx - \frac{\lambda}{q}\int_\Omega (u^+)^q dx - \frac{1}{p^*}\int_\Omega (u^+)^{p^*} dx - \int f(x)u(x)dx$$

where $\alpha > 0$ and $f \in C^1(\overline{\Omega})$. We give the following new definition of nondegenerate critical point, introduced for the first time in [10].

DEFINITION 3.1. *A critical point u of $J_{\alpha,f}$ is nondegenerate if*

$$J''_{\alpha,f}(u) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \quad \textit{is injective.}$$

In [11], using this new notion, we obtain critical groups estimates in the spirit of differential Morse relation (cf. Theorem 2.6). More precisely:

THEOREM 3.2. *Let $p > 2$, $\lambda > 0$, $\alpha > 0$, $f \in C^1(\overline{\Omega})$ and u be a nondegenerate critical point of $J_{\alpha,f}$. Then u is isolated, the Morse index $m = m(J_{\alpha,f}, u)$ is finite and*

$$C_q(J_{\alpha,f}, u) \cong \delta_{q,m}\mathbf{K}.$$

So, in particular, if u is nondegenerate, then its multiplicity is 1.
Moreover, if u is nondegenerate, then u isolated. Hence, from this new definition, we can infer something that Definition 2.7 needed to assume.

We remark that in 1969 Smale, as reported by Uhlenbeck in [18], conjectured that mere injectivity could be enough for developing Morse theory in some Banach settings. So the previous result proves that, as regards $J_{\alpha,f}$, Smale's conjecture is true.

In order to give an interpretation of *multiplicity* for a solution to (P_λ), we need a deep insight into this notion. We do it taking advantage of the following abstract result, proved by Cingolani, Lazzo and Vannella in [9].

Theorem 3.3. *Let A be an open subset of a Banach space Y. Let I be a C^1 functional on A and $z \in A$ be an isolated critical point of I. Assume that there exists an open neighborhood U of z such that $\overline{U} \subset A$, z is the only critical point of I in $\overline{U}$ and I satisfies the Palais–Smale condition in $\overline{U}$.*
Then, there exists $\mu > 0$ such that, for any $J \in C^1(A, \mathbf{R})$ such that

- $\|I - J\|_{C^1(A)} < \mu$,
- *J satisfies the Palais–Smale condition in $\overline{U}$,*
- *J has a finite number $\{u_1, u_2, \dots, u_m\}$ of critical points in U,*

we have

$$\sum_{j=1}^{m} \mathcal{P}_t(J, u_j) = \mathcal{P}_t(I, z) + (1 + t)Q(t),$$

where $Q(t)$ is a formal series with coefficients in $\mathbf{N} \cup \{+\infty\}$.
So, in particular,

$$\sum_{j=1}^{m} \textit{multiplicity}\ (J, u_j) \geq \textit{multiplicity}\ (I, z).$$

In what follows, we say that $\partial\Omega$ satisfies the interior sphere condition if for each $x_0 \in \partial\Omega$ there exists a ball $B_R(x_1) \subset \Omega$ such that $\overline{B_R(x_1)} \cap \partial\Omega = \{x_0\}$.

Due to the previous abstract theorem, considering also that, if B is bounded in $W_0^{1,p}(\Omega)$, then

$$\lim_{\alpha \to 0^+} \|J_{\alpha,0} - I_\lambda\|_{C^1(B)} = 0 \quad \text{and} \quad \lim_{\|f\|_{C^1(\overline{\Omega})} \to 0} \|J_{\alpha,f} - J_{\alpha,0}\|_{C^1(B)} = 0,$$

in [12] we proved the following result.

Theorem 3.4. *Assume that $\partial\Omega$ satisfies the interior sphere condition and that $N \geq p^2$, $2 < p \leq q < p^*$, $p^* = Np/(N-p)$. There exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, either (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, for any sequence (α_n), with $\alpha_n > 0$, $\alpha_n \to 0$, there exists a sequence (f_n) with $f_n \in C^1(\overline{\Omega})$, $\|f_n\|_{C^1} \to 0$ such that problem*

$$(P_n) \begin{cases} -div\big((|\nabla u|^2 + \alpha_n)^{(p-2)/2} \nabla u\big) = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct solutions, for n large enough. Moreover, if $p = 2$, the statement holds also if $\alpha_n \geq 0$.

The previous theorem is a sharp interpretation of the multiplicity of a critical point of (P_λ) in terms of approximating elliptic problems. We remark that this approach is new also for the case $p = 2$. Indeed the perturbation results by Marino and Prodi furnish an interpretation of the multiplicity in terms of C^1 locally approximating functionals, which can not be, in general, the Euler functionals of some semilinear problems.

4. A new approach: II) case $1 < p < 2$.

When we consider (P_λ) in the case $p \in (1, 2)$, an additional difficulty arises, as I_λ is just in $C^1(W_0^{1,p}(\Omega), \mathbf{R})$ and not in $C^2(W_0^{1,p}(\Omega), \mathbf{R})$. So it seems even not possible to give a notion of Morse index, and, more in general, the approach used in [12], where $p \geq 2$, fails.

In order to face this further challenge, we used suitable approximations of (P_λ), suggested by recent results given in [7] and [8]. In these papers we considered a class of functionals including

$$
\begin{aligned}
J_{\alpha,f}(u) = {} & \frac{1}{p}\int_\Omega \Big(\big(\alpha^2 + |\nabla u|^2\big)^{\frac{p}{2}} \Big) dx - \frac{\lambda}{q}\int_\Omega \big(\alpha + (u^+)^s\big)^{\frac{q}{s}} dx \\
& - \frac{1}{p^*}\int_\Omega \big(\alpha + (u^+)^s\big)^{\frac{p^*}{s}} dx - \int f(x)u(x)dx,
\end{aligned}
$$

where $\alpha > 0$, $s > 2$ and $f \in C^1(\overline{\Omega})$. Functionals $J_{\alpha,f}$ are still just in $C^1(W_0^{1,p}(\Omega), \mathbf{R})$ and not in $C^2(W_0^{1,p}(\Omega), \mathbf{R})$.

However, if u_0 is a critical point of $J_{\alpha,f}$, it can be proved that $u_0 \in C^1(\overline{\Omega})$, so we introduce a suitable quadratic form Q_{u_0} defined on $W_0^{1,2}(\Omega)$ (which is embedded in $W_0^{1,p}(\Omega)$, as $p < 2$) by

$$
\begin{aligned}
Q_{u_0}(z) = {} & \int_\Omega (\alpha + |\nabla u_0|^2)^{\frac{p-2}{2}} |\nabla z|^2 + (p-2)\int_\Omega (\alpha + |\nabla u_0|^2)^{\frac{p-4}{2}} (\nabla u_0 / \nabla z)^2 \\
& - \lambda \int_\Omega \left(\big((s-1)\alpha(u_0^+)^{s-2} + (q-1)(u_0^+)^{2s-2}\big) \big(\alpha + (u_0^+)^s\big)^{\frac{q-2s}{s}} \right) z^2 \\
& - \int_\Omega \left(\big((s-1)\alpha(u_0^+)^{s-2} + (p^*-1)(u_0^+)^{2s-2}\big) \big(\alpha + (u_0^+)^s\big)^{\frac{p^*-2s}{s}} \right) z^2.
\end{aligned}
$$

Through this quadratic form, we give the following generalized definition of Morse indices.

DEFINITION 4.1. *We denote by $m(J_{\alpha,f}, u_0)$, the (generalized) Morse index of $J_{\alpha,f}$ at u_0, defined as the supremum of the dimensions of the linear subspaces of $W_0^{1,2}(\Omega)$ where Q_{u_0} is negative definite.*

In a similar way, we denote by $m^*(J_{\alpha,f}, u_0)$ the (generalized) large Morse index of $J_{\alpha,f}$ at u_0, defined as the supremum of the dimensions of the linear subspaces of $W_0^{1,2}(\Omega)$ where Q_{u_0} is negative semidefinite.

We remark that

$$m(J_{\alpha,f}, u_0) \leq m^*(J_{\alpha,f}, u_0) < +\infty.$$

Moreover these generalized Morse indices coincide with the usual ones when $p \geq 2$.

In [8] we proved that even in this case a critical group estimate result holds.

THEOREM 4.2. *Let $p \in (1,2)$, $q \in [p, p^*)$, $\lambda > 0$, $f \in C^1(\overline{\Omega})$ and $\alpha > 0$. If u_0 is a critical point of $J_{\alpha,f}$ and*

$$m(J_{\alpha,f}, u_0) = m^*(J_{\alpha,f}, u_0) = m$$

then u_0 is an isolated critical point of $J_{\alpha,f}$ and

$$C_q(J_{\alpha,f}, u_0) \cong \delta_{q,m}\, \mathbf{K}$$

In particular, if $m(J_{\alpha,f}, u_0) = m^*(J_{\alpha,f}, u_0)$, then multiplicity of u_0 is 1.

Exploiting in a suitable way Theorem 3.3, in [13] we prove the following result.

THEOREM 4.3.
Assume that $\partial\Omega$ satisfies the interior sphere condition and that $N \geq p^2$, $p \in (1,2)$, $p \leq q < p^*$, $p^* = Np/(N-p)$. There exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, either (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, there exists $s > 2$ such that, for any sequence (α_n), with $\alpha_n > 0$, $\alpha_n \to 0$, there exists a sequence (f_n) with $f_n \in C^1(\overline{\Omega})$, $\|f_n\|_{C^1} \to 0$, such that problem

$$(P_n)\ \begin{cases} -div\big((|\nabla u|^2 + \alpha_n)^{(p-2)/2} \nabla u\big) & \\ = \lambda u^{s-1}(\alpha_n + u^s)^{\frac{q-s}{s}} + u^{s-1}(\alpha_n + u^s)^{\frac{p^*-s}{s}} + f_n & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct *solutions, for n large enough.*

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