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CONTINUOUS DEPENDENCE FOR BV-ENTROPY SOLUTIONS TO STRONGLY DEGENERATE PARABOLIC EQUATIONS WITH VARIABLE COEFFICIENTS *

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Abstract. We consider the Cauchy problem for degenerate parabolic equations with variable coefficients. The equation has nonlinear convective term and degenerate diffusion term which depends on the spatial and time variables. In this paper, we prove the continuous dependence for entropy solutions in the space BV to the problem not only initial function but also all coefficients.

Key words. strongly degenerate parabolic, continuous dependence, BV-entropy solution

AMS subject classifications. 35K65, 35K55, 35L65

1. Introduction. Let $0 < T < \infty$ and $N \in \mathbb{N}$ be constants. We consider the Cauchy problem for a degenerate parabolic equation of the form

$$\begin{cases} \partial_t u + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(x,t,u), & (x,t) \in \mathbb{R}^N_T := \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N). \end{cases} \tag{P}$$

Here $\partial_t := \partial/\partial t$, $\nabla := (\partial/\partial x_1, \dots, \partial/\partial x_N)$ and $\Delta := \sum_{i=1}^N \partial^2/\partial x_i^2$ are the spatial nabla and the laplacian in $\mathbb{R}^N$, respectively. $A(x,t,\xi) = (A^1, ..., A^N)(x,t,\xi)$ is an $\mathbb{R}^N$-valued function on $\mathbb{R}^N \times [0,T] \times \mathbb{R}$ and $B(x,t,\xi)$ and $\beta(x,t,\xi)$ are $\mathbb{R}$-valued functions on $\mathbb{R}^N \times [0,T] \times \mathbb{R}$. The function $\beta(x,t,\xi)$ is supposed to be monotone nondecreasing and locally Lipschitz continuous with respect to ξ for any $(x,t) \in \mathbb{R}^N_T$. Therefore, the set of points ξ where $\partial_\xi \beta(x,t,\xi) = 0$ may have a positive measure for any $(x,t) \in \mathbb{R}^N_T$. In this sense, we say that the equation posed as (P):

$$\partial_t u + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(x,t,u) \tag{1.1}$$

is a *strongly degenerate parabolic equation*. The equation (1.1) can be applied to several mathematical models; hyperbolic conservation laws, porous medium, Stefan problem, filtration problem, sedimentation process, traffic flow, and so on. Moreover, (1.1) is regarded as a linear combination of the time dependent conservation laws (quasilinear hyperbolic equation) and the porous medium equation (nonlinear degenerate parabolic equation). Thus, (1.1) has both properties of hyperbolic equations and those of parabolic equations. In particular, up to the assumptions on β, "parabolicity" of (1.1) and "hyperbolicity" of it are not necessarily comparable.

Our mathematical treatment of the equation (1.1) is L^1-framework. More specifically, we consider (1.1) in the space $L^1(\mathbb{R}^N)$ and construct solutions to (1.1) in the space $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Here, solutions to (1.1) should be defined in generalized sense. To ensure the existence and uniqueness of it, it is necessitate to consider distributional solutions satisfying a special condition. This framework was first treated by Vol'pert-Hudjaev [9]. In fact, it is well known that the Fréchet-Kolmogorov compactness theorem in the space BV and the Kružkov's doubling variable method [8]

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are available. Under the above framework, the existence and uniqueness of entropy solutions to (1.1) are given ([3, 4, 5, 7, 10, 11, 12, 13]). Here, entropy solutions are weak solutions satisfying an entropy inequality which is derived by Kružkov [8]. In particular, Watanabe [11] proved the existence and uniqueness of entropy solutions to (P) in the space BV.

In this paper, we prove the continuous dependence of the BV-entropy solution to (P) not only initial data but also coefficients A, B and β. Feature of the present paper is to consider the equation (1.1) with variable coefficients. In particular, the equation with variable diffusion coefficients is treated in few literatures. For example, Chen-Karlsen [4] considered the equation with a separation variable type convective term and a quasi-linear type diffusion term. Notice that, these coefficients do not depend on time variable. In this article, we consider the time dependent nonlinear type convection $\nabla \cdot A(x,t,u)$ and diffusion $\Delta\beta(x,t,u)$. To prove desired estimate, we modify the choice of entropy triplet and the calculation in [4].

Throughout this paper, we employ the following notations and terminologies. For $1 \le p \le \infty$, the Lebesgue space of real-valued Lebesgue-measurable functions on $\mathbb{R}^N$ equipped with the usual norm $||\cdot||_p$ is denoted by $L^p(\mathbb{R}^N)$. The space of functions of bounded variation in $\mathbb{R}^N$ is denoted by $BV(\mathbb{R}^N)$ and the total variation on $\mathbb{R}^N$ is denoted by $TV(\cdot)$ (cf. [2, 6, 14]). The space $C_0^\infty(\mathbb{R}^N)^+$ is the class of nonnegative valued $C_0^\infty(\mathbb{R}^N)$-functions. The function $\mathrm{sgn}(\xi)$ means the usual *signum* function.

2. Assumptions and the main result. In this section, we present some assumptions and the main result. Before that, we write the nabla of the function $A(x,t,u)$ and the laplacian of the function $\beta(x,t,u)$ as follows:

$$\nabla \cdot A(x,t,u) = [\nabla \cdot A](x,t,u) + [\partial_\xi A](x,t,u) \cdot \nabla u$$

and

$$\begin{aligned}\Delta\beta(x,t,u) &= \nabla \cdot ([\nabla\beta](x,t,u) + [\partial_\xi\beta](x,t,u) \cdot \nabla u) \\ &= [\Delta\beta](x,t,u) + 2[\partial_\xi\nabla\beta](x,t,u) \cdot \nabla u + [\partial_\xi^2\beta](x,t,u)|\nabla u|^2 + [\partial_\xi\beta](x,t,u)\Delta u\end{aligned}$$

for $(x,t) \in \mathbb{R}_T^N$ and some regular function u. These are based on the chain rule formulas in [1] (see also [2, Theorem 3.99], [14, Theorem 2.1.11]).

Throughout this paper, we impose the following assumptions on the functions A, B, β and u_0. Here, we write $\partial_{x_i} := \partial/\partial_{x_i}$ for $i = 1, \ldots, N$, $\partial_{x_{N+1}} := \partial_t$, $\widehat{\nabla} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}, \partial_{x_{N+1}})$ and $\mathcal{U} := [-U, U]$ for any $U > 0$. For any $U > 0$, the following conditions hold:

{A0} $u_0(x) \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$;

{A1} $\begin{cases} A \in L^1(\mathbb{R}_T^N \times \mathcal{U})^N \cap L^\infty(\mathbb{R}_T^N \times \mathcal{U})^N \cap L^\infty(\mathcal{U}; L^2(\mathbb{R}_T^N))^N, \\ \partial_\xi A \in L^1(\mathbb{R}_T^N \times \mathcal{U})^N \cap L^\infty(\mathbb{R}_T^N \times \mathcal{U})^N, \;\; \nabla \cdot A,\ \partial_\xi \nabla \cdot A \in L^1(\mathbb{R}_T^N \times \mathcal{U}); \end{cases}$

{A2} $\begin{cases} B \in L^1(\mathbb{R}_T^N \times \mathcal{U}) \cap L^\infty(\mathbb{R}_T^N \times \mathcal{U}) \cap L^\infty(\mathcal{U}; L^1(\mathbb{R}_T^N)), \\ |\widehat{\nabla} B| \in L^\infty(\mathcal{U}; L^1(\mathbb{R}_T^N)), \;\; \partial_\xi B \in L^\infty(\mathbb{R}_T^N \times \mathcal{U}); \end{cases}$

{A3} $\begin{cases} \beta \in L^1(\mathbb{R}_T^N \times \mathcal{U}) \cap L^\infty(\mathbb{R}_T^N \times \mathcal{U}), \\ \nabla\beta \in L^1(\mathbb{R}_T^N \times \mathcal{U})^N \cap L^\infty(\mathcal{U}; L^2(\mathbb{R}_T^N))^N,\ \partial_t\beta \in L^\infty(\mathcal{U}; L^1(\mathbb{R}_T^N)), \\ \partial_\xi\beta \in L^\infty(\mathbb{R}_T^N \times \mathcal{U}),\ \partial_\xi\nabla\beta \in L^1(\mathbb{R}_T^N \times \mathcal{U})^N,\ \Delta\beta,\ \partial_\xi\Delta\beta \in L^1(\mathbb{R}_T^N \times \mathcal{U}); \end{cases}$

{A4} $B(x,t,0) = 0$ and $\nabla\beta(x,t,0) - A(x,t,0) = \vec{0}$ for $(x,t) \in \mathbb{R}_T^N$;

{A5} Let $\Psi(x,t,\xi) := \nabla \cdot A(x,t,\xi) - \Delta\beta(x,t,\xi) + B(x,t,\xi)$. Then, there exist positive constants c_0, c_1 such that

$$\sup_{(x,t)\in\mathbb{R}_T^N} |\Psi(x,t,0)| \le c_0, \qquad \sup_{(x,t,\xi)\in\mathbb{R}_T^N\times\mathbb{R}} (-\partial_\xi\Psi(x,t,\xi)) \le c_1;$$

{A6} For $i = 1, 2, \ldots, N+1$,
$$\begin{cases} \partial_\xi \partial_{x_i}(\nabla\beta - A) \in L^\infty(\mathbb{R}_T^N \times \mathcal{U})^N, \\ \nabla\cdot(\nabla\beta - A),\ |\widehat{\nabla}(\Delta\beta - \nabla\cdot A)| \in L^\infty(\mathcal{U}; L^1(\mathbb{R}_T^N)); \end{cases}$$

{A7} For $(x,t,\xi) \in \mathbb{R}_T^N \times \mathcal{U}$ and $\lambda = (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$, there exists a constant $\kappa > 0$ such that
$$\textstyle\sum_{i,j=1}^{N+1}(\partial_\xi\beta(x,t,\xi)\lambda_i\lambda_j - \kappa(\partial_{x_i}\partial_\xi\beta(x,t,\xi)\lambda_j)^2) \geq 0.$$

The conditions $\{A1\}$-$\{A3\}$ are regularity assumptions for the functions A, B and β with respect to x, t and ξ. {A4}-{A6} are used to prove L^∞, L^1 and BV-estimates for approximate solutions. {A6} is also interpreted the regularity assumptions for the flux $A(x,t,\xi) - \nabla\beta(x,t,\xi)$ to (1.1). The condition {A7} fulfills to get a BV-estimate with respect to x and t for approximate solutions to (P).

Remark 1. *By the assumption* $\{A7\}$, *it is deduced that*
$$\partial_\xi\beta(x,t,\xi) \geq 0 \quad \textit{for } (x,t,\xi) \in \mathbb{R}_T^N \times \mathcal{U}. \tag{2.1}$$
More specifically, $\beta(x,t,\xi)$ *degenerate on nondegenerate intervals with respect to* ξ. *In particular, if* $\beta(x,t,\xi) \equiv \beta(\xi)$, *then* $\{A7\}$ *is equivalent to (2.1).*

We also impose an additional regularity assumption to prove the uniqueness of BV-entropy solution: for any $i, j = 1, \ldots, N$,

{A8} $$\begin{cases} \partial_\xi\partial_{x_i}A^j,\ \partial_\xi\partial_{x_i}\partial_{x_j}\beta \in L^\infty(\mathbb{R}_T^N \times \mathcal{U}), \quad \sqrt{\partial_\xi\beta},\ \partial_{x_i}\sqrt{\partial_\xi\beta} \in L^1(\mathbb{R}_T^N \times \mathcal{U}), \\ \partial_{x_i}\partial_\xi\beta,\ \partial_{x_i}\sqrt{\partial_\xi\beta},\ \partial_{x_i}\partial_{x_j}\sqrt{\partial_\xi\beta} \in L^\infty(\mathbb{R}_T^N \times \mathcal{U}). \end{cases}$$

Next, we introduce generalized solutions to (P). Usually, the weak solution is interpreted as a generalized solution to equations with divergence form. Then, the existence and uniqueness of it may be shown. However, we can not prove the uniqueness of weak solutions to (P) in general. Because, discontinuities break out from the nonlinear convective term $\nabla\cdot A(x,t,u)$ and the uniqueness of weak solutions are possibly broken because of it. Therefore, we formulate the weak solution satisfying a special condition. It is called by the name entropy solution. To define it, we state the concept of entropy:

Definition 2.1. *Let* $\eta(\xi) \in C^2(\mathbb{R})$ *and* $q(x,t,\xi), r(x,t,\xi) \in L^1(\mathbb{R}_T^N \times \mathbb{R})^N \cap L^\infty(\mathbb{R}_T^N \times \mathbb{R})^N$ *satisfying* $q(x,t,\cdot), r(x,t,\cdot) \in C^2(\mathbb{R})^N$ *for* $(x,t) \in \mathbb{R}_T^N$. *A triplet* (η, q, r) *is entropy triplet to (P) if it satisfies*
$$\partial_\xi q(x,t,\xi) = \eta'(\xi)\partial_\xi A(x,t,\xi), \quad \partial_\xi r(x,t,\xi) = \eta'(\xi)\partial_\xi\nabla\beta(x,t,\xi)$$
for a.e. $(x,t,\xi) \in \mathbb{R}_T^N \times \mathbb{R}$. *Then,* η *is called entropy and* (q,r) *is called entropy flux.*

Definition 2.2. *Let* $u_0 \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$. *A function* $u \in L^\infty(\mathbb{R}_T^N) \cap BV(\mathbb{R}_T^N)$ *is called an* BV*-entropy solution to (P), if it satisfies the two conditions below:*

(I) $u \in C([0,T]; L^1(\mathbb{R}^N))$ *and* L^1-$\lim_{t\downarrow 0} u(t) = u_0$;

(II) $\beta(x,t,u) \in L^2(0,T; H^1(\mathbb{R}^N))$, *and for all* $\varphi \in C_0^\infty(\mathbb{R}_T^N)^+$,
$$\begin{aligned} &\int_{\mathbb{R}_T^N}\{\eta(u)\partial_t\varphi + (q(x,t,u) - r(x,t,u))\cdot\nabla\varphi + ([\nabla\cdot q](x,t,u) - [\nabla\cdot r](x,t,u))\varphi \\ &\qquad -\eta'(u)(\nabla\beta(x,t,u) - [\nabla\beta](x,t,u))\cdot\nabla\varphi \\ &\qquad -\eta'(u)([\nabla\cdot A](x,t,u) - [\Delta\beta](x,t,u) + B(x,t,u))\varphi\}dxdt \\ &\geq \int_{\mathbb{R}_T^N}\eta''(u)|\sqrt{\partial_\xi\beta(x,t,u)}Du|^2\varphi dxdt. \end{aligned}$$

The existence and uniqueness of the BV-entropy solution to (P) is given by Watanabe [11] as follows:

THEOREM 2.3 (Watanabe [11]). *We assume the conditions $\{A0\}$-$\{A7\}$. Then, the following statements hold:*

(I) There exists a BV-entropy solution u to (P). Moreover, if we take $U > 0$ satisfying $(||u_0||_{L^\infty(\mathbb{R}^N)} + c_0T)e^{c_1T} < U$ for the positive constants c_0 and c_1 in $\{A5\}$, then it follows that $u(x,t) \in \mathcal{U}$ for $(x,t) \in \mathbb{R}^N_T$. Additionally, there exist positive constants C_0 and C_1 which depend on T such that $TV(u(\cdot,t)) \leq e^{C_0t}(TV(u_0) + C_1)$ for $t \in (0,T)$;

(II) We additionally impose the assumption $\{A8\}$. Let u, v be a pair of BV-entropy solutions to (P) with initial values u_0 and v_0, respectively. Then, there exist a positive constant C_2 which depend on T such that

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)|dx \leq e^{(\alpha+C_2)t} \int_{\mathbb{R}^N} |u_0(x) - v_0(x)|dx,$$

where $\alpha := ||\partial_\xi B||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})}$ for $t \in (0,T)$. In particular, for each initial value u_0, a BV-entropy solution is uniquely determined.

In the above result, we give the assumptions {A0}-{A7} to prove the existence of BV-entropy solutions. In this paper, we prove the continuous dependence of the BV-entropy solution to the function u_0, A, B and β under the additional assumption: for any $i, j = 1, \ldots, N$,

$$\textbf{\{A8\}'} \begin{cases} \sqrt{\partial_\xi \beta},\ \partial_{x_i}\sqrt{\partial_\xi \beta} \in L^1(\mathbb{R}^N_T \times \mathcal{U}), \quad \partial_{x_i}\partial_\xi \beta,\ \partial_{x_i}\sqrt{\partial_\xi \beta} \in L^\infty(\mathbb{R}^N_T \times \mathcal{U}), \\ \partial_{x_i}B,\ \nabla \cdot A,\ \partial_{x_j}\partial_{x_i}A^i,\ \Delta\beta,\ \partial_\xi\partial_{x_i}\partial_{x_j}\beta,\ \partial_{x_j}\partial^2_{x_i}\beta \in L^\infty(L^1), \end{cases}$$

where $L^\infty(L^1) := L^\infty((0,T) \times \mathcal{U}; L^1(\mathbb{R}^N))$. Notice that, since we consider nonlinear type coefficients, we need stronger regularity assumption than separation variable and quasilinear diffusion case [4].

THEOREM 2.4. *Let u_i be the BV-entropy solution to (P) with initial functions $u_{i,0}$ and coefficients A_i, B_i, β_i satisfying the assumptions $\{A0\}$-$\{A7\}$ and $\{A8\}'$ for $i = 1, 2$, respectively. For any $t \in (0,T)$, the following inequality holds:*

$$\begin{aligned} ||u_1(x,t) - u_2(x,t)||_{L^1(\mathbb{R}^N)} &\leq e^{\alpha' t}||u_{1,0} - u_{2,0}||_{L^1(\mathbb{R}^N)} + \tfrac{e^{\alpha' t}-1}{\alpha'}\{||B_1 - B_2||_{L^\infty(L^1)} \\ &+ ||[\nabla_x \cdot A_1] - [\nabla_x \cdot A_2]||_{L^\infty(L^1)} + ||[\Delta_x\beta_1] - [\Delta_x\beta_2]||_{L^\infty(L^1)} \\ &+ e^{C_0t}(TV(u_0) + C_1)(||[\partial_\xi\nabla_x\beta_1] - [\partial_\xi\nabla_x\beta_2]||_{(L^\infty)^N} + ||[\partial_\xi A_1] - [\partial_\xi A_2]||_{(L^\infty)^N} \\ &\qquad + 2\left|\left|\nabla_x\sqrt{[\partial_\xi\beta_2]}\right|\right|_{(L^\infty)^N} \left|\left|\sqrt{[\partial_\xi\beta_1]} - \sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty})\} \\ &+ \widehat{C}\sqrt{t}e^{\alpha' t}\left|\left|\sqrt{[\partial_\xi\beta_1]} - \sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty} \end{aligned}$$

for some positive constants $\widehat{C}$ and $\alpha' := \max_{i=1,2}\{||\partial_\xi B_i||_{L^\infty}\}$. Here, $TV(u_0) := \max_{i=1,2}\{TV(u_{i,0})\}$, $L^\infty := L^\infty(\mathbb{R}^N \times (0,T) \times \mathcal{U})$, $L^\infty(L^1) := L^\infty((0,T) \times \mathcal{U}; L^1(\mathbb{R}^N))$ and C_0, C_1 are positive constants in Theorem 2.3 (I).

3. Proof of Main Theorem. *Step 0.* Let $\varphi \in C_0^\infty(\mathbb{R}^N_T)^+$. In addition, we introduce a symmetric function $\theta \in C_0^\infty(\mathbb{R})^+$ satisfying $\int_{\mathbb{R}} \theta(t)dt = 1$ and $\text{supp}[\theta(t)] \subset \{|t| \leq 1\}$. Similarly, we use a spherically symmetric function $\omega \in C_0^\infty(\mathbb{R}^N)^+$ satisfying $\int_{\mathbb{R}^N} \omega(x)dx = 1$ and $\text{supp}[\omega(x)] \subset \{|x| \leq 1\}$. Let $\delta_0, \delta > 0$ and define $\theta_{\delta_0}(t) = (1/\delta_0)\theta(t/\delta_0)$ and $\omega_\delta(x) = (1/\delta^N)\omega(x/\delta)$. These are smooth functions on $\mathbb{R}$ and $\mathbb{R}^N$, respectively, and satisfy

$$\lim_{\delta_0 \downarrow 0} \int_0^T \theta_{\delta_0}(t)\varphi(x,t)dt = \varphi(x,0), \quad \lim_{\delta \downarrow 0} \int_{\mathbb{R}^N} \omega_\delta(x)\varphi(x,t)dx = \varphi(0,t)$$

for $(x,t) \in \mathbb{R}^N_T$. Moreover, let $\nu, \tau \in (0,T)$ with $\nu < \tau$. For any $\alpha_0 > 0$, we define

$$\varphi_{\alpha_0}(t) := H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau), \quad H_{\alpha_0}(t) := \int_{-\infty}^{t} \theta_{\alpha_0}(\sigma) d\sigma.$$

Then, we now employ the test function $\phi_\delta^{\delta_0,\alpha_0}$ defined by

$$\phi_\delta^{\delta_0,\alpha_0}(x,y,t,s) := \varphi_{\alpha_0}(t)\omega_\delta(x-y)\theta_{\delta_0}(t-s) \tag{3.1}$$

for $0 < \alpha_0 < \min(\nu, T-\tau)$ and $(x,t,y,s) \in (\mathbb{R}^N_T)^2$. Then, the following property holds

$$\lim_{\delta \downarrow 0} \int_{(\mathbb{R}^N)^2} |x-y| \left|\omega_\delta\left(\tfrac{x-y}{2}\right)\right| dxdy = 0$$

and there exists a constant $C > 0$ such that

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \int_{\mathbb{R}^N_T \times \mathbb{R}^N} |x-y| \left|\partial_{x_i}\omega_\delta\left(\tfrac{x-y}{2}\right)\right| \varphi\left(\tfrac{x+y}{2}, t\right) dxdydt \le C \int_{\mathbb{R}^N_T} \varphi(x,t) dxdt, \\
&\lim_{\delta \downarrow 0} \int_{\mathbb{R}^N_T \times \mathbb{R}^N} |x-y|^2 \left|\partial_{x_i}\partial_{x_j}\omega_\delta\left(\tfrac{x-y}{2}\right)\right| \varphi\left(\tfrac{x+y}{2}, t\right) dxdydt \le C \int_{\mathbb{R}^N_T} \varphi(x,t) dxdt
\end{aligned}$$

for $1 \le i,j \le N$. Moreover, it follows that

$$(\partial_t + \partial_s)\phi_\delta^{\delta_0,\alpha_0} = (\theta_{\alpha_0}(t-\nu) - \theta_{\alpha_0}(t-\tau))\theta_{\delta_0}(t-s)\omega_\delta(x-y), \quad (\nabla_x + \nabla_y)\phi_\delta^{\delta_0,\alpha_0} = 0.$$

In this section, the proof of Theorem 2.4 is presented. Hereafter, we give the entropy triplet in the following concrete form:

$$\begin{aligned}
\eta(u) = \eta_\rho(u) &:= \int_k^u \operatorname{sgn}_\rho(\xi-k) d\xi, \\
q(x,t,u) = q_\rho(x,t,u) &:= \int_k^u \operatorname{sgn}_\rho(\xi-k)[\partial_\xi A](x,t,\xi) d\xi, \\
r(x,t,u) = r_\rho(x,t,u) &:= \int_k^u \operatorname{sgn}_\rho(\xi-k)[\partial_\xi \nabla\beta](x,t,\xi) d\xi
\end{aligned} \tag{3.2}$$

for $k \in \mathbb{R}$. Here, we use the approximated signum function $\operatorname{sgn}_\rho(\xi)$ for $\rho > 0$ by $\operatorname{sgn}_\rho(\xi) = \operatorname{sgn}(\xi)$ for $|\xi| \ge \rho$ and $\operatorname{sgn}_\rho(\xi) = \sin\left(\frac{\pi}{2\rho}\xi\right)$ for $|\xi| < \rho$. Then, it can be seen that

$$\begin{aligned}
\eta_\rho(u) \to |u-k|, \quad q_\rho(x,t,u) &\to \operatorname{sgn}(u-k)(A(x,t,u) - A(x,t,k)), \\
r_\rho(x,t,u) &\to \operatorname{sgn}(u-k)([\nabla\beta](x,t,u) - [\nabla\beta](x,t,k))
\end{aligned} \tag{3.3}$$

as $\rho \to 0$. Moreover, we set

$$\begin{aligned}
[\nabla \cdot q_\rho](x,t,u) &:= \int_k^u \operatorname{sgn}_\rho(\xi-k)[\partial_\xi \nabla \cdot A](x,t,\xi) d\xi, \\
[\nabla \cdot r_\rho](x,t,u) &:= \int_k^u \operatorname{sgn}_\rho(\xi-k)[\partial_\xi \Delta\beta](x,t,\xi) d\xi.
\end{aligned} \tag{3.4}$$

Then, it can be also seen that

$$\begin{aligned}
[\nabla \cdot q_\rho](x,t,u) &\to \operatorname{sgn}(u-k)([\nabla \cdot A](x,t,u) - [\nabla \cdot A](x,t,k)), \\
[\nabla \cdot r_\rho](x,t,u) &\to \operatorname{sgn}(u-k)([\Delta\beta](x,t,u) - [\Delta\beta](x,t,k))
\end{aligned} \tag{3.5}$$

as $\rho \to 0$. Then, the entropy inequality in the definition of BV-entropy solutions implies that

$$\begin{aligned}\int_{\mathbb{R}^N_T} \{&\eta_\rho(u)\partial_t\varphi + (q_\rho(x,t,u) - r_\rho(x,t,u))\cdot\nabla\varphi + ([\nabla\cdot q_\rho](x,t,u) - [\nabla\cdot r_\rho](x,t,u))\varphi \\ &- \mathrm{sgn}_\rho(u-k)(\nabla\beta(x,t,u) - [\nabla\beta](x,t,u))\cdot\nabla\varphi \\ &- \mathrm{sgn}_\rho(u-k)([\nabla\cdot A](x,t,u) - [\Delta\beta](x,t,u) + B(x,t,u))\varphi\}dxdt \\ \geq \int_{\mathbb{R}^N_T} &\mathrm{sgn}'_\rho(u-k)|\sqrt{[\partial_\xi\beta](x,t,u)}Du|^2\varphi dxdt.\end{aligned}$$

Step 1. Let u_i be the BV-entropy solution to (P) with $u_{i,0}$, A_i, B_i, β_i satisfying {A0}-{A7} and {A8}$'$. We put $k = u_2(y,s)$ and $\varphi = \phi_\delta^{\delta_0,\alpha_0}(x,y,t,s)$ (see (3.1)) in the definition of BV-entropy solution u_1. Integrating the inequality on $\mathbb{R}^N_T$ with respect to (y,s), then it follows that

$$\begin{aligned}\int_{(\mathbb{R}^N_T)^2} \{&\eta_\rho(u_1)\partial_t\phi_\delta^{\delta_0,\alpha_0} + (q_{\rho,1}(x,t,u_1) - r_{\rho,1}(x,t,u_1))\cdot\nabla_x\phi_\delta^{\delta_0,\alpha_0} \\ &+([\nabla_x\cdot q_{\rho,1}](x,t,u_1) - [\nabla_x\cdot r_{\rho,1}](x,t,u_1))\phi_\delta^{\delta_0,\alpha_0} \\ &- \mathrm{sgn}_\rho(u_1(x,t)-u_2(y,s))((\nabla_x\beta_1(x,t,u_1) - [\nabla_x\beta_1](x,t,u_1))\cdot\nabla_x\phi_\delta^{\delta_0,\alpha_0} \\ &\quad -([\nabla_x\cdot A_1](x,t,u_1) - [\Delta_x\beta_1](x,t,u_1) + B_1(x,t,u_1))\phi_\delta^{\delta_0,\alpha_0})\}d\mathbf{x}d\mathbf{y} \\ \geq \int_{(\mathbb{R}^N_T)^2} &\mathrm{sgn}'_\rho(u_1(x,t)-u_2(y,s))|\sqrt{[\partial_\xi\beta_1](x,t,u_1)}D_xu_1|^2\phi_\delta^{\delta_0,\alpha_0}d\mathbf{x}d\mathbf{y}.\end{aligned} \tag{3.6}$$

Here, we write that $d\mathbf{x} = dxdt$ and $d\mathbf{y} = dyds$. Moreover, $q_{\rho,1}$, $[\nabla_x\cdot q_{\rho,1}]$, $r_{\rho,1}$ and $[\nabla_x\cdot r_{\rho,1}]$ are defined in (3.2) and (3.4) using A_1 and β_1, respectively. Similarly, we define the inequality (3.6)$'$ using the definition of another BV-entropy solution u_2. Moreover, we then set (EI) := (3.6) + (3.6)$'$ in what follows. The desired result is obtained by combining the estimates for (EI). In fact, using the same way in [11, Section 4] (see also [4, Section 4]), (EI) implies that

$$\begin{aligned}\int_{(\mathbb{R}^N_T)^2} &\mathrm{sgn}(u_1-u_2)\{(u_1-u_2)(\partial_t+\partial_s)\phi_\delta^{\delta_0,\alpha_0} + (B_1(x,t,u_1) - B_2(y,s,u_2))\phi_\delta^{\delta_0,\alpha_0} \\ &+((A_1(x,t,u_1) - A_1(x,t,u_2)) + (A_2(y,s,u_2) - A_2(y,s,u_1)))\cdot\nabla_x\phi_\delta^{\delta_0,\alpha_0} \\ &-([\nabla_x\cdot A_1](x,t,u_2) - [\nabla_y\cdot A_2](y,s,u_1))\phi_\delta^{\delta_0,\alpha_0} \\ &-([\nabla_x\beta_1](x,t,u_1) - [\nabla_x\beta_1](x,t,u_2))\cdot(\nabla_x\omega_\delta)\varphi_{\alpha_0}\theta_{\delta_0} \\ &+([\nabla_y\beta_2](y,s,u_1) - [\nabla_y\beta_2](y,s,u_2))\cdot(\nabla_x\omega_\delta)\varphi_{\alpha_0}\theta_{\delta_0} \\ &-([\Delta_y\beta_2](y,s,u_1) - [\Delta_x\beta_1](x,t,u_2))\phi_\delta^{\delta_0,\alpha_0}\}d\mathbf{x}d\mathbf{y} \\ \geq &\int_{(\mathbb{R}^N_T)^2}\left(\int_{u_2}^{u_1}\mathrm{sgn}(\xi-u_2)\varepsilon(x,t,y,s,\xi)d\xi\right)\varphi_{\alpha_0}\theta_{\delta_0}(\nabla_x\cdot\nabla_y\omega_\delta)d\mathbf{x}d\mathbf{y} \\ &+\int_{(\mathbb{R}^N_T)^2}\left(\int_{u_2}^{u_1}\mathrm{sgn}(\xi-u_2)\nabla_y\varepsilon(x,t,y,s,\xi)d\xi\right)\cdot(\nabla_x\omega_\delta)\varphi_{\alpha_0}\theta_{\delta_0}d\mathbf{x}d\mathbf{y} \\ &+\int_{(\mathbb{R}^N_T))^2}\left(\int_{u_2}^{u_1}\mathrm{sgn}(\xi-u_2)\nabla_x\varepsilon(x,t,y,s,\xi)d\xi\right)\cdot(\nabla_y\omega_\delta)\varphi_{\alpha_0}\theta_{\delta_0}d\mathbf{x}d\mathbf{y} \\ &+\int_{(\mathbb{R}^N_T)^2}\left(\int_{u_2}^{u_1}\mathrm{sgn}(\xi-u_2)\nabla_x\cdot\nabla_y\varepsilon(x,t,y,s,\xi)d\xi\right)\varphi_{\alpha_0}\theta_{\delta_0}\omega_\delta d\mathbf{x}d\mathbf{y}\end{aligned} \tag{3.7}$$

by (3.2)-(3.5) and the properties of $\phi_\delta^{\delta_0,\alpha_0}$. Here, we set:

$$\varepsilon(x,t,y,s,\xi) := [\partial_\xi\beta_1](x,t,\xi) - 2\sqrt{[\partial_\xi\beta_1](x,t,\xi)}\sqrt{[\partial_\xi\beta_2](y,s,\xi)} + [\partial_\xi\beta_2](y,s,\xi).$$

To derive (3.7), we make the following terms:

$$\begin{aligned}
&\{-(\nabla_x\beta_1(x,t,u_1) - [\nabla_x\beta_1](x,t,u_1)) \\
&\quad +(\nabla_y\beta_2(y,s,u_2) - [\nabla_y\beta_2](y,s,u_2))\}\cdot(\nabla_x+\nabla_y)\phi_\delta^{\delta_0,\alpha_0} \\
&+(\nabla_x\beta_1(x,t,u_1) - [\nabla_x\beta_1](x,t,u_1))\cdot\nabla_y\phi_\delta^{\delta_0,\alpha_0} \\
&-(\nabla_y\beta_2(y,s,u_2) - [\nabla_y\beta_2](y,s,u_2))\cdot\nabla_x\phi_\delta^{\delta_0,\alpha_0}.
\end{aligned} \tag{3.8}$$

After that, we move the last two terms in (3.8) to the right-hand side in (EI) and use the notation $\varepsilon(x,t,y,s,\xi)$. Detailed calculation is referred to [4, 11].

Step 2. We investigate the diffusion terms in (3.7). First, the right-hand side of (3.7) is equal to

$$\begin{aligned}
&-\int_{(\mathbb{R}_T^N)^2} \operatorname{sgn}(u_1-u_2)\varepsilon(x,t,y,s,u_1)\varphi_{\alpha_0}(t)\theta_{\delta_0}(t-s)\nabla_y\omega_\rho\cdot Du_1\,dtd\mathbf{y} \\
&-\int_{(\mathbb{R}_T^N)^2} \operatorname{sgn}(u_1-u_2)\varphi_{\alpha_0}(t)\theta_{\delta_0}(t-s)\omega_\rho\nabla_y\varepsilon(x,t,y,s,u_1)\cdot Du_1\,dtd\mathbf{y} =: R_\delta^{\delta_0,\alpha_0},
\end{aligned}$$

using the Gauss divergence theorem. Therefore, it is deduced that

$$\begin{aligned}
\lim_{\alpha_0\to 0}\lim_{\delta_0\to 0} R_\delta^{\delta_0,\alpha_0} &\ge -\int_\nu^\tau\int_{(\mathbb{R}^N)^2} |\varepsilon(x,t,y,s,u_1)||\nabla_y\omega_\delta(x-y)||Du_1|\,dydt \\
&\quad -\int_\nu^\tau\int_{(\mathbb{R}^N)^2} |\omega_\delta(x-y)||\nabla_y\varepsilon(x,t,y,s,u_1)||Du_1|\,dydt =: R_\delta^1 + R_\delta^2.
\end{aligned}$$

By $\int_{\mathbb{R}^N}|\nabla_y\omega_\delta(x-y)|dy \le \frac{C}{\delta}$ for some constant $C>0$ independent of δ, we have

$$\begin{aligned}
R_\delta^1 &\ge -2\int_\nu^\tau\int_{(\mathbb{R}^N)^2}\{\Big(\sqrt{[\partial_\xi\beta_1](x,t,u_1)} - \sqrt{[\partial_\xi\beta_2](x,t,u_1)}\Big)^2 \\
&\qquad + \Big(\sqrt{[\partial_\xi\beta_2](x,t,u_1)} - \sqrt{[\partial_\xi\beta_2](y,t,u_1)}\Big)^2\}|\nabla_y\omega_\delta(x-y)||Du_1|\,dydt \\
&\ge -2\{\tfrac{C(\tau-\nu)}{\delta}\sup_{t\in(\nu,\tau)} TV(u_1(\cdot,t))\left|\left|\sqrt{[\partial_\xi\beta_1]} - \sqrt{[\partial_\xi\beta_2]}\right|\right|^2_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} \\
&\qquad + C\delta(\tau-\nu)\sup_{t\in(\nu,\tau)} TV(u_1(\cdot,t))\textstyle\sum_{i=1}^N\left|\left|\partial_{x_i}\sqrt{[\partial_\xi\beta_2]}\right|\right|^2_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})}\}.
\end{aligned}$$

Here, we set $\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)} \equiv \mathbb{R}^N\times(\nu,\tau)\times\mathcal{U}$. In addition, it follows that

$$\begin{aligned}
R_\delta^2 &\ge -2\int_\nu^\tau\int_{(\mathbb{R}^N)^2}\left|\nabla_y\sqrt{[\partial_\xi\beta_2](y,s,u_1)}\right| \\
&\qquad\qquad \times\left|\sqrt{[\partial_\xi\beta_1](x,t,u_1)} - \sqrt{[\partial_\xi\beta_2](y,t,u_1)}\right||\omega_\delta(x-y)||Du_1|\,dydt \\
&\ge -2(\tau-\nu)\sup_{t\in(\nu,\tau)} TV(u_1(\cdot,t))\left|\left|\nabla_y\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N} \\
&\qquad \times\Big(\left|\left|\sqrt{[\partial_\xi\beta_1]} - \sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} + \delta\textstyle\sum_{i=1}^N\left|\left|\partial_{x_i}\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})}\Big).
\end{aligned}$$

On the other hand, the diffusion terms of the right-hand side in (3.7) are calculated as follows:

$$\begin{aligned}
&\int_{(\mathbb{R}^N_T)^2} \mathrm{sgn}(u_1-u_2)(\{-([\nabla_x\beta_1](x,t,u_1)-[\nabla_x\beta_1](x,t,u_2))\\
&\qquad +([\nabla_y\beta_2](y,s,u_1)-[\nabla_y\beta_2](y,s,u_2))\}\cdot\nabla_x\omega_\delta(x-y)\varphi_{\alpha_0}(t)\theta_{\delta_0}(t-s)\\
&\qquad -([\Delta_y\beta_2](y,s,u_1)-[\Delta_x\beta_1](x,t,u_2))\phi_\delta^{\delta_0,\alpha_0})d\mathbf{x}d\mathbf{y}\\
&=\int_{(\mathbb{R}^N_T)^2}\mathrm{sgn}(u_1-u_2)\phi_\delta^{\delta_0,\alpha_0}([\partial_\xi\nabla_x\beta_1](x,t,u_1)-[\partial_\xi\nabla_y\beta_2](y,s,u_1))\cdot Du_1dtd\mathbf{y}\\
&\quad-\int_{(\mathbb{R}^N_T)^2}\mathrm{sgn}(u_1-u_2)\{([\Delta_y\beta_2](y,s,u_1)-[\Delta_y\beta_1](y,s,u_1))\\
&\qquad +([\Delta_y\beta_1](y,s,u_1)-[\Delta_x\beta_1](x,t,u_1))\}\phi_\delta^{\delta_0,\alpha_0}d\mathbf{x}d\mathbf{y}=:L_{\delta,\beta}^{\delta_0,\alpha_0}.
\end{aligned}$$

Then, we can see that

$$\begin{aligned}
\lim_{\alpha_0\to0}\lim_{\delta_0\to0}L_{\delta,\beta}^{\delta_0,\alpha_0}&\le(\tau-\nu)||[\Delta_y\beta_2]-[\Delta_y\beta_1]||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))}\\
&+\delta(\tau-\nu)\textstyle\sum_{i,j=1}^N||[\partial_{x_j}\partial_{x_i}^2\beta_1]||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))}\\
&+(\tau-\nu)\sup_{t\in(\nu,\tau)}TV(u_1(\cdot,t))||[\partial_\xi\nabla_x\beta_1]-[\partial_\xi\nabla_x\beta_2]||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N}\\
&+\delta(\tau-\nu)\sup_{t\in(\nu,\tau)}TV(u_1(\cdot,t))\textstyle\sum_{i,j=1}^N||[\partial_{x_i}\partial_\xi\partial_{x_j}\beta_2]||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})}.
\end{aligned}$$

Step 3. We investigate the convection terms in (3.7) as follows:

$$\begin{aligned}
&\int_{(\mathbb{R}^N_T)^2}\mathrm{sgn}(u_1-u_2)\{((A_1(x,t,u_1)-A_1(x,t,u_2))\\
&\qquad +(A_2(y,s,u_2)-A_2(y,s,u_1)))\cdot\nabla_x\phi_\delta^{\delta_0,\alpha_0}\\
&\qquad -([\nabla_x\cdot A_1](x,t,u_2)-[\nabla_yA_2](y,s,u_1))\phi_\delta^{\delta_0,\alpha_0}\}d\mathbf{x}d\mathbf{y}\\
&=-\int_{(\mathbb{R}^N_T)^2}\mathrm{sgn}(u_1-u_2)([\partial_\xi A_1](x,t,u_1)-[\partial_\xi A_2](y,s,u_1))\phi_\delta^{\delta_0,\alpha_0}Du_1dtd\mathbf{y}\\
&\quad+\int_{(\mathbb{R}^N_T)^2}\mathrm{sgn}(u_1-u_2)\{([\nabla_y\cdot A_2](y,s,u_1)-[\nabla_y\cdot A_1](y,s,u_1))\\
&\qquad +([\nabla_y\cdot A_1](y,s,u_1)-[\nabla_x\cdot A_1](x,t,u_1))\}\phi_\delta^{\delta_0,\alpha_0}d\mathbf{x}d\mathbf{y}=:L_{\delta,A}^{\delta_0,\alpha_0}.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\lim_{\alpha_0\to0}\lim_{\delta_0\to0}L_{\delta,A}^{\delta_0,\alpha_0}&\le(\tau-\nu)\sup_{t\in(\nu,\tau)}TV(u_1(\cdot,t))||[\partial_\xi A_1]-[\partial_\xi A_2]||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N}\\
&+(\tau-\nu)||[\nabla_y\cdot A_2]-[\nabla_y\cdot A_1]||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))}\\
&+\delta(\tau-\nu)\textstyle\sum_{i,j=1}^N||\partial_{x_j}\partial_{x_i}A_1^i||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))}.
\end{aligned}$$

On the other hand, it can be seen that

$$\begin{aligned}
&\int_{(\mathbb{R}^N_T)^2}|u_1-u_2|(\partial_t+\partial_s)\phi_\delta^{\delta_0,\alpha_0}d\mathbf{x}d\mathbf{y}\\
&=\int_{(\mathbb{R}^N_T)^2}(|u_1(x,t)-u_1(y,t)|+|u_1(y,t)-u_2(y,t)|+|u_2(y,t)-u_2(y,s)|)\\
&\qquad\times(\theta_{\alpha_0}(t-\nu)-\theta_{\alpha_0}(t-\tau))\theta_{\delta_0}(t-s)\omega_\delta(x-y)d\mathbf{x}d\mathbf{y}=:L_\delta^{\delta_0,\alpha_0}.
\end{aligned}$$

Then, it is deduced that

$$
\begin{aligned}
&\lim_{\alpha_0\to 0}\lim_{\delta_0\to 0} L_\delta^{\delta_0,\alpha_0} \\
&= \int_{(\mathbb{R}^N)^2} (|u_1(x,\nu)-u_1(y,\nu)| - |u_1(x,\tau)-u_1(y,\tau)|)\omega_\delta(x-y)dxdy \\
&\quad + \int_{(\mathbb{R}^N)^2} (|u_1(y,\nu)-u_2(y,\nu)| - |u_1(y,\tau)-u_2(y,\tau)|)\omega_\delta(x-y)dxdy \\
&\le 2\delta \sup_{t\in(\nu,\tau)} TV(u_1(\cdot,t)) + ||u_1(y,\nu)-u_2(y,\nu)||_{L^1(\mathbb{R}^N)} \\
&\quad - ||u_1(y,\tau)-u_2(y,\tau)||_{L^1(\mathbb{R}^N)}.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&\lim_{\alpha_0\to 0}\lim_{\delta_0\to 0} \int_{(\mathbb{R}^N_T)^2} \operatorname{sgn}(u_1-u_2)(B_1(x,t,u_1)-B_2(y,s,u_2))\phi_\delta^{\delta_0,\alpha_0} d\mathbf{x}d\mathbf{y} \\
&\le (\tau-\nu)||B_1-B_2||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} + \delta(\tau-\nu)\textstyle\sum_{i=1}^N ||\partial_{x_i}B_2||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} \\
&\quad + ||\partial_\xi B_2||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} \int_\nu^\tau \int_{\mathbb{R}^N} |u_1-u_2| d\mathbf{x}.
\end{aligned}
$$

Step 4. According to the above estimates and Theorem 2.3 (I), we see that

$$
\begin{aligned}
&||u_1(y,\tau)-u_2(y,\tau)||_{L^1(\mathbb{R}^N)} \le ||u_1(y,\nu)-u_2(y,\nu)||_{L^1(\mathbb{R}^N)} + (\alpha_1^{\nu,\tau}+\alpha_2^{\nu,\tau}\delta)(\tau-\nu) \\
&\quad + ||\partial_\xi B_2||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} \int_\nu^\tau \int_{\mathbb{R}^N} |u_1-u_2| d\mathbf{x} + 2\delta e^{C_0\tau}(TV(u_{1,0})+C_1) \\
&\quad + \tfrac{C(\tau-\nu)}{\delta} e^{C_0\tau}(TV(u_{1,0})+C_1) \left|\left| \sqrt{[\partial_\xi\beta_1]} - \sqrt{[\partial_\xi\beta_2]} \right|\right|^2_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})},
\end{aligned}
$$

where $\alpha_1^{\nu,\tau}$ and $\alpha_2^{\nu,\tau}$ are constants depending ν and τ which are defined as follows

$$
\begin{aligned}
\alpha_1^{\nu,\tau} := &||B_1-B_2||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} + ||[\nabla_y\cdot A_1]-[\nabla_y\cdot A_2]||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} \\
&+ ||[\Delta_y\beta_1]-[\Delta_y\beta_2]||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} + e^{C_0\tau}(TV(u_{1,0})+C_1) \\
&\times\{||[\partial_\xi\nabla_x\beta_1]-[\partial_\xi\nabla_x\beta_2]||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N} + ||[\partial_\xi A_1]-[\partial_\xi A_2]||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N} \\
&+2\left|\left|\nabla_y\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N} \left|\left|\sqrt{[\partial_\xi\beta_1]}-\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_2^{\nu,\tau} := &\textstyle\sum_{i=1}^N ||\partial_{x_i}B_2||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} + \sum_{i,j=1}^N ||\partial_{x_j}\partial_{x_i}A_1^i||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} \\
&+ \textstyle\sum_{i,j=1}^N ||\partial_{x_j}\partial^2_{x_i}\beta_1||_{L^\infty((\nu,\tau)\times\mathcal{U};L^1(\mathbb{R}^N))} + e^{C_0\tau}(TV(u_{1,0})+C_1) \\
&\times\{\textstyle\sum_{i,j=1}^N ||[\partial_{x_i}\partial_\xi\partial_{x_j}\beta_2]||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} + \sum_{i=1}^N \left|\left|\partial_{x_i}\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} \\
&\times(2\left|\left|\nabla_y\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})^N} + C\textstyle\sum_{i=1}^N \left|\left|\partial_{x_i}\sqrt{[\partial_\xi\beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})})\}.
\end{aligned}
$$

Here, we set $\delta = \sqrt{\tau - \nu}\left|\left|\sqrt{[\partial_\xi \beta_1]} - \sqrt{[\partial_\xi \beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})}$. Then, we obtain

$$\begin{aligned}
&||u_1(y,\tau) - u_2(y,\tau)||_{L^1(\mathbb{R}^N)} \leq ||u_1(y,\nu) - u_2(y,\nu)||_{L^1(\mathbb{R}^N)} \\
&\quad + ||\partial_\xi B_2||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})} \int_\nu^\tau \int_{\mathbb{R}^N} |u_1 - u_2| dx dt + \alpha_1^{\nu,\tau}(\tau - \nu) + \sqrt{\tau - \nu} \\
&\quad\quad \times (\alpha_2^{\nu,\tau}(\tau - \nu) + (C+2)e^{C_0 \tau}(TV(u_{1,0}) + C_1)) \left|\left|\sqrt{[\partial_\xi \beta_1]} - \sqrt{[\partial_\xi \beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(\nu,\tau)})}.
\end{aligned}$$

Consequently, we conclude that

$$\begin{aligned}
||u_1(y,t) - u_2(y,t)||_{L^1(\mathbb{R}^N)} &\leq e^{\alpha' t}||u_1(y,0) - u_2(y,0)||_{L^1(\mathbb{R}^N)} + \tfrac{(e^{\alpha' t} - 1)}{\alpha'} \alpha_1^{0,T} \\
&\quad + \widehat{C}\sqrt{t} e^{\alpha' t} \left|\left|\sqrt{[\partial_\xi \beta_1]} - \sqrt{[\partial_\xi \beta_2]}\right|\right|_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(0,T)})},
\end{aligned}$$

where $\alpha' := \max_{i=1,2}\{||\partial_\xi B_i||_{L^\infty(\mathbb{R}^{N,\mathcal{U}}_{(0,T)})}\}$ and $\widehat{C} := \alpha_2^{0,T} T + (C+2)e^{C_0 T}(TV(u_{1,0}) + C_1)$.

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