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CONVERSE PROBLEM FOR THE TWO-COMPONENT RADIAL GROSS-PITAEVSKII SYSTEM WITH A LARGE COUPLING PARAMETER

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Abstract. We consider strongly coupled competitive elliptic systems that arise in the study of two-component Bose-Einstein condensates. As the coupling parameter tends to infinity, solutions that remain uniformly bounded are known to converge to a segregated limiting profile, with the difference of its components satisfying a limit scalar PDE. In the case of radial symmetry, under natural non-degeneracy assumptions on a solution of the limit problem, we establish by a perturbation argument its persistence as a solution to the elliptic system.

Key words. Singular perturbation, competitive elliptic system, segregation

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1. Introduction.

We consider coupled elliptic systems of the form

$$\Delta u_i = f_i(u_i) + g u_i \sum_{j\neq i} a_{ij} u_j^2, \text{ in } \Omega;\ u_i = 0 \text{ on } \partial\Omega, \tag{1.1}$$

$i = 1, \cdots, m$, where f_i are smooth functions with

$$f_i(0) = 0, \tag{1.2}$$

g is a real parameter, a_{ij} are nonnegative constants such that $a_{ii} > 0$, $a_{ij} = a_{ji}$, $i, j = 1, \cdots, m$, and Ω is a bounded smooth N-dimensional domain. Systems of this form arise in the study of multi-component Bose-Einstein condensates. In this context, the reaction terms are typically

$$f_i(u) = g_i u^3 - \mu_i u,\ g_i, \mu_i \in (-\infty, +\infty). \tag{1.3}$$

The coupling parameter g measures the interaction between the different components in the mixture: if $g < 0$ they attract each other, whereas if $g > 0$ they repel each other. On the other hand, the coefficients g_i in (1.3) measure the interaction between atoms in the same i-th component: if $g_i < 0$ there is attraction, whereas if $g_i > 0$ there is repulsion.

The function u_i represents the density corresponding to the i-th component in the mixture, and thus is naturally assumed to be positive. Nevertheless, the mathematical interest to (1.1) also extends to sign-changing solutions. In passing, we note that (1.1) has variational structure as it comes from a Gross-Pitaevskii energy.

In the following, we will consider the case of strong repulsion (or competition), that is $g \gg 1$. Moreover, we will focus on the case of two components, but first let us recall some of the main known results for the case of m components.

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1.1. Known results. In the seminal paper [13] (see also [8] for the corresponding parabolic problem), it was shown that if a family of solutions $\mathbf{u}_g = (u_1^g, \cdots, u_m^g)$ of (1.1) remains bounded in $L^\infty(\Omega)$ as $g \to +\infty$, then it also remains bounded in $C^\alpha(\bar\Omega)$ for any $\alpha \in (0,1)$. We also refer to [25] for a related result in planar domains. Hence, thanks to a well known compact imbedding, possibly up to a subsequence $g_n \to +\infty$, such a family converges in $C^\alpha(\bar\Omega)$ for any $\alpha < 1$ to some limiting configuration $\mathbf{u}_\infty = (u_1^\infty, \cdots, u_m^\infty)$. In fact, it was shown in [13] that the limiting profile has Lipschitz regularity up to the boundary of Ω. Furthermore, the limiting components are segregated, that is their supports are disjoint. In its respective support, the limiting component u_i^∞ satisfies the following elliptic problem

$$\Delta u_i^\infty = f_i(u_i^\infty). \tag{1.4}$$

In the language of singular perturbations, the above limit problem is called the outer limit problem.

More recently, it was shown in [18] that such families $\mathbf{u}_g$ remain bounded, uniformly in g, even in the Lipschitz norm, at least away from the boundary of the domain and for positive solutions

The regularity properties of the sharp interface

$$\Gamma = \left\{x \in \bar\Omega \ : \ u_1^\infty(x) = \cdots = u_m^\infty(x) = 0\right\}$$

were subsequently studied in [22]. It was shown there that Γ has properties analogous to the nodal set of eigenfunctions of the Laplacian: there exists $\Sigma \subset \Gamma$ with $\mathcal{H}_{dim}(\Sigma) \leq N-2$ such that $\Gamma \setminus \Sigma$ is a finite union of smooth manifolds (we refer to [23] for a detailed description of Σ). The set Σ is referred to as the singular part of the interface Γ, whereas $\Gamma \setminus \Sigma$ as the regular part. On each side of a smooth manifold M that composes the regular part of the interface there is only one nontrivial limiting component. Moreover, across M the corresponding limiting components, say $u_\infty = u_i^\infty$ and $v_\infty = u_j^\infty$ (it holds $i \neq j$, see [9]), satisfy the following reflection law:

$$|\nabla u_\infty| = |\nabla v_\infty| \ \text{ on } M. \tag{1.5}$$

We note that the above normal derivatives are nonzero by (1.2), (1.4) and Hopf's boundary point lemma.

More refined estimates for the convergence as $g \to +\infty$ have recently been obtained in [20] and [24]. In particular, it was shown in the former reference that near a point p of M, the two corresponding components $u_g = u_i^g$, $v_g = u_j^g$ ($i \neq j$) that survive as $g_n \to +\infty$ should behave, to main order, in the following self-similar fashion:

$$u_g(x) \sim g^{-\frac14} U\left(g^{\frac14}\mathrm{dist}(x,M)\right), \quad v_g(x) \sim g^{-\frac14} V\left(g^{\frac14}\mathrm{dist}(x,M)\right), \tag{1.6}$$

where $\mathrm{dist}(\cdot, M)$ stands for the signed distance to M, while the one-dimensional profiles $U(t), V(t)$ depend only on the point p and satisfy

$$\begin{cases} U'' = UV^2 \\ V'' = VU^2 \end{cases} \tag{1.7}$$

in the entire real line. It was shown in [4, 5] that the above problem has just a 2-parameter family of positive solutions given by

$$\mu U(\mu t + \tau), \ \mu V(\mu t + \tau),$$

with scaling parameter $\mu > 0$ and translation $\tau \in (-\infty, +\infty)$, for some fixed solution pair (U, V) which satisfies the mirror reflection symmetry

$$U(-t) \equiv V(t), \tag{1.8}$$

and enjoys the following asymptotic behaviour at respective infinities:

$$U(t) \to 0 \text{ as } t \to -\infty; \quad U'(t) \to |\nabla u_\infty(p)| > 0 \text{ as } t \to +\infty.$$

Notice that the convergence in the previous limits is super-exponentially fast. In fact, it was observed in [1] that there is an asymptotic phase $k = k(p) > 0$ in the asymptotic behaviour of U at $+\infty$. Combining all the previous information, we deduce that, for $t > 0$ large enough,

$$U(t) = |\nabla u_\infty(p)|t + k + O(e^{-c_1 t^2}) \text{ and } V(t) = O(e^{-c_2 t^2}), \tag{1.9}$$

for some positive constants c_1 and c_2. The above relations can be differentiated and, via (1.8), provide the corresponding asymptotic behaviour as $t \to -\infty$.

One also expects that the behaviour of solutions for large g near Σ should be governed by an equivariant entire solution with polynomial growth of the PDE version of system (1.7), see [5, 19], which is usually called the inner (or blow-up) limit problem.

1.2. The problem with two-components. ¿From now on, we will consider the special case of problem (1.1) with $m = 2$, which (after a rescaling) we can write as

$$\begin{cases} -\Delta u + f(u) + guv^2 = 0 & \\ & \text{in } \Omega; \\ -\Delta v + h(v) + gvu^2 = 0 & \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

for some smooth functions f and h such that $f(0) = h(0) = 0$ and Ω still a bounded, smooth N-dimensional domain.

We note that the reflection law (1.5) implies that the difference

$$w = u_\infty - v_\infty$$

is smooth across the regular part of the interface. In fact, it was shown in [9] that this difference is a classical solution of the following limit problem

$$\Delta w = f(w^+) - h(-w^-) \text{ in } \Omega; \; w = 0 \text{ on } \partial\Omega, \tag{1.11}$$

where one writes

$$w = w^+ + w^- \text{ with } w^+ \geq 0 \text{ and } w^- \leq 0.$$

It is worthwhile mentioning that in the special case where $f \equiv h$ is odd, the above limit problem reduces to

$$\Delta w = f(w) \text{ in } \Omega; \; w = 0 \text{ on } \partial\Omega. \tag{1.12}$$

1.3. The converse problem. So far we have discussed how one can reach the limit problem (1.11) (and also (1.7)) starting from an appropriate family of solutions to (1.10) for large g. It is also of interest whether one can go in the opposite direction, that is under which conditions do solutions of the limit problem (1.11) generate corresponding solutions of (1.10) for large values of g.

In [10], Dancer considered (1.10) for nonlinearities as in (1.3) with $g_1, g_2 > 0$ (with the obvious correspondence with (1.1)). It was shown by variational methods that, under appropriate restrictions on μ_1, μ_2, a certain type of nodal least energy solutions of (1.11) generate corresponding solutions with positive components to (1.10) for large g. On the other hand, the authors of [26] considered the case where $g_1 = g_2 < 0$ (say -1) and $\mu_1 = \mu_2 > 0$ (say 1) in a ball in two or three dimensions. In this case, it is well known that, for any integer $m \geq 1$, the (reduced) limit problem (1.12) admits a radial nodal solution w_m with exactly an m number of sign changes. Using variational methods, they were able to show that each w_m produces a corresponding radial solution of (1.10) with positive components that shadow respectively $(w_m)^+$ and $-(w_m)^-$ as $g \to +\infty$.

At this point let us make a small detour and discuss briefly the analogous elliptic system modeling two competing populations that arises in spatial ecology. In that context, the coupling terms in both equations of (1.10) are guv, while the nonlinearities f, h are usually of logistic type. Remarkably, uniformly bounded families of solutions to both systems share essentially the same regularity properties (with respect to large g), see [6]. In particular, they have the same (outer) limit problem (1.11). For the population problem, it was shown in [7] by means of a topological degree theoretic argument that non-degenerate (in the sense that the linearized operator does not have a kernel) nodal solutions w of (1.11) give corresponding solutions (u_g, v_g) with positive components for the system with large g. The key idea for proving this is to consider the difference $u - v$ and note that this leads to a system with only one singularly perturbed equation (a standard slow-fast system in the language of dynamical systems). Interestingly enough, this result was established without making use of the analogous blow-up limit problem to (1.7). In light of the aforementioned common features of the two systems, it is natural to expect that an analogous converse result should also hold for the condensate problem (1.10), see [11].

2. Main result. We show that an analogous converse result holds for the condensate problem (1.10), provided that we restrict to the radial setting and we impose some extra but milder non-degeneracy assumptions on the solution of the limit problem (1.11).

THEOREM 2.1. *Let Ω be an N-dimensional ball or annulus, $N \geq 1$, and let $f, h \in C^4[0,\infty)$ be such that $f(0) = h(0) = 0$. Suppose that w is a radial nodal solution of the limit problem (1.11) with one sign change, which is non-degenerate in the radial class in the sense that the associated linearization does not have a nontrivial radially symmetric element in its kernel. Moreover, assume that $-w^-$ and w^+ are also non-degenerate in the radial class as solutions of (1.11) in their respective supports. Then, if g is sufficiently large, there exists a radial solution (u_g, v_g) of (1.10) with positive components such that*

$$\|v_g + w^-\|_{L^\infty(\Omega)} \leq Cg^{-\frac{1}{4}}, \quad \|u_g - w^+\|_{L^\infty(\Omega)} \leq Cg^{-\frac{1}{4}},$$

where the constant $C > 0$ is independent of g.

If r_0 denotes the radius of the sphere where w vanishes, and $(r - r_0)w(r) > 0$ for

$r \neq r_0$, it holds

$$\begin{cases} u_g(r) = g^{-\frac{1}{4}} U\left(g^{\frac{1}{4}}(r-r_0)\right) + O\left(g^{-\frac{1}{2}} + (r-r_0)^2\right) \\ v_g(r) = g^{-\frac{1}{4}} V\left(g^{\frac{1}{4}}(r-r_0)\right) + O\left(g^{-\frac{1}{2}} + (r-r_0)^2\right) \end{cases}$$

for $|r - r_0| \leq (\ln g) g^{-\frac{1}{4}}$, as $g \to +\infty$, where the pair (U, V) is the unique solution of (1.7) satisfying (1.8) and (1.9) with $u_\infty = w^+$ and $|p| = r_0$.

As we will describe in more detail in the sequel, our proof relies on a perturbative method. We first combine the outer and inner problems, (1.11) and (1.7) respectively, to construct a sufficiently good approximate solution to (1.10) for large g that is valid in the whole domain. Then, we can capture a genuine solution nearby by a fixed point argument owing to appropriate invertibility properties of the associated linearized operator between carefully chosen weighted spaces.

We point out that the separate non-degeneracy assumptions on $-w^-$ and w^+ were not present in the previously mentioned result of [7] for the population system. As will become apparent shortly, the underline reason for imposing them is the presence of the positive asymptotic phase k in the asymptotic behaviour of the blow-up profile (recall (1.9)). We point out that there was no such phase present in the analogous blow-up limits for the population problem. Loosely speaking, the outer and inner approximate solutions, given to main order by $\pm w^\pm$ and (1.6) with $M = \{|x| = r_0\}$, respectively, do not have the phase $k > 0$ in common (in the intermediate zone where they must match). Therefore, we need to move the outer solutions towards the inner one by a regular perturbation to compensate for the gap caused by $k > 0$ (in principle, the inner solution should control the outer ones). To be able to do so, by means of the implicit function theorem, we need to impose these non-degeneracy assumptions on $-w^-$ and w^+. We remark that the non-degeneracy assumptions for $\pm w^\pm$ are much easier to verify in practice (see for instance [16]) in comparison to that for w which is a sign-changing solution (see [21]); see also Section 4 below.

We believe that an analogous result still holds when w changes sign an arbitrary number of times, provided one imposes further analogous non-degeneracy assumptions to take into account the interaction created by adjacent zeros of $w(r)$ for $1 \ll g < \infty$.

3. Sketch of the proof. In this section, we describe briefly the main steps in the proof of Theorem 2.1. For simplicity, we will do this in a one-dimensional setting where $\Omega = (a, b)$ and $r_0 = 0$. The general radial case can be treated in a completely analogous manner.

We write v_0 instead of $-w^-$, u_0 instead of w^+, and set

$$\psi_0 = -v_0'(0) = u_0'(0) > 0.$$

3.1. Construction of the approximate solution (u_{ap}, v_{ap}). Firstly, around the origin we consider a two-parameter family of first order inner approximate solutions of the form

$$u_{in}(x) = \mu g^{-\frac{1}{4}} U(t), \ v_{in}(x) = \mu g^{-\frac{1}{4}} V(t), \text{ where } t = \mu g^{\frac{1}{4}}(x - \xi), \tag{3.1}$$

with $\mu > 0$ and $\xi \in (-\infty, \infty)$. The remainder left by this approximation in (1.10) is of order $|x| + g^{-\frac{1}{4}}$, therefore we will use it for $|x| \leq |\ln g| g^{-\frac{1}{4}}$ (keep in mind also the super-exponential rate of convergence in (1.9)).

In $(a,0)$ and $(0,b)$ we consider one-parameter family of outer approximate solutions of the form $\left(0, v_{\tilde{\delta}}\right)$ and $(u_\delta, 0)$, respectively, through the following boundary value problems:

$$
\begin{cases} v_{\tilde{\delta}}'' = h(v_{\tilde{\delta}}), & x \in (a,0), \\ \\ v_{\tilde{\delta}}(a) = 0, & v_{\tilde{\delta}}(0) = \tilde{\delta}, \end{cases} \qquad \begin{cases} u_\delta'' = f(u_\delta), & x \in (0,b), \\ \\ u_\delta(0) = \delta, & u_\delta(b) = 0, \end{cases} \tag{3.2}
$$

for $0 \le \tilde{\delta}, \delta \ll 1$. We point out that such $v_{\tilde{\delta}}, u_\delta$ exist and depend smoothly on $\tilde{\delta}, \delta \ge 0$ thanks to the implicit function theorem and the assumption that v_0 and u_0 are non-degenerate solutions of the above problems for $\tilde{\delta} = 0$ and $\delta = 0$, respectively. In fact, the following asymptotic expansion holds:

$$
u_\delta = u_0 + \delta u_1 + \delta^2 u_2 + \delta^3 u_3 + O(\delta^4)
$$

where the u_i for $i \ge 1$ are given as solutions of linear inhomogeneous problems (which are solvable thanks to the aforementioned non-degeneracy of u_0). In particular, we have

$$
-u_1'' + f'(u_0)u_1 = 0,\ x \in (0,b);\ u_1(0) = 1,\ u_1(b) = 0.
$$

Naturally, an analogous expansion holds also for $v_{\tilde{\delta}}$. The outer approximate solution, made up by $(0, v_{\tilde{\delta}})$ and $(u_\delta, 0)$, will be used for $|x| \ge |\ln g| g^{-\frac{1}{4}}$. In fact, it solves (1.10) exactly except from $x = 0$. As a first order outer approximate solution (u_{out}, v_{out}) we take the pairs

$$
\left(0, v_0 + \tilde{\delta}_1 v_1\right) \ \text{and} \ (u_0 + \delta_1 u_1, 0) \tag{3.3}
$$

in $\left(a, -|\ln g| g^{-\frac{1}{4}}\right)$ and $\left(|\ln g| g^{-\frac{1}{4}}, b\right)$, respectively, with $\tilde{\delta}_1, \delta_1$ free parameters.

The main effort is placed in adjusting conveniently the four free parameters $\mu, \xi, \delta_1, \tilde{\delta}_1$ so that the above first order inner and outer approximate solutions match in an appropriate intermediate zone, which we can take as $|\ln g| g^{-\frac{1}{4}} \le |x| \le 2|\ln g| g^{-\frac{1}{4}}$. On the one hand, from (3.1), by virtue of (1.9) with asymptotic slope $\psi_0 > 0$ and asymptotic phase $k > 0$, the first component of the first order inner approximate solution behaves essentially as a linear function of $t = \mu g^{\frac{1}{4}}(x - \xi) \gg 1$. On the other hand, we see from (3.3) that the corresponding component of the outer approximate solution has, to main order, a linear behaviour in x near $x = 0^+$. By comparing these (say equating the powers x^0 and x^1), we get two equations to be satisfied. We point out that powers of x^2 are not present in neither the first order outer or inner approximation. We stress that an analogous property propagates to higher order powers of x when matching higher order inner and outer approximate solutions, merely by equating the powers x^0 and x^1 at each step. Doing the same on the other side for the second components, gives two more equations. The resulting system of four equations and! four unknowns, after setting $\mu = 1 + \mu_1$, reads as follows:

$$
\begin{cases} \delta_1 & = g^{-\frac{1}{4}} k - \xi \psi_0, \\ \delta_1 u_1'(0) & = 2\psi_0 \mu_1, \\ \tilde{\delta}_1 & = g^{-\frac{1}{4}} k + \xi \psi_0, \\ \tilde{\delta}_1 v_1'(0) & = -2\psi_0 \mu_1. \end{cases}
$$

The above system has the following unique solution, provided that $v_1'(0) \neq u_1'(0)$:

$$\mu_1 = -\frac{g^{-\frac{1}{4}} k u_1' v_1'}{\psi_0(u_1' - v_1')},\ \xi = \frac{g^{-\frac{1}{4}} k (u_1' + v_1')}{\psi_0(u_1' - v_1')},\ \delta_1 = -\frac{2g^{-\frac{1}{4}} k v_1'}{(u_1' - v_1')},\ \tilde{\delta}_1 = \frac{2g^{-\frac{1}{4}} k u_1'}{(u_1' - v_1')},$$

where here u_1', v_1' are evaluated at zero. Observe that thanks to the non-degeneracy assumption on w, we always have $v_1'(0) \neq u_1'(0)$ (otherwise, the union of v_1 and u_1 would be an element of the kernel of the linearization of (1.11)).

To improve the remainder left by (3.1) in (1.10), we consider a more refined inner approximate solution of the form

$$u_{in}(x) = \mu g^{-\frac{1}{4}} U(t) + \varphi(t),\ v_{in}(x) = \mu g^{-\frac{1}{4}} V(t) + \tilde{\varphi}(t), \tag{3.4}$$

for fluctuations $\varphi, \tilde{\varphi}$ of higher order. We point out that we will not adjust further μ and ξ, analogous parameters will appear shortly. In order to choose corrections $\varphi, \tilde{\varphi}$ for a second order inner approximate solution, we have to try (3.4) in (1.10), and then take into account the matching with the corresponding second order outer approximate solution. The latter is comprised of

$$\left(0, v_0 + (\tilde{\delta}_1 + \tilde{\delta}_2) v_1 + (\tilde{\delta}_1 + \tilde{\delta}_2)^2 v_2\right),\ \left(u_0 + (\delta_1 + \delta_2) u_1 + (\delta_1 + \delta_2)^2 u_2, 0\right) \tag{3.5}$$

in $\left(a, -|\ln g| g^{-\frac{1}{4}}\right)$ and $\left(|\ln g| g^{-\frac{1}{4}}, b\right)$, respectively, with $\delta_1, \tilde{\delta}_1$ as above and $\delta_2, \tilde{\delta}_2$ are higher order corrections to be chosen.

At first sight it seems that, to main order, the inner corrections should satisfy the following inhomogeneous linear problem in $(-\infty, +\infty)$:

$$\begin{cases} -\varphi'' + V^2 \varphi + 2UV \tilde{\varphi} = -\mu^{-1} g^{-3/4} f'(0) U, \\ -\tilde{\varphi}'' + U^2 \tilde{\varphi} + 2UV \varphi = -\mu^{-1} g^{-3/4} h'(0) V. \end{cases} \tag{3.6}$$

We note that the linear operator in the left side is precisely the linearization of the blow-up problem (1.7) about (U, V). It is important to note that this operator includes in its kernel the pairs (U', V') and $(tU' + U, tV' + V)$ due to the translation and scaling invariance of (1.7). In fact, it was shown in [4] that the only bounded elements in the kernel are constant multiples of (U', V'). By setting

$$(\varphi, \tilde{\varphi}) = \mu^{-1} g^{-\frac{3}{4}} \left((Z, \tilde{Z}) + (\varphi_1, \tilde{\varphi}_1) \right),$$

where $Z, \tilde{Z}$ are fixed, smooth functions such that

$$\begin{cases} Z(t) = 0,\ t \leq -1, \quad Z(t) = f'(0) \left(k \frac{t^2}{2} + \psi_0 \frac{t^3}{6} \right),\ t \geq 1, \\ \tilde{Z}(t) = h'(0) \left(k \frac{t^2}{2} - \psi_0 \frac{t^3}{6} \right),\ t \leq -1, \quad \tilde{Z}(t) = 0,\ t \geq 1, \end{cases}$$

we can transform (3.6) to an equivalent problem for $(\varphi_1, \tilde{\varphi}_1)$ with the same linear operator on the left side but with righthand side that decays super-exponential fast as $t \to \pm\infty$ and is independent of g. By the linear theory developed in [1], the resulting problem has a solution such that, for any $M > 1$, it holds

$$\begin{gathered} \varphi_1(t) = a_+ t + b + O(e^{-Mt}),\ \tilde{\varphi}_1(t) = O(e^{-Mt}) \text{ as } t \to +\infty, \\ \varphi_1(t) = O(e^{Mt}),\ \tilde{\varphi}_1(t) = a_- t + b + O(e^{Mt}) \text{ as } t \to -\infty, \end{gathered}$$

for some constants $a_\pm, b$. Therefore, we seek corrections $(\varphi,\tilde\varphi)$ in (3.4) in the form

$$(3.7)\quad (\varphi,\tilde\varphi)=\mu^{-1}g^{-\frac{3}{4}}\left((Z,\tilde Z)+(\varphi_1,\tilde\varphi_1)+A(U',V')+B(tU'+U,tV'+V)\right),$$

with A, B free parameters to be determined through the matching with the outer approximation in (3.5). As before, by looking at the powers x^0, x^1, the matching amounts to solving a 4×4 linear system for $A, B, \delta_2, \tilde\delta_2$ which is again possible thanks to the non-degeneracy condition on w. More precisely, we find that $A=O(g^{\frac14}), B=O(1), \delta_2=O(g^{-\frac12}), \tilde\delta_2=O(g^{-\frac12})$. However, it turns out that $A=O(g^{\frac14})$ causes the second order inner approximate solution to leave a remainder of the same order as the first order one. This suggests that there should be a quasi-second order inner approximate solution given by

$$(\psi,\tilde\psi)=\mu^{-1}g^{-\frac34}\left(A_1(U',V')+B_1(tU'+U,tV'+V)\right)$$

as the main correction in (3.4) for some appropriate $A_1=O(g^{\frac14})$ and $B_1=O(1)$. It turns out that a successful way to go about this issue is to determine at the same time (through the previous matching considerations) the above quasi-second order inner solution, the quasi-second order outer (3.5), the genuine second order inner solution that is given by (3.7), writing $A=A_2$, $B=B_2$, with $(\varphi_1,\tilde\varphi_1)$ satisfying the inhomogeneous problem (3.6) with the addition of some super-exponential decaying terms of the same order in the righthand side involving A_1, B_1, and the genuine second order outer solution

$$(3.8)\qquad \left(0,\ \sum_{i=0}^{3}(\tilde\delta_1+\tilde\delta_2+\tilde\delta_3)^i v_i\right),\ \left(\sum_{i=0}^{3}(\delta_1+\delta_2+\delta_3)^i u_i,\ 0\right)$$

where $\delta_3,\tilde\delta_3$ are higher order corrections. We are led to two 4×4 linear systems for $(A_1,B_1,\delta_2,\tilde\delta_2)$ and the corresponding $(A_2,B_2,\delta_3,\tilde\delta_3)$, that are again solvable, with the flexibility of rearranging conveniently their right hand sides so that we get solutions of the desired order in g.

Finally, we can smoothly patch the (genuine) second order outer and inner approximate solutions using cutoff functions in the intermediate zone, and get a smooth global approximate solution (u_{ap},v_{ap}) that leaves a remainder in (1.10) of order $|\ln g|^4 g^{-\frac12}$.

3.2. The fixed point argument. We can perturb the approximate solution to a genuine one by applying the contraction mapping theorem, based on the following a-priori estimates for the associated linearized operator, expanding on ideas from [1].

PROPOSITION 3.1. *Suppose that*

$$\mathcal{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}F\\ H\end{pmatrix},\ x\in(a,b);\quad \phi(a)=\phi(b)=0,\ \psi(a)=\psi(b)=0,$$

where $F,H\in C[a,b]$ *and*

$$\mathcal{L}\begin{pmatrix}\phi\\ \\ \psi\end{pmatrix}\equiv\begin{pmatrix}-\phi''+f'(u_{ap})\phi+gv_{ap}^2\phi+2gu_{ap}v_{ap}\psi\\ \\ -\psi''+h'(v_{ap})\psi+gu_{ap}^2\psi+2gu_{ap}v_{ap}\phi\end{pmatrix}.$$

Then, given $\gamma\in(0,1),\rho>0$, *there exist* $C, g_0>0$, *independent of* (F,H) *and* (ϕ,ψ), *such that*

$$\|(\phi,\psi)\|_1\le Cg^{-\frac14}\|(F,H)\|_2,$$

$$\|(\phi,\psi)\|_1 \le Cg^{-\frac{1}{4}}\|(F,H)\|_0 + Cg^{\rho-\frac{1}{2}}\|(F,H)\|_2,$$

where

$$\|(\Phi,\Psi)\|_i = \|w_i(x)\Phi\|_{L^\infty(a,b)} + \|w_i(-x)\Psi\|_{L^\infty(a,b)}, \quad i=0,1,2,$$

with

$$w_0(x) = \begin{cases} 1+|g^{\frac{1}{4}}x|^{1+\gamma}, & x\in[0,b), \\ 1, & x\in(a,0). \end{cases} \qquad w_1(x) = \begin{cases} 1, & x\in[0,b), \\ e^{g^{\frac{1}{4}}|x|}, & x\in(a,0), \end{cases}$$

$$w_2(x) = \begin{cases} 1+|g^{\frac{1}{4}}x|^{1+\gamma}, & x\in[0,b), \\ e^{g^{\frac{1}{4}}|x|}, & x\in(a,0), \end{cases}$$

provided that $g \ge g_0$.

4. Applications of the main result. Let us now give briefly some applications of Theorem 2.1. As it was already pointed out earlier, in the case $f\equiv h$ and f is odd the limit problem becomes (1.12). It is known that when $f(u)=\lambda u - u^{2p+1}$, $\lambda\ge 0$ and p is such that

$$1 < 2p+1 < \frac{N+2}{N-2} \text{ if } N\ge 3,\ p>0 \text{ if } N=2, \tag{4.1}$$

then a radial solution w to (1.12) is unique and non-degenerate in the radial class provided that

- w is positive, $\lambda\neq 0$ and Ω is an annulus or the exterior of a ball, see [12];
- w is positive, $\lambda=0$ and Ω is a ball or an annulus, see [14];
- w is positive, $\lambda\neq 0$ and Ω is a ball, see [2];
- w is a nodal solution with two nodal regions, $\lambda=0$, see [15].

We also refer to [17] for more general results concerning the function f. We point out that such solutions can be shown to exist by variational methods.

Thanks to these previous results, we see that our result applies in the case $f(u)=-u^{2p+1}$ with p as in (4.1), and Ω a ball or an annulus. In a related topic, let us point out that when Ω is the whole N-dimensional space, $N\ge 3$, and $f(u)=u-|u|^{p-1}u$ with $1<p<\frac{N+2}{N-2}$ sufficiently close to $\frac{N+2}{N-2}$, Ao, Wei and Yao [3] constructed radial solutions with $k\ge 1$ nodes to (1.12) that tend to zero as $r\to\infty$. Moreover, they established that their solutions are unique and non-degenerate. Our theorem, with only minor modifications in the proof, can produce a corresponding solution to (1.10) for large g, starting from such a one-node solution.

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