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UPPER HAUSDORFF DIMENSION ESTIMATES FOR INVARIANT SETS OF EVOLUTIONARY SYSTEMS ON HILBERT MANIFOLDS

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Abstract. We prove a generalization of the Douady-Oesterlé theorem on the upper bound of the Hausdorff dimension of an invariant set of a smooth map on an infinite dimensional manifold. It is assumed that the linearization of this map is a noncompact linear operator. A similar estimate is given for the Hausdorff dimension of an invariant set of a dynamical system generated by a differential equation on a Hilbert manifold.

Key words. Hilbert manifold, Hausdorff dimension, singular value

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1. Basic notation of manifold theory. Let us shortly introduce some definitions and properties for manifolds over a Hilbert space ([1, 8]). Suppose $\mathbb{H}$ is a Hilbert space and $\mathcal{M}$ is a set. A *chart* on $\mathcal{M}$ is a bijection $x : \mathcal{D}(x) \subset \mathcal{M} \to \mathcal{R}(x) \subset \mathbb{H}$, where $\mathcal{R}(x)$ is an open set. An *atlas* A of class $C^k (k \geq 1)$ on $\mathcal{M}$ is a set of charts, such that:
(AT1) $\cup_{x \in A} \mathcal{D}(x) = \mathcal{M}$;
(AT2) For arbitrary $x, y \in A$, such that $\mathcal{D}(y) \cap \mathcal{D}(x) \neq \emptyset$, the set $x(\mathcal{D}(x) \cap \mathcal{D}(y))$ is an open subset in $\mathbb{H}$;
(AT3) For arbitrary $x, y \in A$ the map $y \circ x^{-1} : x(\mathcal{D}(x) \cap \mathcal{D}(y)) \to y(\mathcal{D}(x) \cap \mathcal{D}(y))$ is a C^k diffeomorphism.

A pair $(\mathcal{M}, A)$ where $\mathcal{M}$ is a set and A is a C^k-atlas on $\mathcal{M}$, is called C^k *-manifold over the Hilbert space* $\mathbb{H}$.

Let x and y be two arbitrary charts on $\mathcal{M}$ around the point $u \in \mathcal{M}$. Let $\xi, \eta \in \mathbb{H}$ be arbitrary. Introduce the equivalence relation

$$(u, x, \xi) \sim (u, y, \eta) \Leftrightarrow \eta = (y \circ x^{-1})'(x(u))\xi.$$

The equivalence class

$$[u, x, \xi] = \{(u, y, \eta) | u \in \mathcal{D}(x) \cap \mathcal{D}(y), (u, y, \eta) \sim (u, x, \xi)\},$$

is called *tangent vector* at u. The *tangent space* of $\mathcal{M}$ at u is the set $T_u\mathcal{M}$ of all equivalence classes $[u, x, \xi]$ such that x is a chart, $u \in \mathcal{D}(x)$ and $\xi \in \mathbb{H}$. It is equipped with a vector space structure on $T_u\mathcal{M}$ given by:

$$\begin{aligned} [u, x, \xi] + [u, x, \eta] &= [u, x, \xi + \eta], \forall \xi \in \mathbb{H}, \eta \in \mathbb{H} \\ \lambda[u, x, \xi] &= [u, x, \lambda\xi], \qquad \forall \lambda \in \mathbb{R}, \xi \in \mathbb{H}. \end{aligned}$$

The *tangent bundel* $T\mathcal{M}$ of $\mathcal{M}$ is defined by $T\mathcal{M} = \cup_{u \in \mathcal{M}} T_u\mathcal{M}$.

Suppose that $\mathcal{M}$ is a C^k-manifold over the Hilbert space $\mathbb{H}$. The map $\varphi : \mathcal{U} \subset \mathcal{M} \to \mathcal{M}$ is said to be C^r *-differentiable* ($r \leq k$) at $u \in \mathcal{M}$ if there are charts x around u and y around $\varphi(u)$ such that the map $y \circ \varphi \circ x^{-1}$ is C^r-differentiable in $x(u)$ in the sense of Fréchet.

The *differential* of φ at $u \in \mathcal{U}$ is the linear map $d_u\varphi : T_u\mathcal{M} \to T_{\varphi(u)}\mathcal{M}$, given by

$$d_u\varphi([u, x, \xi]) = [\varphi(u), y, (y \circ \varphi \circ x^{-1})'(x(u))\xi], \tag{1.1}$$

where x, y are charts around u and $\varphi(u)$, respectively, and $\xi \in \mathbb{H}$ is arbitrary.

Let a *Riemannian metric* of class C^{k-1} be defined on the connected C^k-manifold $\mathcal{M}(k \geq 2)$ over the Hilbert space $\mathbb{H}$. Suppose that at every point $u \in \mathcal{M}$ and for every chart x around u there is given a symmetric positive definite operator $G_x : \mathbb{H} \to \mathbb{H}$ with the following properties

(RM1) The map $G_x : \mathcal{D}(x) \to \mathcal{L}(\mathbb{H})$ is $\mathcal{C}^k$-smooth.

(RM2) $[(y \circ x^{-1})'(x(u))]^* G_y(u)[(y \circ x^{-1})'(x(u))] = G_x(u)$ for any two charts x, y around u.

Let $(\mathcal{M}, g)$ be a Riemannian C^r-manifold $(r \geq 3)$ over the Hilbert space $\mathbb{H}$. For any $u \in \mathcal{M}$ and any $v \in T_u\mathcal{M}$ there exists a unique geodesic $\varphi(\cdot, u, v)$ with $\varphi(0, u, v) = u$, $\dot{\varphi}(0, u, v) = v$. Then $(t, u, v) \mapsto \varphi(t, u, v)$ is a C^{r-2}-map.

Definition 1.1. *The map $v \mapsto \exp_u v = \varphi(1, u, v)$ is called exponential map of class C^{r-2} around $0 \in T_u\mathcal{M}$.*

Let $\mathcal{V}$ be a sufficiently small neighborhood of $0 \in T_u\mathcal{M}$. Then the map $\exp_u : \mathcal{V} \to \exp_u \mathcal{V}$ is a C^{r-2} - diffeomorphism.

It follows for any $u \in \mathcal{M}$ and any sufficiently small number $\varepsilon > 0$ the map $\exp_u$ is a C^{r-2} -diffeomorphism on $\mathcal{B}_\varepsilon(0_u) \subset T_u\mathcal{M}$.

For any $v \in \mathcal{B}_\varepsilon(0_u)$ the map $t \mapsto c(t) = \exp_u(t, v)$ with $t \in [0, 1]$ is a geodesic on $\mathcal{M}$.

Let us define a dynamical system and an associated global attractor on the Riemannian manifold ([1, 8]). Let $(\mathcal{M}, \rho)$ be the metric space generated on the Riemannian manifold $(\mathcal{M}, G)$ and let $\{\varphi^t\}_{t \in \mathcal{J}}$ be a family of maps $\varphi^t : \mathcal{M} \to \mathcal{M}$, where $\mathcal{J} \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+\}$. The pair $(\{\varphi^t\}_{t \in \mathcal{J}}, (\mathcal{M}, \rho))$ is called a *dynamical system* on the metric space $(\mathcal{M}, \rho)$ if the following holds:

1. $\varphi^0 = \mathrm{id}_\mathcal{M}$;
2. $\varphi^{t+s} = \varphi^t \circ \varphi^s$ for all $s, t \in \mathcal{J}$;
3. $\varphi^{(\cdot)}(\cdot) : \mathcal{J} \times \mathcal{M} \to \mathcal{M}$ is smooth if $\mathcal{J} \in \{\mathbb{R}, \mathbb{R}_+\}$. The family $\varphi^t : \mathcal{M} \to \mathcal{M}$ of maps with $t \in \mathcal{J}$ is smooth if $\mathcal{J} \in \{\mathbb{Z}, \mathbb{Z}_+\}$

Let $(\{\varphi^t\}_{t \in \mathcal{J}}, (\mathcal{M}, \rho))$ be a dynamical system. A set $\mathcal{A} \subset \mathcal{M}$ is called a *global $\mathcal{B}$-attractor* for the dynamical system if the following conditions are satisfied:

(CM1) $\mathcal{A}$ is compact;

(CM2) $\mathcal{A}$ is an invariant set in the sense that $\varphi^t(\mathcal{A}) = \mathcal{A}, \forall t \in \mathcal{J}$;

(CM3) $\mathcal{A}$ attracts any bounded set $\mathcal{B} \subset \mathcal{M}$ under $\{\varphi^t\}_{t \in \mathcal{J}}$, i.e.

$$\mathrm{dist}(\varphi^t(\mathcal{B}), \mathcal{A}) \to 0 \qquad \text{for} \quad t \to \infty \tag{1.2}$$

$$\text{where} \quad \mathrm{dist}(\mathcal{Z}_1, \mathcal{Z}_2) = \sup_{u \in \mathcal{Z}_1} \inf_{v \in \mathcal{Z}_2} \rho(u, v) \tag{1.3}$$

for any nonempty subsets $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{M}$ is the Hausdorff semidistance.

2. Hausdorff dimension and singular values. In the following we introduce some basic definitions and propositions of singular values for noncompact linear operators. Consider the linear not compact operator $T : \mathbb{K} \to \mathbb{K}'$, where $(\mathbb{K}, (\cdot, \cdot)_\mathbb{K})$ and $(\mathbb{K}', (\cdot, \cdot)_{\mathbb{K}'})$ are Hilbert spaces. (The case when $\mathbb{K} = \mathbb{K}'$ is considered in [[10]].) The *adjoint operator* $T^{[*]} : \mathbb{K}' \to \mathbb{K}$, is defined by the relation $(T\xi, \eta)_{\mathbb{K}'} = (\xi, T^{[*]}\eta)_\mathbb{K}$, $\forall \xi \in \mathbb{K}, \forall \eta \in \mathbb{K}'$.

The *singular values* of T, denoted by $\alpha_i(T)$, are given by

$$\alpha_k(T) = \sup_{\substack{\mathbb{L} \subset \mathbb{K} \\ \dim \mathbb{L} = k}} \inf_{\substack{\xi \in \mathbb{L} \\ |\xi|_\mathbb{K} = 1}} |T\xi|_{\mathbb{K}'}, \qquad k = 1, 2, \ldots. \tag{2.1}$$

Let $T^{\wedge k} : \mathbb{K}^{\wedge k} \to \mathbb{K}'^{\wedge k}$ and let consider $\omega_k(T) = \alpha(T^{\wedge k})$. The function

$$\omega_d(T) = \begin{cases} \omega_{d_0}^{1-s}(T) \cdot \omega_{d_0+1}^{s}(T), & d > 0 \\ 1, & d = 0 \end{cases}$$

is called *the singular value function* of T. Here $d \geq 0$ is written in the form $d = d_0 + s$, $d_0 \in \mathbb{N}_0$, $s \in (0, 1]$.

Let $\{\xi_i\}_{i\in\mathcal{I}}$ be an orthonormal basis of $\mathbb{K}$ such that ξ_i is an eigenvector of $T^{[*]}T$ corresponding to the eigenvalue $\alpha_i(T)$, $i \in \mathcal{I}$. Then there exists an orthonormal basis $\{\eta_i\}_{i\in\mathcal{I}}$ in $\mathbb{K}'$ with $\eta_i = \frac{1}{\alpha_i} T\xi_i$ for any $i \in \mathcal{I}$ and $\alpha_i > 0$. The image of the unit ball $B_1(0) \subset \mathbb{K}$ under the map T is the set

$$\left\{ \sum_{i\in\mathcal{I},\alpha_i(T)\neq 0} c_i\eta_i \in \mathbb{K}' | \sum_{i\in\mathcal{I},\alpha_i(T)\neq 0} \left(\frac{c_i}{\alpha_i(T)}\right)^2 \leq 1 \right\}.$$

The operator $\tilde{T} = T^{[*]}T$ is positive, self-adjoint, and continuous but no longer compact. We introduce the sequence of numbers $\beta_n(\tilde{T})$, $n \geq 1$, defined by

$$\beta_n(\tilde{T}) = \inf_{\substack{\mathbb{L} \subset \mathbb{K} \\ \dim \mathbb{L} = k}} \sup_{\substack{\xi \in \mathbb{L} \\ |\xi|_{\mathbb{K}} = 1}} (\tilde{T}\xi, \xi)_{\mathbb{K}}. \tag{2.2}$$

The sequence $\{\beta_n(\tilde{T})\}$ is nonincreasing and we can easily see that the definition of $\beta_n(\tilde{T})$ is unchanged if we replace the infimum in (2.2) by the infimum for $\mathbb{L} \subset \mathbb{K}$. If $\tilde{T}$ is compact then, according to the well known min-max principle $\beta_n(\tilde{T})$ would be the eigenvalues of $\tilde{T}$.

We set

$$\beta_\infty(\tilde{T}) = \lim_{n\to\infty} \beta_n(\tilde{T}) = \inf_{n\geq 1} \beta_n(\tilde{T}). \tag{2.3}$$

The sequence is stationary at some stage:

$$\beta_1(\tilde{T}) \geq \ldots \geq \beta_{n_0}(\tilde{T}) > \beta_{n_0+1}(\tilde{T}) = \beta_m(\tilde{T}) = \beta_\infty(\tilde{T}), \qquad \forall m \geq n_0 + 1 \tag{2.4}$$

or

$$\beta_m(\tilde{T}) > \beta_\infty(\tilde{T}), \quad \forall m \in N. \tag{2.5}$$

In the first case it follows from the above result that $\beta_1, \ldots \beta_{n_0}$, are eigenvalues of $\tilde{T}$, while in the second case each β_m is an eigenvalue of $\tilde{T}$. In both cases we decompose $\mathbb{K}$ into the direct sum $\mathbb{K}_v \oplus \mathbb{K}_v^\perp$, where $\mathbb{K}_v$ is the space spanned by the eigenvectors of $\tilde{T}, e_i, i \in I$, which we suppose orthonormalized ($I = (1, ..., n_o)$ when (2.3) occurs, $I = \mathbb{N}$ when (2.4) holds). Of course, it may happen that $\mathbb{K}_v = \{O\}$ or $\mathbb{K}_v = \mathbb{K}$.

Let $\mathbb{K} = \mathbb{K}_v \oplus \mathbb{K}_v^\perp$ denote the decomposition of $\mathbb{K}$, where $\mathbb{K}_v$ and $\mathbb{K}_v^\perp$ are orthogonal. In the same way let us introduce the decomposition $\mathbb{K}' = \mathbb{K}'_v \oplus {\mathbb{K}'_v}^\perp$. Let $\{\xi_i\}_{i\in I}$ be an orthonormal basis of $\mathbb{K}_v$ such that ξ_i is an eigenvector of $T^{[*]}T$ corresponding to the eigenvalue $\alpha_i(T)$, $i \in I$. Then there exists an orthonormal basis $\{\eta_i\}_{i\in I}$ in $\mathbb{K}'$ with $\eta_i = \frac{1}{\alpha_i} T\xi_i$ for any $i \in I$ and $\alpha_i \neq 0$. We observe that the vectors $Te_i, i \in I$ are orthogonal, i. e.

$$(Te_i, Te_j)_{\mathbb{K}'} = (T^{[*]}Te_i, e_j)_{\mathbb{K}} = \beta_i(e_i, e_j)_{\mathbb{K}} = \beta_i\delta_{ij} \quad \forall i, j \in I, \tag{2.6}$$

where $\delta_{ij} = (e_i, e_j), \forall i, j \in I$.

The image of the unit ball $B_1(0) \subset \mathbb{K}$ under the map T is included in the sum of the ellipsoid $\sum_{i \in I} \frac{1}{\alpha_i^2} \left(\xi, \frac{Te_i}{\alpha_i} \right)^2 \leq 1$ of $\mathbb{K}'_v$ and of the ball of ${\mathbb{K}'_v}^{\perp}$ centered at 0 of radius $\alpha_\infty(T)$.

The next proposition is a generalization of a result of [10]

PROPOSITION 2.1. *Let $\mathbb{K}$ be a Hilbert space and B its unit ball. Let $T : \mathbb{K} \to \mathbb{K}'$ be a linear continuous operator and, if T is not compact, let $\mathbb{K}_v$ be defined as above. Then $T(B)$ is included in an ellipsoid $\mathcal{E}$:*
(i) If T is not compact, but $\mathbb{K}'_v = \mathbb{K}'$, the axes of $\mathcal{E}$ are directed along the vectors Te_i and their length is $\alpha_i(T)$, the e_i being the eigenvectors of $T^{[*]}T$.
(ii) If T is not compact and $\mathbb{K}'_v \neq \mathbb{K}'$, $\mathcal{E}$ is the product of the ball centered at 0 of radius α_∞ in ${\mathbb{K}'_v}^{\perp}$, and of the ellipsoid of $\mathbb{K}'_v$ whose axes are directed along the vectors Te_i with lengths $\alpha_i(T)$, the e_i being the eigenvectors of T spanning $\mathbb{K}'_v$.

Let $\mathcal{E}$ be an ellipsoid in the Hilbert space $\mathbb{H}'$ and let $a_1(\mathcal{E}) \geq a_2(\mathcal{E}) \geq \ldots$ denote the lengths of the half-axes. For any $j \in \mathbb{N}_0$ we define

$$\omega_j(\mathcal{E}) = \begin{cases} a_1(\mathcal{E}) \cdot \ldots \cdot a_j(\mathcal{E})), & j \in \mathbb{N} \\ 1, & j = 0 \end{cases}.$$

For any $d > 0$ of the form $d = d_0 + s$ with $d_0 \in \mathbb{N}_0$ and $s \in (0, 1]$ we define

$$\omega_d(\mathcal{E}) = \omega_{d_0}^{1-s}(\mathcal{E}) \cdot \omega_{d_0}^{s}(\mathcal{E}).$$

Let $(\mathcal{M}, G)$ be a Riemannian manifold over the Hilbert space $\mathbb{H}$ and $\mathcal{K} \subset \mathcal{M}$ be a subset.

For arbitrary real numbers $\epsilon > 0$ and $d \geq 0$ we consider the *d-dimensional Hausdorff outer premeasure* at level ϵ of $\mathcal{K}$ given by

$$\mu_{\mathrm{H}}(\mathcal{K}, d, \epsilon) := \inf \sum_i r_i^d, \tag{2.7}$$

where the infimum is taken over all countable covers of $\mathcal{K}$ by balls $\mathcal{B}_{r_i}(u_i) = \{v \in \mathcal{M} | \rho(u_i, v) \leq r_i\}$ of radius $r_i \leq \epsilon$ and outer $u_i \in \mathcal{M}$. For fixed d and $\mathcal{K}$ the function $\mu_{\mathrm{H}}(\mathcal{K}, d, \epsilon)$ is monotone decreasing in $\mathcal{E}$.

Hence the limit

$$\mu_{\mathrm{H}}(\mathcal{K}, d) = \lim_{\epsilon \to 0+0} \mu_{\mathrm{H}}(\mathcal{K}, d, \epsilon) \tag{2.8}$$

exists and is called *d-dimensional Hausdorff outer measure* of $\mathcal{K}$.

For every subset $\mathcal{K} \subset \mathcal{M}$ there exists a critical number d^* with

$$\mu_{\mathrm{H}}(\mathcal{K}, d) = \begin{cases} \infty & \text{for any } 0 \leq d < d^*, \\ 0 & \text{for any } d > d^*. \end{cases} \tag{2.9}$$

This critical number can be characterized as

$$d^* = \sup\{d \geq 0 | \ \mu(\mathcal{K}, d) = \infty\}. \tag{2.10}$$

It is called *Hausdorff dimension* of $\mathcal{K}$ and denoted by $\dim_{\mathrm{H}} \mathcal{K}$.

Introduce *the global Lyapunov exponents* $\nu_1^u \geq \nu_2^u \geq \ldots$ by

$$\nu_1^u + \nu_2^u + \ldots + \nu_m^u = \lim_{t \to \infty} \frac{1}{t} \log \max_{p \in \mathcal{K}} \omega_m(d_p \varphi^t), \quad m = 1, 2, \ldots.$$

The *upper Lyapunov dimension* of φ^t on $\mathcal{K}$ with respect to the global Lyapunov exponents is

$$\dim_L^u(\varphi^t, \mathcal{K}) \le N + \frac{\nu_1^u + \cdots \nu_N^u}{\nu_{N+1}^u},$$

where $N \ge 0$ denotes the smallest number satisfying $\nu_1^u + \nu_2^u + \cdots + \nu_N^u + \nu_{N+1}^u < 0$

3. Hausdorff dimension bounds for invariant sets of maps on Hilbert manifolds. Let $(\mathcal{M}, G)$ be a Riemannian manifold, let $\mathcal{U} \subset \mathcal{M}$ be an open subset and let us consider the map $\varphi : \mathcal{U} \to \mathcal{M}$ of class C^1. The tangent map of φ at a point $u \in \mathcal{U}$ is denoted by $d_u\varphi : T_u\mathcal{M} \to T_{\varphi(u)}\mathcal{M}$.

Let $u \in \mathcal{U}$ be an arbitrary point and consider charts x and x' at u and $\varphi(u)$, respectively. We introduce the operators $G_x(u) : \mathbb{H} \to \mathbb{H}$ and $G'_{x'}(\varphi(u))$ that realizes the metric fundamental tensor G in the canonical bases of $T_u\mathcal{M}$ and $T_{\varphi(u)}\mathcal{M}$, respectively. The tangent map of φ at u written in coordinates of the charts x and x' is given by the operator $\Phi = D(x' \circ \varphi \circ x^{-1})(x(u))$. The singular values of the tangent map $d_u\varphi : T_u\mathcal{M} \to T_{\varphi(u)}\mathcal{M}$ coincide with the singular values of the operator $\sqrt{G'}\Phi\sqrt{G^{-1}}$.

Let $\mathcal{K} \subset \mathcal{U}$ is a compact set and the tangent map $d_u\varphi$ be uniformly differentiable in the sense of Fréchet on the open set $\mathcal{U}$.

Let us consider the exponential map $\exp_u : T_u\mathcal{M} \to \mathcal{M}$.

By τ_v^u we denote the isometry between $T_u\mathcal{M}$ and $T_v\mathcal{M}$ defined by parallel transport along the geodesic for points lying sufficiently near to each other.

Let us fix a finite cover with balls $\mathcal{B}(u_i, r_i)_i$ of radius $r_i \le \varepsilon$ of $\mathcal{K}$. The Taylor formula for differentiable maps provides that for every $v \in \mathcal{B}(u_i, r_i)$

$$\begin{aligned} &||\exp_{\varphi(u_i)}^{-1}\varphi(v) - d_{u_i}\varphi(\exp_{u_i}^{-1}(v))|| \le \\ \sup_{w \in \mathcal{B}(u_i, r_i)} &||\tau_{\varphi(w)}^{\varphi(u_i)} d_w\varphi\tau_{u_i}^w - d_{u_i}\varphi|| \cdot ||\exp_{u_i}^{-1}(w)||. \end{aligned} \tag{3.1}$$

THEOREM 3.1. *Let $d > 0$ be a real number and $\mathcal{K} \subset \mathcal{U}$ a compact set which is negatively invariant with respect to φ, i.e. $\varphi(\mathcal{K}) \supset \mathcal{K}$. If the inequality*

$$\sup_{u \in \mathcal{K}} \omega_d(d_u\varphi) < 1 \tag{3.2}$$

holds, then $\dim_{\mathrm{H}} \mathcal{K} < d$.

In difference to the paper [7] we consider here the case when the linearization of the map φ may be a noncompact linear operator.

COROLLARY 3.2. *Let $\mathcal{K} \subset \mathcal{U} \subset \mathcal{M}$ be a compact set satisfying $\varphi(\mathcal{K}) \supset \mathcal{K}$. If for some continuous function $\kappa : \mathcal{U} \to \mathbb{R}_+$ and for some number $d > 0$ the inequality*

$$\sup_{u \in \mathcal{K}} \left(\frac{\kappa(\varphi(u))}{\kappa(u)} \omega_d(d_u\varphi) \right) < 1 \tag{3.3}$$

holds, then $\dim_{\mathrm{H}} \mathcal{K} < d$.

Let us describe the main ideas which are used in the proof of Theorem 3.1. Consider the exponential map

$$\exp_u : T_u\mathcal{M} \to \mathcal{M}, \tag{3.4}$$

where $u \in \mathcal{M}$ is an arbitrary point. Then the set $\exp_u(\mathcal{E})$ is the image of an ellipsoid $\mathcal{E}$ in the tangent space $T_u\mathcal{M}$ centered at 0 under the map $\exp_u$. Let $\mathcal{K} \subset \mathcal{U}$ be a compact set, let $\varepsilon > 0$ be a sufficiently small number and let us fix a number $d > 0$. The *outer ellipsoid premeasure* at level ε and of order d of $\mathcal{K}$ is given by

$$\tilde{\mu}_H(\mathcal{K}, d, \varepsilon) = \inf \left\{ \sum_i \omega_d(\mathcal{E}_i) \right\}, \tag{3.5}$$

where the infimum is taken over all finite covers $\cup_i \exp_{u_i}(\mathcal{E}_i) \subset \mathcal{K}$, where $u_i \in \mathcal{M}$, $\mathcal{E}_i \subset T_{u_i}\mathcal{M}$ are ellipsoids satisfying $\omega_d(\mathcal{E}_i)^{1/d} \leq \varepsilon$.

The following two lemmas for the compact case of the differentional are proved in [1]. The proof for the noncompact case can be done using Proposition 2.1. The use of the two lemmas is an essential part in the proof of Theorem 3.1.

LEMMA 3.3. *For an arbitrary number $d > 0$, $d = d_0 + s$, $s \in (0, 1]$, $d_0 \in \mathbb{N}_0$ we define the numbers $\lambda = \sqrt{d_0 + 1}$ and $C_d \geq 2^{d_0}(d_0 + 1)^{d/2}$. Then for a compact set $\mathcal{K} \subset \mathcal{U}$ and for every sufficiently small $\varepsilon > 0$ the inequality*

$$\mu_H(\mathcal{K}, d, \varepsilon) \geq \tilde{\mu}_H(\mathcal{K}, d, \varepsilon) \geq C_d^{-1} \mu_H(\mathcal{K}, d, \lambda\varepsilon) \qquad \textit{holds.} \tag{3.6}$$

LEMMA 3.4. *Let $\mathcal{K} \subset \mathcal{U}$ be a compact set and consider a map $\varphi : \mathcal{U} \to \mathcal{M}$ of class C^1. For a number $d > 0$, we assume that $\sup_{u \in \mathcal{K}} \omega_d(d_u\varphi) \leq k$. Then, for every $l > k$ there exists a number $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$*

$$\mu_H(\varphi(\mathcal{K}), d, \lambda l^{1/d}\varepsilon) \leq C_d l \mu_H(\mathcal{K}, d, \varepsilon) \tag{3.7}$$

holds, where $\lambda = \sqrt{d_0 + 1}$, $C_d \geq 2^{d_0}(d_0 + 1)^{d/2}$, $d = d_0 + s$, $s \in (0, 1]$, $d_0 \in \mathbb{N}_0$.

4. Hausdorff dimension bounds for invariant sets of vector fields on Hilbert manifolds. Let $(\mathcal{M}, G)$ be a Riemannian manifold, let $\mathcal{U} \subset \mathcal{M}$ be an open subset and $\mathcal{I}_1 \subset \mathbb{R}$ be an open interval with 0. We consider a time-dependent vector field $F : \mathcal{I}_1 \times \mathcal{U} \to T\mathcal{U}$ of class C^1 and the corresponding differential equation

$$\dot{u} = F(t, u). \tag{4.1}$$

Suppose, that for a point $(t, u) \in \mathcal{I}_1 \times \mathcal{U}$ the *covariant derivative* of the vector field F is $\nabla F(t, u) : T_u\mathcal{M} \to T_u\mathcal{M}$ and ∇F is a compact operator. The case when ∇F is noncompact can be also considered with the help of Section 3.

Let $\mathcal{D} \subset \mathcal{U}$ be an open set and $\mathcal{I} \subset \mathcal{I}_1$ be an open interval such that the solution $\varphi(\cdot, u)$ with $\varphi(0, u) = u$, $u \in \mathcal{D}$ of equation (15) exists everywhere on $\mathcal{I}$.

For every $t \in \mathcal{I}$ there exists the operator $\varphi^t : \mathcal{D} \to \mathcal{U}$ such that $\varphi^t(u) = \varphi(t, u)$.

Since the vector field F is continuously differentiable, the same holds for the operator $\{\varphi^t\}_{t \in \mathcal{I}}$. For an arbitrary point $u \in \mathcal{D}$, the tangent map $d_u\varphi^t$ solves the variation equation

$$y' = \nabla F(t, \varphi^t(u))y \tag{4.2}$$

with initial condition $d_u\varphi^t|_{t=0} = \mathrm{id}_{T_u\mathcal{M}}$.

Here the *absolute derivative* y' is taken along the integral curve $t \mapsto \varphi^t(u)$ in the direction of the vector field F.

Let us denote the eigenvalues of the symmetric part of the covariant derivative ∇F, i.e., of the operator

$$S(t,u) = \frac{1}{2}[\nabla F(t,u) + \nabla F(t,u)^{[*]}], \tag{4.3}$$

by $\lambda_i(t,u)$, $i = 1, 2, \ldots$ and order them with respect to its size and multiplicity, i.e., $\lambda_1(t,u) \geq \lambda_2(t,u) \geq \ldots$.

Let us introduce on $\mathcal{U}$ a new metric tensor $\tilde{g}_{|u} = \kappa^2(u) g_{|u}$ by means of a function $\kappa : \mathcal{U} \to \mathbb{R}_+$ of class C^1. Let $u \in \mathcal{U}$ be a fixed point and consider the chart x around u. Let $V : \mathcal{U} \to \mathbb{R}$ be a differentiable function and the map $\dot{V} : \mathcal{I} \times \mathcal{U} \to \mathbb{R}$ be defined by $\dot{V}(t,u) = \langle d_u V, F(t,u) \rangle$. The symmetric part of the covariant derivative $\tilde{\nabla} F(t,u)$ at $u \in \mathcal{U}$ with respect to the new metric is given by

$$\frac{1}{2}[G^{-1}\Phi^T G + \Phi] + \frac{\dot{\kappa}}{\kappa}\mathrm{Id}, \tag{4.4}$$

where $\Phi = D(\tilde{x} \circ \varphi \circ x^{-1})(x(u))$ and the operator G represents $g_{|u}$.

If

$$\kappa(u) = e^{\frac{V(u)}{d}} \quad (u \in \mathcal{U}) \tag{4.5}$$

then $\dot{\kappa}(u) = \kappa(u)\frac{\dot{V}(u)}{d}$ implies that the eigenvalues $\tilde{\lambda}_i$ of (4.4) are related to the eigenvalues with respect to the original metric g by $\tilde{\lambda}_i = \lambda_i + \frac{\dot{V}}{d}$, $i = 1, 2, \ldots$.

The next theorems are corollaries of Theorem 3.1.

Theorem 4.1. *Let $d > 0$, be a real number written in the form $d = d_0 + s$ with $d_0 \in \mathbb{N}_0$, $s \in (0,1]$ and let $\mathcal{K} \subset \mathcal{D}$ be a compact set satisfying $\varphi^\tau(\mathcal{K}) \supset \mathcal{K}$ for a certain $\tau \in \mathcal{I} \cap \mathbb{R}_+$. If the condition*

$$\sup_{u \in \mathcal{K}} \int_0^\tau [\lambda_1(t, \varphi^t(u)) + \lambda_2(t, \varphi^t(u)) + \ldots + \lambda_{d_0}(t, \varphi^t(u)) + s\lambda_{d_0+1}(t, \varphi^t(u))]dt < 0$$

holds, then $\dim_{\mathrm{H}} \mathcal{K} \leq d$.

Theorem 4.2. *Let $\mathcal{K} \subset \mathcal{D}$ be a compact set such that $\varphi^\tau(\mathcal{K}) \supset \mathcal{K}$ is true for some $\tau \in \mathcal{I} \cap \mathbb{R}_+$. Let $V : \mathcal{U} \to \mathbb{R}$ be a differentiable function and denote by $\lambda_1(t,u) \geq \lambda_2(t,u) \geq \ldots$ the eigenvalues of $S(t,u)$. If for a real number $d > 0$ $d = d_0 + s$ with $d_0 \in \mathbb{N}_0$ and $s \in (0,1]$ the condition*

$$\begin{aligned} \sup_{u \in \mathcal{K}} \int_0^\tau [\lambda_1(t, \varphi^t(u)) + \lambda_2(t, \varphi^t(u)) + \ldots \\ + \lambda_{d_0}(t, \varphi^t(u)) + s\lambda_{d_0+1}(t, \varphi^t(u)) + \dot{V}(t, \varphi^t(u))]dt < 0 \end{aligned} \tag{4.6}$$

holds, then $\dim_{\mathrm{H}} \mathcal{K} \leq d$.

The application of the Theorem 4.1 and Theorem 4.2 for the compact case to the sine-Gordon equation given on the cylinder was considered in the paper [7]. The non-compact version of these theorems can be applied to estimate the Hausdorff dimension of an attractor for the Ginzburg-Landau equation [3] using a nontrivial metric tensor instead of the Lyapunov function used in this paper. Thus it is possible to calculate the Lyapunov dimension $\dim_L^u(\varphi^t, \mathcal{K})$, introduced in Section 2, for this equation.

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