UPPER HAUSDORFF DIMENSION ESTIMATES FOR INVARIANT SETS OF EVOLUTIONARY SYSTEMS ON HILBERT MANIFOLDS

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Abstract. We prove a generalization of the Douady-Oesterlé theorem on the upper bound of the Hausdorff dimension of an invariant set of a dynamical system generated by a differential equation on a Hilbert manifold. A similar estimate is given for the Hausdorff dimension of an invariant set of a smooth map on an infinite dimensional manifold. It is assumed that the linearization of this map is a noncompact linear operator. A similar estimate is for the Hausdorff dimension of an invariant set of a dynamical system generated by a differential equation on a Hilbert manifold.

Key words. Hilbert manifold, Hausdorff dimension, singular value

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1. Basic notation of manifold theory. Let us shortly introduce some definitions and properties for manifolds over a Hilbert space ([1, 8]). Suppose \( H \) is a Hilbert space and \( M \) is a set. A chart on \( M \) is a bijection \( x : D(x) \subset M \to \mathcal{R}(x) \subset H \), where \( \mathcal{R}(x) \) is an open set. An atlas \( A \) of class \( C^k (k \geq 1) \) on \( M \) is a set of charts, such that:

\((\text{AT1})\) \( \cup_{x \in A} D(x) = M; \)

\((\text{AT2})\) For arbitrary \( x, y \in A \), such that \( D(y) \cap D(x) \neq \emptyset \), the set \( x(D(x) \cap D(y)) \) is an open subset in \( H; \)

\((\text{AT3})\) For arbitrary \( x, y \in A \) the map \( y \circ x^{-1} : x(D(x) \cap D(y)) \to y(D(x) \cap D(y)) \) is a \( C^k \) diffeomorphism.

A pair \((M, A)\) where \( M \) is a set and \( A \) is a \( C^k \)-atlas on \( M \), is called a \( C^k \)-manifold over the Hilbert space \( H \).

Let \( x \) and \( y \) be two arbitrary charts on \( M \) around the point \( u \in M \). Let \( \xi, \eta \in \mathbb{H} \) be arbitrary. Introduce the equivalence relation

\[(u, x, \xi) \sim (u, y, \eta) \iff \eta = (y \circ x^{-1})'(x(u))\xi.\]

The equivalence class

\[\{u, x, \xi\} = \{(u, y, \eta) | u \in D(x) \cap D(y), (u, y, \eta) \sim (u, x, \xi)\},\]

is called tangent vector at \( u \). The tangent space of \( M \) at \( u \) is the set \( T_uM \) of all equivalence classes \([u, x, \xi]\) such that \( x \) is a chart, \( u \in D(x) \) and \( \xi \in \mathbb{H} \). It is equipped with a vector space structure on \( T_uM \) given by:

\[\lambda[u, x, \xi] = [u, x, \lambda \xi], \quad \forall \lambda \in \mathbb{R}, \xi \in \mathbb{H}.\]

The tangent bundle \( TM \) of \( M \) is defined by \( TM = \cup_{u \in M} T_uM. \)

Suppose that \( M \) is a \( C^k \)-manifold over the Hilbert space \( H \). The map \( \varphi : U \subset M \to M \) is said to be \( C^r \)-differentiable \((r \leq k)\) at \( u \in M \) if there are charts \( x \) around \( u \) and \( y \) around \( \varphi(u) \) such that the map \( y \circ \varphi \circ x^{-1} \) is \( C^r \)-differentiable in \( x(u) \) in the sense of Fréchet.

The differential of \( \varphi \) at \( u \in U \) is the linear map \( d_u\varphi : T_uM \to T_{\varphi(u)}M \), given by

\[d_u\varphi([u, x, \xi]) = [\varphi(u), y, (y \circ \varphi \circ x^{-1})'(x(u))\xi], \quad (1.1)\]
where \( x, y \) are charts around \( u \) and \( \varphi(u) \), respectively, and \( \xi \in \mathbb{H} \) is arbitrary.

Let a Riemannian metric of class \( C^{k-1} \) be defined on the connected \( C^k \)-manifold \( \mathcal{M}(k \geq 2) \) over the Hilbert space \( \mathbb{H} \). Suppose that at every point \( u \in \mathcal{M} \) and for every chart \( x \) around \( u \) there is given a symmetric positive definite operator \( G_x : \mathbb{H} \to \mathbb{H} \) with the following properties

\(\text{(RM1)}\) The map \( G_x : D(x) \to L(\mathbb{H}) \) is \( C^k \)-smooth.

\(\text{(RM2)}\) \((y \circ x^{-1})(x(u)) \rangle G_y(u) \langle (y \circ x^{-1})(x(u)) \rangle = G_x(u) \) for any two charts \( x, y \) around \( u \).

Let \( (\mathcal{M}, g) \) be a Riemannian \( C^r \)-manifold \((r \geq 3)\) over the Hilbert space \( \mathbb{H} \). For any \( u \in \mathcal{M} \) and any \( v \in T_u \mathcal{M} \) there exists a unique geodesic \( \varphi(\cdot, u, v) \) with \( \varphi(0, u, v) = u, \varphi(0, u, v) = v \). Then \((t, u, v) \mapsto \varphi(t, u, v)\) is a \( C^{r-2} \)-map.

**Definition 1.1.** The map \( v \mapsto \exp_u v = \varphi(1, u, v) \) is called exponential map of class \( C^{r-2} \) around \( 0 \in T_u \mathcal{M} \).

Let \( \mathcal{V} \) be a sufficiently small neighborhood of \( 0 \in T_u \mathcal{M} \). Then the map \( \exp_u : \mathcal{V} \to \exp_u \mathcal{V} = \{ \varphi(1, u, v) \mid v \in \mathcal{V} \} \) is a \( C^{r-2} \)-diffeomorphism.

It follows for any \( u \in \mathcal{M} \) and any sufficiently small number \( \varepsilon > 0 \) the map \( \exp_u \) is a \( C^{r-2} \)-diffeomorphism on \( B_u(0) \subset T_u \mathcal{M} \).

For any \( v \in B_u(0_u) \) the map \( t \mapsto c(t) = \exp_u(t, v) \) with \( t \in [0, 1] \) is a geodesic on \( \mathcal{M} \).

Let us define a dynamical system and an associated global attractor on the Riemannian manifold \((1, 8)\). Let \( (\mathcal{M}, \rho) \) be the metric space generated on the Riemannian manifold \((\mathcal{M}, G)\) and let \( \{ \varphi^t \}_{t \in \mathcal{J}} \) be a family of maps \( \varphi^t : \mathcal{M} \to \mathcal{M} \), where \( \mathcal{J} = \{ \mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+ \} \). The pair \((\{ \varphi^t \}_{t \in \mathcal{J}}, (\mathcal{M}, \rho))\) is called a dynamical system on the metric space \((\mathcal{M}, \rho)\) if the following holds:

1. \( \varphi^0 = \text{id}_\mathcal{M} \);  
2. \( \varphi^{t+s} = \varphi^t \circ \varphi^s \) for all \( s, t \in \mathcal{J} \);  
3. \( \varphi^t(\cdot) : \mathcal{J} \times \mathcal{M} \to \mathcal{M} \) is smooth if \( \mathcal{J} \in \{ \mathbb{R}, \mathbb{R}_+ \} \). The family \( \varphi^t : \mathcal{M} \to \mathcal{M} \) of maps with \( t \in \mathcal{J} \) is smooth if \( \mathcal{J} \in \{ \mathbb{Z}, \mathbb{Z}_+ \} \).

Let \( \{ \varphi^t \}_{t \in \mathcal{J}}, (\mathcal{M}, \rho) \) be a dynamical system. A set \( \mathcal{A} \subset \mathcal{M} \) is called a global \( B \)-attractor for the dynamical system if the following conditions are satisfied:

\(\text{(CM1)}\) \( \mathcal{A} \) is compact;

\(\text{(CM2)}\) \( \mathcal{A} \) is an invariant set in the sense that \( \varphi^t(\mathcal{A}) = \mathcal{A}, \forall t \in \mathcal{J} \);

\(\text{(CM3)}\) \( \mathcal{A} \) attracts any bounded set \( \mathcal{B} \subset \mathcal{M} \) under \( \{ \varphi^t \}_{t \in \mathcal{J}} \), i.e.

\[
\text{dist}(\mathcal{A}, \mathcal{B}) \to 0 \quad \text{for} \quad t \to \infty \quad (1.2)
\]

where

\[
\text{dist}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{u \in \mathcal{B}_1} \inf_{v \in \mathcal{B}_2} \rho(u, v) \quad (1.3)
\]

for any nonempty subsets \( \mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B} \) is the Hausdorff semidistance.

**2. Hausdorff dimension and singular values.** In the following we introduce some basic definitions and propositions of singular values for noncompact linear operators. Consider the linear not compact operator \( T : \mathbb{K} \to \mathbb{K}' \), where \((\mathbb{K}, (\cdot, \cdot)_\mathbb{K})\) and \((\mathbb{K}', (\cdot, \cdot)_{\mathbb{K}'})\) are Hilbert spaces. (The case when \( \mathbb{K} = \mathbb{K}' \) is considered in \([10]\).) The adjoint operator \( T^* : \mathbb{K}' \to \mathbb{K} \), is defined by the relation \((T^* \xi, \eta)_{\mathbb{K}'} = (\xi, T \eta)_{\mathbb{K}}, \forall \xi \in \mathbb{K}, \forall \eta \in \mathbb{K}'\).

The singular values of \( T \), denoted by \( \alpha_i(T) \), are given by

\[
\alpha_k(T) = \sup_{L \subset \mathbb{K}} \inf_{\xi \in L} |T^* \xi|_{\mathbb{K}'}, \quad k = 1, 2, \ldots \quad (2.1)
\]
Let $T^{\wedge k} : \mathbb{K}^{\wedge k} \to \mathbb{K}^{\wedge k}$ and let consider $\omega_k(T) = \alpha(T^{\wedge k})$. The function

$$\omega_d(T) = \begin{cases} \omega_{d_0}^{1-s}(T) \cdot \omega_{d_0 + 1}^s(T), & d > 0 \\ 1, & d = 0 \end{cases}$$

is called the singular value function of $T$. Here $d \geq 0$ is written in the form $d = d_0 + s$, $d_0 \in \mathbb{N}_0$, $s \in (0, 1]$.

Let $\{\xi_i\}_{i \in I}$ be an orthonormal basis of $\mathbb{K}$ such that $\xi_i$ is an eigenvector of $T^{[i]}T$ corresponding to the eigenvalue $\alpha_i(T)$, $i \in I$. Then there exists an orthonormal basis $\{\eta_i\}_{i \in I}$ in $\mathbb{K}'$ with $\eta_i = \frac{1}{\alpha_i} T \xi_i$ for any $i \in I$ and $\alpha_i > 0$. The image of the unit ball $B_1(0) \subset \mathbb{K}$ under the map $T$ is the set

$$\left\{ \sum_{i \in I, \alpha_i(T) \neq 0} c_i \eta_i \in \mathbb{K}' \left| \sum_{i \in I, \alpha_i(T) \neq 0} \left( \frac{c_i}{\alpha_i(T)} \right)^2 \leq 1 \right. \right\}.$$ 

The operator $\tilde{T} = T^{[i]}T$ is positive, self-adjoint, and continuous but no longer compact. We introduce the sequence of numbers $\beta_n(\tilde{T})$, $n \geq 1$, defined by

$$\beta_n(\tilde{T}) = \inf_{L \subset \mathbb{K}} \sup_{\xi \in L} \frac{\langle \tilde{T} \xi, \xi \rangle_{\mathbb{K}}}{\dim L} = k,$$

(2.2)

The sequence $\{\beta_n(\tilde{T})\}$ is nonincreasing and we can easily see that the definition of $\beta_n(\tilde{T})$ is unchanged if we replace the infimum in (2.2) by the infimum for $L \subset \mathbb{K}$. If $\tilde{T}$ is compact then, according to the well known min-max principle $\beta_n(\tilde{T})$ would be the eigenvalues of $\tilde{T}$.

We set

$$\beta_\infty(\tilde{T}) = \lim_{n \to \infty} \beta_n(\tilde{T}) = \inf_{n \geq 1} \beta_n(\tilde{T}).$$

(2.3)

The sequence is stationary at some stage:

$$\beta_1(\tilde{T}) \geq \ldots \geq \beta_{n_0}(\tilde{T}) > \beta_{n_0 + 1}(\tilde{T}) = \beta_m(\tilde{T}) = \beta_\infty(\tilde{T}), \quad \forall m \geq n_0 + 1$$

(2.4)

or

$$\beta_m(\tilde{T}) > \beta_\infty(\tilde{T}), \quad \forall m \in \mathbb{N}.$$ 

(2.5)

In the first case it follows from the above result that $\beta_1, \ldots, \beta_{n_0}$, are eigenvalues of $\tilde{T}$, while in the second case each $\beta_m$ is an eigenvalue of $\tilde{T}$. In both cases we decompose $\mathbb{K}$ into the direct sum $\mathbb{K}_v \oplus \mathbb{K}_v^\perp$, where $\mathbb{K}_v$ is the space spanned by the eigenvectors of $\tilde{T}, e_i, i \in I$, which we suppose orthonormalized $(I = (1, \ldots, n_0)$ when (2.3) occurs, $I = \mathbb{N}$ when (2.4) holds). Of course, it may happen that $\mathbb{K}_v = \{0\}$ or $\mathbb{K}_v = \mathbb{K}$.

Let $\mathbb{K} = \mathbb{K}_v \oplus \mathbb{K}_v^+$ denote the decomposition of $\mathbb{K}$, where $\mathbb{K}_v$ and $\mathbb{K}_v^+$ are orthogonal. In the same way let us introduce the decomposition $\mathbb{K}' = \mathbb{K}_v' \oplus \mathbb{K}_v'^\perp$. Let $\{\xi_i\}_{i \in I}$ be an orthonormal basis of $\mathbb{K}_v$ such that $\xi_i$ is an eigenvector of $T^{[i]}T$ corresponding to the eigenvalue $\alpha_i(T)$, $i \in I$. Then there exists an orthonormal basis $\{\eta_i\}_{i \in I}$ in $\mathbb{K}'$ with $\eta_i = \frac{1}{\alpha_i} T \xi_i$ for any $i \in I$ and $\alpha_i \neq 0$. We observe that the vectors $T e_i, i \in I$ are orthogonal, i. e.

$$(T e_i, T e_j)_{\mathbb{K}'} = (T^{[i]} T e_i, e_j)_{\mathbb{K}} = \beta_i'(e_i, e_j)_{\mathbb{K}} = \beta_i' \delta_{ij} \quad \forall i, j \in I,$$

(2.6)
where $\delta_{ij} = (e_i, e_j), \forall i, j \in I$.

The image of the unit ball $B_1(0) \subset \mathbb{K}$ under the map $T$ is included in the sum of the ellipsoid $\sum_{\alpha \in I} \frac{1}{\alpha^2} \left( \xi, \frac{T \alpha}{\alpha} \right)^2 \leq 1$ of $\mathbb{K}'_\alpha$ and of the ball of $\mathbb{K}'_\alpha^\perp$ centered at 0 of radius $\alpha_\infty(T)$.

The next proposition is a generalization of a result of [10]

**Proposition 2.1.** Let $K$ be a Hilbert space and $B$ its unit ball. Let $T: K \to K'$ be a linear continuous operator and, if $T$ is not compact, let $K'_c$ be defined as above. Then $T(B)$ is included in an ellipsoid $E$:

(i) If $T$ is not compact, but $K'_c = K'$, the axes of $E$ are directed along the vectors $Te_i$ and their length is $\alpha_i(T)$, the $e_i$ being the eigenvectors of $T^{[*]}T$.

(ii) If $T$ is not compact and $K'_c \neq K'$, $E$ is the product of the ball centered at 0 of radius $\alpha_\infty$ in $K'_c$, and of the ellipsoid of $K'_c$ whose axes are directed along the vectors $Te_i$ with lengths $\alpha_i(T)$, the $e_i$ being the eigenvectors of $T$ spanning $K'_c$.

Let $E$ be an ellipsoid in the Hilbert space $\mathbb{H}$ and let $a_1(E) \geq a_2(E) \geq \ldots$ denote the lengths of the half-axes. For any $j \in \mathbb{N}$ we define

$$\omega_j(E) = \left\{ \begin{array}{ll}
a_1(E) \cdot \ldots \cdot a_j(E), & j \in \mathbb{N} \\
1, & j = 0 \end{array} \right.$$ 

For any $d > 0$ of the form $d = d_0 + s$ with $d_0 \in \mathbb{N}$ and $s \in (0, 1]$ we define

$$\omega_d(E) = \omega_{d_0}^{1-s}(E) \cdot \omega_d(E).$$

Let $(M, G)$ be a Riemannian manifold over the Hilbert space $\mathbb{H}$ and $K \subset M$ be a subset.

For arbitrary real numbers $\epsilon > 0$ and $d \geq 0$ we consider the $d$-dimensional Hausdorff outer premeasure at level $\epsilon$ of $K$ given by

$$\mu_h(K, d, \epsilon) := \inf \sum_i r_i^d, \quad (2.7)$$

where the infimum is taken over all countable covers of $K$ by balls $B_{r_i}(u_i) = \{ v \in M | r_i(v) \leq r_i \}$ of radius $r_i \leq \epsilon$ and outer $u_i \in M$. For fixed $d$ and $K$ the function $\mu_h(K, d, \epsilon)$ is monotone decreasing in $\epsilon$.

Hence the limit

$$\mu_h(K, d) = \lim_{\epsilon \to 0^+} \mu_h(K, d, \epsilon) \quad (2.8)$$

exists and is called $d$-dimensional Hausdorff outer measure of $K$.

For every subset $K \subset M$ there exists a critical number $d^*$ with

$$\mu_h(K, d) = \left\{ \begin{array}{ll}
\infty & \text{for any } 0 \leq d < d^*; \\
0 & \text{for any } d > d^*. \end{array} \right. \quad (2.9)$$

This critical number can be characterized as

$$d^* = \sup\{ d \geq 0 | \mu(K, d) = \infty \}. \quad (2.10)$$

It is called Hausdorff dimension of $K$ and denoted by $\dim_h K$.

Introduce the global Lyapunov exponents $\nu_1^u \geq \nu_2^u \geq \ldots$ by

$$\nu_1^u + \nu_2^u + \ldots + \nu_m^u = \lim_{t \to \infty} \frac{1}{t} \log \max_{p \in K} \omega_m(d_p, \nu^u), \quad m = 1, 2, \ldots$$
The upper Lyapunov dimension of \( \varphi^t \) on \( K \) with respect to the global Lyapunov exponents is

\[
\dim_{\text{Fr}}^u(\varphi^t, K) \leq N + \frac{\nu_1^u + \cdots + \nu_{\nu_{N+1}}^u}{\nu_{\nu_{N+1}}^u},
\]

where \( N \geq 0 \) denotes the smallest number satisfying \( \nu_1^u + \nu_2^u + \cdots + \nu_\nu_{N+1}^u < 0 \).

### 3. Hausdorff dimension bounds for invariant sets of maps on Hilbert manifolds

Let \((M, G)\) be a Riemannian manifold, let \( U \subset M \) be an open subset and let us consider the map \( \varphi : U \to M \) of class \( C^1 \). The tangent map of \( \varphi \) at a point \( u \in U \) is denoted by \( d_u \varphi : T_u M \to T_{\varphi(u)} M \).

Let \( u \in U \) be an arbitrary point and consider charts \( x \) and \( x' \) at \( u \) and \( \varphi(u) \), respectively. We introduce the operators \( G_x(u) : \mathbb{H} \to \mathbb{H} \) and \( G'_{x'}(\varphi(u)) \) that realizes the metric fundamental tensor \( G \) in the canonical bases of \( T_u M \) and \( T_{\varphi(u)} M \), respectively. The tangent map of \( \varphi \) at \( u \) written in coordinates of the charts \( x \) and \( x' \) is given by the operator \( \Phi = D(x' \circ \varphi \circ x^{-1})(x(u)) \). The singular values of the tangent map \( d_u \varphi : T_u M \to T_{\varphi(u)} M \) coincide with the singular values of the operator \( \sqrt{G'} \Phi \sqrt{G}^{-1} \).

Let \( K \subset U \) be a compact set and the tangent map \( d_u \varphi \) be uniformly differentiable in the sense of Fréchet on the open set \( U \).

Let us consider the exponential map \( \exp_u : T_u M \to M \).

By \( \tau^u \) we denote the isometry between \( T_u M \) and \( T_{\varphi(u)} M \) defined by parallel transport along the geodesic for points lying sufficiently near to each other.

Let us fix a finite cover with balls \( B(u_i, r_i) \) of radius \( r_i \leq \varepsilon \) of \( K \). The Taylor formula for differentiable maps provides that for every \( v \in B(u_i, r_i) \)

\[
|| \exp_{\varphi^{-1}(u_i)}^{-1}(v) - d_{u_i} \varphi(\exp_{\varphi^{-1}(u_i)}^{-1}(v)) || \leq \sup_{w \in B(u_i, r_i)} || \tau^u_{\varphi^{-1}(u_i)} d_{w} \tau^w_{\varphi^{-1}(u_i)} - d_{u_i} \varphi || \cdot || \exp_{\varphi^{-1}(u_i)}^{-1}(w) ||. \tag{3.1}
\]

**Theorem 3.1.** Let \( d > 0 \) be a real number and \( K \subset U \) a compact set which is negatively invariant with respect to \( \varphi \), i.e. \( \varphi(K) \supset K \). If the inequality

\[
\sup_{u \in K} \omega_d(d_u \varphi) < 1 \tag{3.2}
\]

holds, then \( \dim_{\text{H}} K < d \).

In difference to the paper [7] we consider here the case when the linearization of the map \( \varphi \) may be a noncompact linear operator.

**Corollary 3.2.** Let \( K \subset U \subset M \) be a compact set satisfying \( \varphi(K) \supset K \). If for some continuous function \( \kappa : U \to \mathbb{R}_+ \) and for some number \( d > 0 \) the inequality

\[
\sup_{u \in K} \left( \frac{\kappa(\varphi(u))}{\kappa(u)} \omega_d(d_u \varphi) \right) < 1 \tag{3.3}
\]

holds, then \( \dim_{\text{H}} K < d \).

Let us describe the main ideas which are used in the proof of Theorem 3.1. Consider the exponential map

\[
\exp_u : T_u M \to M, \tag{3.4}
\]
where \( u \in \mathcal{M} \) is an arbitrary point. Then the set \( \exp_u(E) \) is the image of an ellipsoid \( E \) in the tangent space \( T_u\mathcal{M} \) centered at 0 under the map \( \exp_u \). Let \( K \subset \mathcal{U} \) be a compact set, let \( \varepsilon > 0 \) be a sufficiently small number and let us fix a number \( d > 0 \). The outer ellipsoid premeasure at level \( \varepsilon \) and of order \( d \) of \( K \) is given by

\[
\mu_H(K, d, \varepsilon) = \inf \left\{ \sum_i \omega_d(E_i) \right\},
\]

where the infimum is taken over all finite covers \( \cup_i \exp_{u_i}(E_i) \subset K \), where \( u_i \in \mathcal{M} \), \( E_i \subset T_{u_i}\mathcal{M} \) are ellipsoids satisfying \( \omega_d(E_i)^{1/d} \leq \varepsilon \).

The following two lemmas for the compact case of the differential are proved in [1]. The proof for the noncompact case can be done using Proposition 2.1. The use of the two lemmas is an essential part in the proof of Theorem 3.1.

**Lemma 3.3.** For an arbitrary number \( d > 0 \), \( d = d_0 + s \), \( s \in (0, 1] \), \( d_0 \in \mathbb{N}_0 \) we define the numbers \( \lambda = \sqrt{d_0 + 1} \) and \( C_d \geq 2^{d_0}(d_0 + 1)^{d/2} \). Then for a compact set \( K \subset \mathcal{U} \) and for every sufficiently small \( \varepsilon > 0 \) the inequality

\[
\mu_H(K, d, \varepsilon) \geq \tilde{\mu}_H(K, d, \varepsilon) \geq C_d^{-1} \mu_H(K, d, \lambda \varepsilon) \quad \text{holds.}
\]

**Lemma 3.4.** Let \( K \subset \mathcal{U} \) be a compact set and consider a map \( \varphi : \mathcal{U} \to \mathcal{M} \) of class \( C^1 \). For a number \( d > 0 \), we assume that \( \sup_{u \in K} \omega_d(d_u \varphi) \leq k. \) Then, for every \( l > k \) there exists a number \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \)

\[
\mu_H(\varphi(K), d, \lambda \varepsilon) \leq C_d \mu_H(K, d, \varepsilon)
\]

holds, where \( \lambda = \sqrt{d_0 + 1} \), \( C_d \geq 2^{d_0}(d_0 + 1)^{d/2} \), \( d = d_0 + s \), \( s \in (0, 1] \), \( d_0 \in \mathbb{N}_0 \).

4. Hausdorff dimension bounds for invariant sets of vector fields on Hilbert manifolds. Let \((\mathcal{M}, G)\) be a Riemannian manifold, let \( \mathcal{U} \subset \mathcal{M} \) be an open subset and \( \mathcal{I}_1 \subset \mathbb{R} \) be an open interval with 0. We consider a time-dependent vector field \( F : \mathcal{I}_1 \times \mathcal{U} \to T\mathcal{U} \) of class \( C^1 \) and the corresponding differential equation

\[
\dot{u} = F(t, u).
\]

Suppose, that for a point \((t, u) \in \mathcal{I}_1 \times \mathcal{U}\) the covariant derivative of the vector field \( F \) is \( \nabla F(t, u) : T_u\mathcal{M} \to T_u\mathcal{M} \) and \( \nabla F \) is a compact operator. The case when \( \nabla F \) is noncompact can be also considered with the help of Section 3.

Let \( \mathcal{D} \subset \mathcal{U} \) be an open set and \( \mathcal{I} \subset \mathcal{I}_1 \) be an open interval such that the solution \( \varphi(\cdot, u) \) with \( \varphi(0, u) = u, u \in \mathcal{D} \) of equation (15) exists everywhere on \( \mathcal{I} \).

For every \( t \in \mathcal{I} \) there exists the operator \( \varphi^t : \mathcal{D} \to \mathcal{U} \) such that \( \varphi^t(u) = \varphi(t, u) \).

Since the vector field \( F \) is continuously differentiable, the same holds for the operator \( \{ \varphi^t \}_{t \in \mathcal{I}} \). For an arbitrary point \( u \in \mathcal{D} \), the tangent map \( d_u \varphi^t \) solves the variation equation

\[
y' = \nabla F(t, \varphi^t(u)) y
\]

with initial condition \( d_u \varphi^t|_{t=0} = \text{id}_{T_u\mathcal{M}} \).

Here the absolute derivative \( y' \) is taken along the integral curve \( t \mapsto \varphi^t(u) \) in the direction of the vector field \( F \).
Let us denote the eigenvalues of the symmetric part of the covariant derivative $\nabla F$, i.e., of the operator
\[
S(t, u) = \frac{1}{2}[\nabla F(t, u) + \nabla F(t, u)[^t]],
\]
by $\lambda_i(t, u), i = 1, 2, \ldots$ and order them with respect to its size and multiplicity, i.e.,
$\lambda_1(t, u) \geq \lambda_2(t, u) \geq \ldots$.

Let us introduce on $\mathcal{U}$ a new metric tensor $\tilde{g}_{\mu} = \kappa^2(u)g_{\mu}$ by means of a function $\kappa : \mathcal{U} \to \mathbb{R}_+$ of class $C^1$. Let $u \in \mathcal{U}$ be a fixed point and consider the chart $x$ around $u$. Let $V : \mathcal{U} \to \mathbb{R}$ be a differentiable function and the map $V : \mathcal{I} \times \mathcal{U} \to \mathbb{R}$ be defined by $V(t, u) = \langle d_u V, F(t, u) \rangle$. The symmetric part of the covariant derivative $\nabla F(t, u)$ at $u \in \mathcal{U}$ with respect to the new metric is given by
\[
\frac{1}{2}[G^{-1}\Phi^T G + \Phi] + \frac{\dot{\kappa}}{\kappa}\text{Id},
\]
where $\Phi = D(\tilde{x} \circ \varphi \circ x^{-1})(x(u))$ and the operator $G$ represents $g_{\mu}$.

If
\[
\kappa(u) = e^{\frac{V(u)}{d}} \quad (u \in \mathcal{U})
\]
then $\dot{\kappa}(u) = \kappa(u)\frac{V(u)}{d}$ implies that the eigenvalues $\tilde{\lambda}_i$ of (4.4) are related to the eigenvalues with respect to the original metric $g$ by $\tilde{\lambda}_i = \lambda_i + \frac{V}{\kappa}, i = 1, 2, \ldots$.

The next theorems are corollaries of Theorem 3.1.

**Theorem 4.1.** Let $d > 0$, be a real number written in the form $d = d_0 + s$ with $d_0 \in \mathbb{N}_0$, $s \in (0, 1]$ and let $\mathcal{K} \subset \mathcal{D}$ be a compact set satisfying $\varphi^\tau(\mathcal{K}) \supset \mathcal{K}$ for a certain $\tau \in \mathcal{I} \cap \mathbb{R}_+$. If the condition
\[
\sup_{u \in \mathcal{K}} \int_0^\tau [\lambda_1(t, \varphi^t(u)) + \lambda_2(t, \varphi^t(u)) + \ldots + \lambda_{d_0}(t, \varphi^t(u)) + s\lambda_{d_0+1}(t, \varphi^t(u))]dt < 0
\]
holds, then $\dim_H \mathcal{K} \leq d$.

**Theorem 4.2.** Let $\mathcal{K} \subset \mathcal{D}$ be a compact set such that $\varphi^\tau(\mathcal{K}) \supset \mathcal{K}$ is true for some $\tau \in \mathcal{I} \cap \mathbb{R}_+$. Let $V : \mathcal{U} \to \mathbb{R}$ be a differentiable function and denote by $\lambda_1(t, u) \geq \lambda_2(t, u) \geq \ldots$ the eigenvalues of $S(t, u)$. If for a real number $d > 0$ $d = d_0 + s$ with $d_0 \in \mathbb{N}_0$ and $s \in (0, 1]$ the condition
\[
\sup_{u \in \mathcal{K}} \int_0^\tau [\lambda_1(t, \varphi^t(u)) + \lambda_2(t, \varphi^t(u)) + \ldots + \lambda_{d_0}(t, \varphi^t(u)) + s\lambda_{d_0+1}(t, \varphi^t(u)) + \dot{V}(t, \varphi^t(u))]dt < 0
\]
holds, then $\dim_H \mathcal{K} \leq d$.

The application of the Theorem 4.1 and Theorem 4.2 for the compact case to the sine-Gordon equation given on the cylinder was considered in the paper [7]. The noncompact version of these theorems can be applied to estimate the Hausdorff dimension of an attractor for the Ginzburg-Landau equation [3] using a nontrivial metric tensor instead of the Lyapunov function used in this paper. Thus it is possible to calculate the Lyapunov dimension $\dim_L^p(\varphi^t, \mathcal{K})$, introduced in Section 2, for this equation.
REFERENCES


