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CLASSICAL AND GENERALIZED JACOBI POLYNOMIALS ORTHOGONAL WITH DIFFERENT WEIGHT FUNCTIONS AND DIFFERENTIAL EQUATIONS SATISFIED BY THESE POLYNOMIALS *

MARIANA MARČOKOVÁ† AND VLADIMÍR GULDAN‡

Abstract. In this contribution we deal with classical Jacobi polynomials orthogonal with respect to different weight functions, their special cases - classical Legendre polynomials and generalized brothers of them. We derive expressions of generalized Legendre polynomials and generalized ultraspherical polynomials by means of classical Jacobi polynomials.

Key words. orthogonal polynomial, weight function, classical Jacobi polynomial, classical Legendre polynomial, generalized orthogonal polynomial, differential equation

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1. Introduction. This paper presents relations of generalized Legendre polynomials of a certain type to classical Jacobi polynomials with some different weight functions. Also generalization to ultraspherical polynomials is given. Further, we deal with influence to Jacobi polynomials, when their weight function is multiplied by even function. In the conclusion we derive the differential equations satisfied by the introduced generalized Legendre polynomials. The motivation for such investigation was obtained when studying the book [2] dealing with physical geodesy and the papers [3], [5], [7], [10], and [11] using Legendre polynomials in applications.

1.1. Definition and basic properties of orthogonal polynomials. We recall the definition and the basic properties of orthogonal polynomials that can be found in the basic literature on orthogonal polynomials (cf. [1], [4], [8], and [9]).

DEFINITION 1.1. *Let $(a, b) \subset R$ be a finite or infinite interval. A function $v(x)$ is called the weight function if at this interval it fulfills the following conditions:*

(i) *$v(x)$ is nonnegative at (a, b), i.e.*

$$v(x) \geq 0,$$

(ii) *$v(x)$ is integrable at (a, b), i.e.*

$$0 < \int_a^b v(x)dx < \infty$$

and

(iii) *for every $n = 0, 1, 2, \dots$*

$$0 < \int_a^b |x|^n v(x)dx < \infty.$$

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†Faculty of Civil Engineering, University of Žilina, Univerzitná 1, Žilina, Slovak Republic and Moscow Power Engineering Institute, Moscow, Russia (`mariana.marcokova@gmail.com`).

‡Faculty of Mechanical Engineering, University of Žilina, Univerzitná 1, Žilina, Slovak Republic (`vladimir.guldan@fstroj.uniza.sk`).

Definition 1.2. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a system of polynomials, where every polynomial* $P_n(x)$ *has the degree* n*. If for all polynomials of this system*

$$\int_a^b P_n(x)P_m(x)v(x)dx = 0,\ \ n \neq m,$$

then the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ *are called orthogonal in* (a,b) *with respect to the weight function* $v(x)$*. If moreover*

$$||P_n(x)||_{v(x)} = \left[\int_a^b P_n^2(x)v(x)dx\right]^{\frac{1}{2}} = 1$$

for every $n = 0, 1, 2, \ldots$*, then the polynomials are called orthonormal in* (a,b)*.*

So the condition of the orthonormality of the system $\{P_n(x)\}_{n=0}^{\infty}$ has the form

$$\int_a^b P_n(x)P_m(x)v(x)dx = \delta_{nm},$$

where δ_{nm} is Kronecker delta.

Theorem 1.3. *For every weight function* $v(x)$ *there exists one and only one system of polynomials* $\{P_n(x)\}_{n=0}^{\infty}$ *orthonormal in* (a,b)*, where*

$$P_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}\ ,\ \ a_0^{(n)} > 0.$$

Theorem 1.4. *A polynomial* $P_n(x)$ *is orthogonal in* (a,b) *with respect to the weight function* $v(x)$*, if and only if for arbitrary polynomial* $S_m(x)$ *of the degree* $m < n$ *the following condition is fulfilled*

$$\int_a^b P_n(x)S_m(x)v(x)dx = 0.$$

Theorem 1.5. *If the interval of orthogonality is symmetric according to the origin of coordinate system and weight function* $v(x)$ *is even function, then every orthogonal polynomial* $P_n(x)$ *fulfils the equality*

$$P_n(-x) = (-1)^n P_n(x).$$

1.2. Classical Jacobi polynomials, classical Legendre polynomials and differential equations satisfied by them. It is well-known that Jacobi polynomials $\{P_n(x;\alpha,\beta)\}_{n=0}^{\infty}$ are orthogonal in the interval $I=(-1,1)$ with respect to the weight function

$$J(x)=(1-x)^{\alpha}(1+x)^{\beta},\ x\in(-1,1), \tag{1.1}$$

where $\alpha>-1,\beta>-1$. Very important special case of Jacobi polynomials are classical Legendre polynomials $\{P_n(x;0,0)\}_{n=0}^{\infty}$, for which $\alpha=\beta=0$ in the weight function $J(x)$. In the next we denote them by $\{P_n(x)\}_{n=0}^{\infty}$. As it is seen the Legendre classical polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are orthogonal in $I=(-1,1)$ with respect to the weight function $L(x)=1$. If $\alpha=\beta$, then polynomials $\{P_n(x;\alpha,\alpha)\}_{n=0}^{\infty}$ are called ultraspherical polynomials.

Classical orthogonal polynomials are solutions of the second order linear homogeneous differential equations of the form (cf. e.g. [4], [8], and [9]):

$$a(x)y_n''(x)+b(x)y_n'(x)+\lambda_n y_n(x)=0,$$

where $a(x)$ is a polynomial of the degree at most 2 , $b(x)$ is a polynomial of the degree 1 and λ_n does not depend of x . For the classical Jacobi polynomials this equation has the form

$$(1-x^2)y_n''(x)+[\beta-\alpha-(\alpha+\beta+2)x]\,y_n'(x)+n(n+\alpha+\beta+1)y_n(x)=0,$$

which in the case of the classical Legendre polynomials is reduced to the equation

$$(1-x^2)y_n''(x)-2xy_n'(x)+n(n+1)y_n(x)=0. \tag{1.2}$$

2. Generalized Legendre polynomials of a certain type and classical Jacobi polynomials with different weight functions. In [6] we introduced the system of polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ which are the polynomials orthonormal in I with respect to the weight function

$$Q(x)=\left(x^2\right)^{\gamma},$$

where $\gamma>0$ and $Q_n(+\infty)>0$. It is clear that these polynomials are generalization of the classical Legendre polynomials, which can be obtained by substituting $\gamma=0$ in the weight function $Q(x)$.

Further in [6] we introduced two classes of orthonormal polynomials:

1. polynomials $\{P_n(x;0,\gamma-\frac{1}{2})\}_{n=0}^{\infty}$ orthonormal in I with respect to the weight function

$$J_1(x)=(1+x)^{\gamma-\frac{1}{2}}$$

and

2. polynomials $\{P_n(x;0,\gamma)\}_{n=0}^{\infty}$ orthonormal in I with respect to the weight function

$$J_2(x)=(1+x)^{\gamma}.$$

In both these cases we have classical Jacobi polynomials orthogonal with the weight function (1.1) for $\alpha = 0$, $\beta = \gamma - \frac{1}{2}$ and $\alpha = 0$, $\beta = \gamma$, respectively. In the next theorem we proved relations between them and the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ (cf. [6]). Here we give this theorem with its proof because it is essential for our further investigation.

THEOREM 2.1. *In the notations introduced in the previous sections we have*

$$Q_{2n}(x) = 2^{\frac{\gamma}{2}-\frac{1}{4}} P_n\left(2x^2-1; 0, \gamma - \frac{1}{2}\right) \tag{2.1}$$

and

$$Q_{2n+1}(x) = 2^{\frac{\gamma}{2}} x P_n\left(2x^2-1; 0, \gamma\right). \tag{2.2}$$

Proof. According to the Theorem 1.5, the function $Q_{2n}(x)$ is even function. Putting $t = x^2$ we denote $W_n(t) = Q_{2n}(x)$. The orthogonality of the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ for $r = 0, 1, \ldots, n-1$ and $n > 0$ yields

$$0 = \int_0^1 x^{2r} Q_{2n}(x) x^{2\gamma} dx = \frac{1}{2}\int_0^1 t^r W_n(t) t^{\gamma-\frac{1}{2}} dt =$$

$$= \frac{1}{2^2}\int_{-1}^1 \left(\frac{\tau+1}{2}\right)^r W_n\left(\frac{\tau+1}{2}\right)\left(\frac{\tau+1}{2}\right)^{\gamma-\frac{1}{2}} d\tau =$$

$$= \frac{1}{2^{\gamma+\frac{3}{2}}}\int_{-1}^1 \left(\frac{\tau+1}{2}\right)^r W_n\left(\frac{\tau+1}{2}\right)(\tau+1)^{\gamma-\frac{1}{2}} d\tau\ .$$

From that it is clear that the polynomials $W_n\left(\frac{x+1}{2}\right)$ are orthogonal in I with respect to the weight function $J_1(x)$. According to the Theorem 1.3, taking into account the uniqueness of these polynomials, we have

$$W_n\left(\frac{x+1}{2}\right) = k\, P_n\left(x; 0, \gamma - \frac{1}{2}\right),$$

where $k > 0$ in consequence of the fact that $P_n\left(\infty; 0, \gamma - \frac{1}{2}\right) > 0$ and $W_n(+\infty) > 0$.

From the orthonormality of the polynomials $W_n(t)$ we derive

$$\frac{1}{2} = \int_0^1 W_n^2(t)\, t^{\gamma-\frac{1}{2}} dt = k^2 \int_{-1}^1 P_n^2\left(\tau; 0, \gamma - \frac{1}{2}\right)\left(\frac{\tau+1}{2}\right)^{\gamma-\frac{1}{2}} \frac{1}{2}\, d\tau =$$

$$= \frac{1}{2^{\gamma+\frac{1}{2}}} k^2 \int_{-1}^1 P_n^2\left(\tau; 0, \gamma - \frac{1}{2}\right)(\tau+1)^{\gamma-\frac{1}{2}} d\tau$$

from where we have $k = 2^{\frac{\gamma}{2}-\frac{1}{4}}$ and the relation (2.1), i.e.

$$Q_{2n}(x) = 2^{\frac{\gamma}{2}-\frac{1}{4}} P_n\left(2t-1; 0, \gamma - \frac{1}{2}\right),\ t = x^2.$$

Now we prove the relation (2.2). Putting $t = x^2$ we have

$$\overline{W}_n(t) = x^{-1} Q_{2n+1}(x),$$

where $\overline{W}_n(t)$ is the polynomial of the degree n and $Q_{2n+1}(x)$ is odd function. For $r = = 0, 1, \ldots, n-1$ and $n > 0$ the orthogonality of the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ yields

$$0 = \int_0^1 x^{2r+1} Q_{2n+1}(x)\, x^{2\gamma} dx = \frac{1}{2}\int_0^1 t^r\, \overline{W}_n(t)\, t^{\gamma+\frac{1}{2}} dt =$$

$$= \frac{1}{2^2}\int_{-1}^1 \left(\frac{\tau+1}{2}\right)^r \overline{W}_n\left(\frac{\tau+1}{2}\right)\left(\frac{\tau+1}{2}\right)^{\gamma+\frac{1}{2}} d\tau =$$

$$= \frac{1}{2^{\gamma+\frac{5}{2}}}\int_{-1}^1 \left(\frac{\tau+1}{2}\right)^r \left(\frac{\tau+1}{2}\right)^{\frac{1}{2}} \overline{W}_n\left(\frac{\tau+1}{2}\right)(\tau+1)^{\gamma} d\tau\ .$$

From there

$$\left(\frac{x+1}{2}\right)^{\frac{1}{2}} \overline{W}_n\left(\frac{x+1}{2}\right) = \overline{k}\, P_n(x; 0, \gamma)\ ,$$

where $\overline{k} > 0$ and from the orthonormality of the polynomials $t^{\frac{1}{2}}\,\overline{W}_n(t)$ we derive

$$\frac{1}{2} = \int_0^1 x^{-2}\, Q_{2n+1}^2(x)\, x^{2\gamma} dx = \int_0^1 t\, \overline{W}_n^2(t)\, t^{\gamma} dt =$$

$$= \frac{1}{2}\int_{-1}^1 \left(\frac{\tau+1}{2}\right) \overline{W}_n^2\left(\frac{\tau+1}{2}\right)\left(\frac{\tau+1}{2}\right)^{\gamma} d\tau = \frac{1}{2^{\gamma+1}}\overline{k}^{\,2}\int_{-1}^1 P_n^2(\tau; 0, \gamma)(\tau+1)^{\gamma} d\tau\ .$$

Finally we get $\overline{k} = 2^{\frac{\gamma}{2}}$ and the relation (2.2) of the theorem. ∎

3. Generalized ultraspherical polynomials and their relation to certain classical Jacobi polynomials. In the next theorem we generalize the relations derived in the Theorem 2.1 for generalized ultraspherical polynomials taking into account polynomials orthonormal in I with respect to the weight function

$$\widetilde{Q}(x) = \left(1 - x^2\right)^{\alpha}\left(x^2\right)^{\gamma}$$

instead of the weight function $Q(x) = \left(x^2\right)^{\gamma}$.

THEOREM 3.1. *Let* $\{\widetilde{Q}_n(x)\}_{n=0}^{\infty}$ *be the polynomials orthonormal in* $I=(-1,1)$ *with the weight function*

$$\widetilde{Q}(x)=\left(1-x^2\right)^{\alpha}\left(x^2\right)^{\gamma},$$

where $\alpha>-1,\gamma>0$ *and* $\widetilde{Q}_n(+\infty)>0$. *Let* $\{P_n\left(x;\alpha,\gamma-\frac{1}{2}\right)\}_{n=0}^{\infty}$ *be the polynomials orthonormal in* I *with the weight function*

$$\widetilde{J}_1(x)=(1-x)^{\alpha}(1+x)^{\gamma-\frac{1}{2}}$$

and $\{P_n(x;\alpha,\gamma)\}_{n=0}^{\infty}$ *be the polynomials orthonormal in* I *with the weight function*

$$\widetilde{J}_2(x)=(1-x)^{\alpha}(1+x)^{\gamma}.$$

Then

$$\widetilde{Q}_{2n}(x)=2^{\frac{\alpha+\gamma}{2}-\frac{1}{4}}P_n\left(2x^2-1;\alpha,\gamma-\frac{1}{2}\right) \tag{3.1}$$

and

$$\widetilde{Q}_{2n+1}(x)=2^{\frac{\alpha+\gamma}{2}}xP_n\left(2x^2-1;\alpha,\gamma\right). \tag{3.2}$$

Proof. Similarly to the proof of the Theorem 2.1 we put the appropriate substitutions to the integrals proving the orthonormality of the polynomials $\{\widetilde{Q}_n(x)\}_{n=0}^{\infty}$. In all the integrals the term $\left(\frac{1-\tau}{2}\right)^{\alpha}$ will appear and after some algebra and integration the term $2^{\frac{\alpha}{2}}$ will appear in the relations (3.1) and (3.2). ∎

4. Even multiple of the weight function of Jacobi polynomials. The result of the following theorem is the analogy of the well-known relation for classical Jacobi polynomials.

THEOREM 4.1. *Let* $\{Q_n(x;\alpha,\beta,\gamma)\}_{n=0}^{\infty}$ *be the polynomials orthonormal in the interval* $I=(-1,1)$ *with the weight function*

$$Q(x;\alpha,\beta,\gamma)=(1-x)^{\alpha}(1+x)^{\beta}\left(x^2\right)^{\gamma}$$

where $\alpha>-1,\beta>-1,\gamma>0$. *Then*

$$Q_n(-x;\alpha,\beta,\gamma)=(-1)^nQ_n(x;\beta,\alpha,\gamma). \tag{4.1}$$

Proof. According to the orthogonality criterion (Theorem 1.4) the necessary and sufficient condition of the orthogonality of the polynomials $\{Q_n(x;\alpha,\beta,\gamma)\}_{n=0}^{\infty}$ has the form

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left(x^2\right)^{\gamma}Q_n(x;\alpha,\beta,\gamma)F_m(x)dx=0,$$

where $F_m(x)$ is an arbitrary polynomial of the degree $m=0,1,\ldots,n-1$. Substituting $x=-t$ this condition will obtain the form

$$\int_{-1}^{1}(1+t)^{\alpha}(1-t)^{\beta}(t^2)^{\gamma}Q_n(-t;\alpha,\beta,\gamma)F_m(-t)dt=0.$$

Because $F_m(t)$ is an arbitrary polynomial of the degree m, then also $F_m(-t)$ is an arbitrary polynomial of the degree m. So, in the consequence of the same theorem, the polynomial $Q_n(-t;\alpha,\beta,\gamma)$ is also orthogonal, but with the weight $Q(t;\beta,\alpha,\gamma)$ and it may differ from the orthogonal polynomial $Q_n(t;\beta,\alpha,\gamma)$ only by constant multiple. So

$$Q_n(-t;\alpha,\beta,\gamma) \equiv c\ Q_n(t;\beta,\alpha,\gamma).$$

Because the polynomials are orthonormal, it yields $|c| = 1$. Comparing the coefficients at the highest powers of these two polynomials, we get $|c| = (-1)^n$ and the relation (4.1). □

It is obvious that the result of this theorem can be generalized for polynomials orthogonal in I with the weight function $J(x)h(x)$, where the factor $h(x)$ is an even function on I.

THEOREM 4.2. *Let* $\{\widetilde{P}_n(x;\alpha,\beta)\}_{n=0}^{\infty}$ *be the polynomials orthonormal in the interval* $I = (-1,1)$ *with the weight function*

$$\widetilde{J}(x;\alpha,\beta) = (1-x)^{\alpha}(1+x)^{\beta}h(x),$$

where $\alpha > -1, \beta > -1, h(x) \geq 0$ *in* I *and* $h(x)$ *is an even function in* I. *Then*

$$\widetilde{P}_n(x;\alpha,\beta) = (-1)^n \widetilde{P}_n(x;\beta,\alpha).$$

Proof. Similar to the proof of the Theorem 4.1. □

5. Consequences of differential equations with generalized Legendre polynomials solutions. Differentiating both sides of (2.1) according to x, then expressing $P_n\left(2x^2-1;0,\gamma-\frac{1}{2}\right)$, $P_n'\left(2x^2-1;0,\gamma-\frac{1}{2}\right)$, and $P_n''\left(2x^2-1;0,\gamma-\frac{1}{2}\right)$ by means of polynomials $Q_{2n}(x)$, $Q_{2n}'(x)$, and $Q_{2n}''(x)$, then substituting the derivatives P_n, P_n', and P_n'' into the differential equation with these Jacobi polynomials solutions, we have the following equation:

$$(1-x^2)Q_{2n}''(x) - \frac{-2\gamma+2x^2+2\gamma x^2}{x}Q_{2n}'(x) = -2n(2n+2\gamma+1)Q_{2n}(x).$$

For $\gamma = 0$ it reduces to the equation

$$(1-x^2)Q_{2n}''(x) - 2xQ_{2n}'(x) = -2n(2n+1)Q_{2n}(x).$$

Comparing it with (1.2) we observe the last equation to be the differential equation for the $(2n)$-th degree Legendre polynomial.

By the similar way from (2.2) we derive the following equation:

$$(1-x^2)Q_{2n+1}''(x) + \frac{-1+2\gamma-x^2-2\gamma x^2}{x}Q_{2n+1}'(x) + \frac{1-2\gamma+x^2+2\gamma x^2}{x^2}Q_{2n+1}(x) =$$

$$= -4n(n+\gamma+1)Q_{2n+1}(x).$$

For $\gamma = \frac{1}{2}$ it reduces to the equation

$$(1-x^2)Q_{2n+1}''(x) - 2xQ_{2n+1}'(x) + 2Q_{2n+1}(x) = -4n\left(n+\frac{3}{2}\right)Q_{2n+1}(x).$$

The last equation is the differential equation for the $(2n+1)$-st degree Legendre polynomial.

In such a way we have proved the following theorem:

Theorem 5.1.

1. *The Jacobi polynomial* $P_n\left(2x^2-1;0,-\frac{1}{2}\right)$ *of the argument* $2x^2-1$ *orthogonal with respect to the weight function* $(1+x)^{-\frac{1}{2}}$ *is the Legendre polynomial of the argument* x *and of the degree* $2n$.

2. *The polynomial* $xP_n\left(2x^2-1;0,\frac{1}{2}\right)$, *where* $P_n\left(2x^2-1;0,\frac{1}{2}\right)$ *is the Jacobi polynomial of the argument* $2x^2-1$, *orthogonal with respect to the weight* $(1+x)^{\frac{1}{2}}$, *is the Legendre polynomial of the argument* x *and of the degree* $2n+1$.

REFERENCES

[1] L. C. Andrews, *Special Functions of Mathematics for Engineers*, Second ed., McGraw-Hill, Inc., 1992.

[2] B. Hofman-Wellenhof and H. Moritz, *Physical Geodesy*, Second ed., Springer, 2006.

[3] D. Janecki and K. Stepien, *Legendre polynomials used for the approximation of cylindrical surfaces*, Communications - Scientific Letters of the University of Žilina, 4 (2005), 59–61.

[4] J. Korous, *Special Parts of Mathematics. Orthogonal Functions and Orthogonal Polynomials*, SNTL, Prague, 1958 (in Czech).

[5] P. W. Livermore, CH. A. Jones, and S. J. Worland, *Spectral radial basis functions for full sphere computations*, J. Comput. Phys., 227 (2007), 1209–1224.

[6] M. Marčoková and V. Guldan, *Jacobi Polynomials and Some Related Functions*, in Mathematical Methods in Engineering, N. M. F. Ferreira and J. A. T. Machado, eds., Springer, 2014, pp. 219–227.

[7] S. P. Mirevski, L. Boyadijev, and R. Scherer, *On the Riemann-Liouville fractional calculus, g-Jacobi functions and F-Gauss functions*, Appl. Math. Comput., 187(1) (2007), 315–325.

[8] P. K. Sujetin, *Classical orthogonal polynomials*, Nauka, Moskva, 1979 (in Russian).

[9] G. Szegö, *Orthogonal polynomials*, Nauka, Moskva, 1969 (in Russian).

[10] R. Tenzer, *Geopotential model of Earth - approximation of Earth shape, geopotential model testing methods*, Communications - Scientific Letters of the University of Žilina, 4 (2001), 50–58.

[11] R. Tenzer, *Methodology for testing of geopotential model*, Studies of University in Žilina, Civil Engineering series, 25 (2002), 113–116.