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KOLMOGOROV'S ε-ENTROPY OF THE ATTRACTOR OF THE STRONGLY DAMPED WAVE EQUATION IN LOCALLY UNIFORM SPACES*

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Abstract. We establish an upper bound on the Kolmogorov's entropy of the locally compact attractor for strongly damped wave equation posed in locally uniform spaces in subcritical case using the method of trajectories.

Key words. Strongly damped wave equation, unbounded domains, locally compact attractor, Kolmogorovs entropy.

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1. Introduction. We are interested in the asymptotic properties of the strongly damped wave equation

$$u_{tt} + \beta u_t - \alpha \Delta u_t - \Delta u + f(u) = g, \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function specified later and α, $\beta > 0$, supplemented by the initial datum

$$u(0) = u_0 \in W_b^{1,2}(\mathbb{R}^d), \qquad u_t(0) = u_1 \in L_b^2(\mathbb{R}^d).$$

The strongly damped wave equation has a number of relevant physical applications, see e.g. [5].

Asymptotic properties of the equation (1.1) in bounded domains have been thoroughly studied. Let us mention some of the results briefly. In [2] the authors establish the existence of global attactor for the critical case. Exponential attractors in the subcritical and critical case have been studied in [11] and [16]. The existence of global attractor for critical and supercritical exponents has been shown for a strongly damped wave equation with memory in [5]. The finite dimensionality of the attractor has been shown in [6]. The situation in supercritical case is studied in detail in [8].

In the non-autonomous case when $g = g(t)$, the resulting uniform attractor might have infinite fractal dimension induced by the time-dependence of the external forces. To measure the complexity of the attractor one can employ Kolmogorov's ε-entropy instead of fractal dimension. In [9] the authors establish an upper bound on Kolmogorov's ε-entropy of the attractor of equation similar to (1.1) in bounded domain and show that if the time-dependent right-hand side is finite-dimensional in the appropriate sense, the resulting attractor is finite dimensional.

In unbounded domains the results are more scarce. In [1] and [4] the authors study the equation (1.1) posed in the classical space $W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and show the existence of a connected universal attractor in the subcritical and critical case. In the context of locally uniform spaces, the non-autonomous wave equation with weak linear damping, i.e. with $\alpha = 0$, has been studied in detail in [17] including an upper bound

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on Kolmogorov's ε-entropy of an attractor reflecting the non-compactness induced both by time-dependent external forces and the unboundedness of the spatial domain. The strongly damped wave equation has been studied in [3], where the well-posedness of the equation in a more regular subspace of locally uniform space $W_b^{2,p}(\mathbb{R}^d)\times L_b^p(\mathbb{R}^d)$, $p > d/2$, $p \geq 2$, and the existence of a locally compact attractor have been shown for the critical case. In [15] the authors generalized these results to the space of locally uniform functions $W_b^{1,2}(\mathbb{R}^d) \times L_b^2(\mathbb{R}^d)$ and obtained a result on the asymptotic regularity of the solutions, cf. the end of this section. In [14] the author studies a variant of the strongly damped wave equation with fractional damping and shows the existence of a locally compact attractor in the critical case together with space-time regularity of the solutions.

The aim of this paper is to establish an upper bound on the Kolmogorov's ε-entropy of the attractor of the equation (1.1) in the subcritical case. To this end we use the method of trajectories and a technique similar to the ones used for a wave equation with nonlinear damping in [12] for bounded domains, resp. in [10] for unbounded domains. However, compared to [10] or [14], solutions of (1.1) do not possess neither a finite speed of propagation nor a smoothing property, and thus the argument must be adapted.

Let ϕ be a weight function, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$. We denote

$$\begin{gathered}
\Phi_{\bar{x},\varepsilon} = W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d) \times L^2_{\bar{x},\varepsilon}(\mathbb{R}^d), \quad W_{\bar{x},\varepsilon} = W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d) \times W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d),\\
\Phi_{b,\phi} = W^{1,2}_{b,\phi}(\mathbb{R}^d) \times L^2_{b,\phi}(\mathbb{R}^d), \quad W_{b,\phi} = W^{1,2}_{b,\phi}(\mathbb{R}^d) \times W^{1,2}_{b,\phi}(\mathbb{R}^d),\\
W_{\mathrm{loc}} = W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d) \times W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d),
\end{gathered}$$

with the convention that we omit the subscript ϕ if $\phi \equiv 1$ and write for example Φ_b instead of $\Phi_{b,1}$. For definitions of weight functions and weighted and locally uniform spaces see Section 2.

For simplicity let us choose $\alpha = \beta = 1$. The nonlinear term $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies the following conditions:

- (*growth condition*) there exist $C > 0$ and $0 \leq q \leq 4/(d-2)$ such that

$$|f(r) - f(s)| \leq C|r-s|\left(1 + |r|^q + |s|^q\right), \qquad \forall r,s \in \mathbb{R}. \tag{1.2}$$

  The nonlinearity is *critical* if $q = 4/(d-2)$ and *subcritical* if $q < 4/(d-2)$.
- (*dissipation condition*) there exist $k \geq 1$ and $\mu_0 > 0$ such that for every $\mu \in (0,\mu_0]$ there exist $C_\mu, C_0 \in \mathbb{R}$ such that

$$kF(s) + \mu s^2 - C_\mu \leq sf(s), \quad -C_0 \leq F(s) \qquad \forall s \in \mathbb{R},$$

  where $F(s) = \int_0^s f(r)\,dr$.

These conditions are the same as in [3] and [15].

The *weak solution* of (1.1) is defined in the sense of distributions on $(0,\infty)\times\mathbb{R}^d$ and has the regularity

$$(u,u_t) \in C([0,T];\Phi_{\bar{x},\varepsilon}), \qquad \|u\|^2_{W^{1,2}_b} + \|u_t\|^2_{L^2_b} \in L^\infty((0,T)),$$

for every $T > 0$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$. Using a standard density argument it can be shown that the equation can be tested by functions

$$\varphi \in L^2(0,T;W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d)) \cap W^{1,2}(0,T;L^2_{\bar{x},\varepsilon}(\mathbb{R}^d))$$

for arbitrary $T > 0$, $\bar{x} \in \mathbb{R}^d$, $\varepsilon > 0$.

The existence and uniqueness of weak solutions has been shown in [15, Section 3] using semigroup theory in the subspace of more regular initial data continuous with respect to spatial to translations. We also have the following dissipative estimates: there exist t_0, $C > 0$ such that for every $t > t_0$ we have

$$\|u\|_{W_b^{1,2}} + \|u_t\|_{W_b^{1,2}} + \|u_{tt}\|_{L_b^2} \leq C. \tag{1.3}$$

For proofs see [15, Section 4]. Let us denote the absorbing set by $\mathcal{B}$ and assume that $\mathcal{B}$ is closed and positively invariant.

In [15], the authors also show the existence of a locally compact attractor in the critical case, namely the existence an invariant set $\mathcal{A} \subseteq \Phi_b$ bounded and closed in $W_b^{2,2}(\mathbb{R}^d) \times W_b^{1,2}(\mathbb{R}^d)$ and compact in W_{loc}, which attracts the bounded sets of Φ_b in the W_{loc}-norm, and the asymptotic regularity, namely the existence of a closed and bounded set $\mathcal{B}_1 \subseteq W_b^{2,2}(\mathbb{R}^d) \times W_b^{1,2}(\mathbb{R}^d)$, a constant $\nu > 0$, and a positive monotonically increasing function $Q(\cdot)$ such that for every bounded $B \subseteq \Phi_b$ we have

$$\operatorname{dist}_{\Phi_b}(S(t)B, \mathcal{B}_1) \leq Q(\|B\|_{\Phi_b}) e^{-\nu t} \qquad \forall t > 0.$$

For proofs see [15, Theorem 1.1 and 1.2]. It is worth noting that the technique presented in this paper do not rely on the asymptotic regularity of the attractor.

This paper is organized as follows: in Section 2 we review the basic definitions of function spaces used in the rest of the paper. In Section 3 we define the trajectory spaces and the trajectory semigroup and show that the trajectory semigroup has a squeezing property which is then used in Section 4 to establish an upper estimate on the locally compact attractor of the equation (1.1).

2. Function spaces. A function $\phi : \mathbb{R}^d \to (0, \infty)$ is called a *weight function* of growth $\mu \geq 0$ if

$$C_\phi^{-1} e^{-\mu|x-y|} \leq \phi(x)/\phi(y) \leq C_\phi e^{\mu|x-y|}, \ |\nabla\phi| \leq \tilde{C}_\phi \mu \phi, \quad \text{for a.e. } x, y \in \mathbb{R}^d, \tag{2.1}$$

for some $C_\phi \geq 1$ and some $\tilde{C}_\phi > 0$. For $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we denote

$$\phi_{\bar{x},\varepsilon}(x) = \exp(-\varepsilon|x - y|).$$

Clearly $\phi_{\bar{x},\varepsilon}$ is a weight function of growth ε.

For $p \in [1, \infty)$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we define the *weighted Lebesgue space* $L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by

$$L^p_{\bar{x},\varepsilon}(\mathbb{R}^d) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d); \|u\|^p_{L^p_{\bar{x},\varepsilon}} = \int_{\mathbb{R}^d} |u(x)|^p \phi_{\bar{x},\varepsilon}(x)\, dx < \infty\}.$$

In the case $p = 2$ we use the notation $\|\cdot\|_{L^2_{\bar{x},\varepsilon}} \equiv \|\cdot\|_{\bar{x},\varepsilon}$ and denote the scalar product in $L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by $(\cdot,\cdot)_{\bar{x},\varepsilon}$. The weighted Sobolev spaces are defined in an obvious manner.

Observe that the space $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ cannot be embedded into $L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$ for any $q > p$. However, assuming that k, $l \in \mathbb{N}_0$ and p, $q \in [1, \infty)$ satisfy $k \geq l$, $q \geq p$ and $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{l,q}(\mathbb{R}^d)$, then for $\tilde{\varepsilon} = \varepsilon q/p$ we have the continuous embedding $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$. Moreover, if the embedding $W^{k,p}(B) \hookrightarrow\hookrightarrow W^{l,q}(B)$ is compact, where $B = B(0,1) \subseteq \mathbb{R}^d$ then for $\tilde{\varepsilon} > \varepsilon q/p$ the embedding $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow\hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$ is compact as well.

Let ϕ be a weight function and $p \in [1,\infty)$. We define the *weighted locally uniform space* $L^p_{b,\phi}(\mathbb{R}^d)$ by

$$L^p_{b,\phi}(\mathbb{R}^d) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d); \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})^{1/p}\|u\|_{L^p(C^1_{\bar{x}})} < \infty\},$$

where C^R_x denotes the cube in $\mathbb{R}^d$ of side $R > 0$ and centred at $x \in \mathbb{R}^d$. We equip the space with a norm equivalent to $\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})^{1/p}\|u\|_{L^p(C^1_{\bar{x}})}$ defined by

$$\|u\|_{L^p_b} = \sup_{k\in\mathbb{Z}^d} \phi(k)^{1/p}\|u\|_{L^p(C^1_k)}. \tag{2.2}$$

Also one can see that if we take any bounded neighbourhood of $\bar{x}$ in (2.2) instead of C^1_k, we again obtain an equivalent norm.

THEOREM 2.1 (see e.g. [7, Theorem 2.1]). *Let $k \in \mathbb{N}_0$, $p \in [1,\infty)$ and $\varepsilon > 0$. Let ϕ be a weight function of growth rate $0 \le \mu < \varepsilon$ and $u \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$. Then $u \in W^{k,p}_{b,\phi}(\mathbb{R}^d)$ if and only if $u \in W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ for every $\bar{x} \in \mathbb{R}^d$ and*

$$\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})^{1/p}\|u\|_{W^{k,p}_{\bar{x},\varepsilon}} < \infty. \tag{2.3}$$

Moreover, the left-hand side of (2.3) defines a norm equivalent to the $W^{k,p}_{b,\phi}(\mathbb{R}^d)$-norm.

For $\mathcal{O} \subseteq \mathbb{R}^d$ denote $\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{Z}^d; C^1_k \cap \mathcal{O} \neq \emptyset\}$ and we define the $W^{k,p}_{b,\phi}(\mathcal{O})$-seminorm by

$$\|u\|_{W^{k,p}_{b,\phi}(\mathcal{O})} = \sup_{l\in\mathbb{I}(\mathcal{O})} \phi(l)^{1/p}\|u\|_{W^{k,p}(C^1_l)}. \tag{2.4}$$

LEMMA 2.2 ([17, Proposition 1.2]). *For $1 \le p < \infty$ and $\varepsilon > 0$ fixed there exist C_1, $C_2 > 0$ such that for $\bar{x} \in \mathbb{R}^d$ and $u \in L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ with we have*

$$C_1\|u\|^p_{L^p_{\bar{x},\varepsilon}} \le \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x)\|u\|^p_{L^p(B(x,1))}\,dx \le C_2\|u\|^p_{L^p_{\bar{x},\varepsilon}}.$$

Let $\ell > 0$ and let ϕ be a weight function. We define the parabolic locally uniform spaces $L^2_{b,\phi}(0,\ell;L^2(\mathbb{R}^d))$, $L^2_{b,\phi}(0,\ell;W^{1,2}(\mathbb{R}^d)) \subseteq L^2_{\mathrm{loc}}((0,\ell)\times\mathbb{R}^d)$ by

$$\begin{aligned} L^2_{b,\phi}(0,\ell;L^2) &= \{u; \|u\|^2_{L^2_{b,\phi}(0,\ell;L^2)} = \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})\|u\|^2_{L^2(0,\ell;L^2(C^1_{\bar{x}}))} < \infty\}, \\ L^2_{b,\phi}(0,\ell;W^{1,2}) &= \{u; \|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2})} = \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})\|u\|^2_{L^2(0,\ell;W^{1,2}(C^1_{\bar{x}}))} < \infty\}. \end{aligned}$$

LEMMA 2.3 ([7, Theorem 2.4]). *Let $\varepsilon > 0$ be fixed and let ϕ be a weight function of growth rate $\mu \in [0,\varepsilon)$. Then*

$$\begin{aligned} \|u\|^2_{L^2_{b,\phi}(0,\ell;L^2)} &\approx \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(x,t)|^2 \phi_{\bar{x},\varepsilon}(x)\,dx\,dt, \\ \|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2})} &\approx \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} \left(|u(x,t)|^2 + |\nabla u(x,t)|^2\right) \phi_{\bar{x},\varepsilon}(x)\,dx\,dt. \end{aligned}$$

In particular the previous lemma implies that for a weight function ϕ of growth rate $\mu \in [0, \min\{\varepsilon_1, \varepsilon_2\})$ for some $\varepsilon_1, \varepsilon_2 > 0$ one has

$$\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(x)|^2 \phi_{\bar{x},\varepsilon_2}(x)\, dx\, dt \approx \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(x)|^2 \phi_{\bar{x},\varepsilon_1}(x)\, dx\, dt$$

and similarly in the case of $L^2_{b,\phi}(0,\ell; W^{1,2})$. For $\mathcal{O} \subseteq \mathbb{R}^d$ we can define the seminorms $L^2_{b,\phi}(0,\ell; L^2(\mathcal{O}))$ and $L^2_{b,\phi}(0,\ell; W^{1,2}(\mathcal{O}))$ similarly as in (2.4).

LEMMA 2.4 (Ehrling's lemma in weighted spaces, see e.g. [13, Lemma 7.6]). *Let p, $q \geq 1$ and ε, $\tilde{\varepsilon} > 0$ be such that the embedding $W^{1,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$ holds. Then for every $\theta > 0$ and $1 \leq \alpha < q$ there exist C, $R > 0$ such that for every $u : (0,\ell) \times \mathbb{R}^d \to \mathbb{R}$ one has*

$$\int_0^\ell \|u(t)\|^\alpha_{L^q_{\bar{x},\tilde{\varepsilon}}}\, dt \leq \theta \int_0^\ell \|u(t)\|^\alpha_{W^{1,p}_{\bar{x},\varepsilon}}\, dt + C \int_0^\ell \int_{B(\bar{x},R)} |u(t,x)|^\alpha\, dxdt. \tag{2.5}$$

3. Squeezing property. We define the energy functional by

$$E[u](t,x) = \frac{1}{2}\left(|u_t(t,x)|^2 + |u(t,x)|^2 + |\nabla u(t,x)|^2\right).$$

Let us define the space of trajectories

$$\mathcal{X} = \{(\chi, \chi_t); \chi \in L^2_{\mathrm{loc}}((0,\ell)\times\mathbb{R}^d) \text{ solves (1.1) on } (0,\ell) \text{ with } (\chi(0), \chi_t(0)) \in \mathcal{B}\}.$$

Let $\ell > 0$ be fixed. The trajectory semigroup $L(t) : \mathcal{X} \to \mathcal{X}$ and the end-point operator $e : \mathcal{X} \to \Phi_b$ are given by

$$(L(t)(\chi,\chi_t))(s) = (S(t)\chi(s), \partial_t S(t)\chi),\ s \in (0,\ell), \qquad e(\chi) = (\chi(\ell), \chi_t(\ell)).$$

Let us also denote $L \equiv L(\ell)$. For a weight function ϕ we also define

$$\Phi^\ell_{b,\phi} = L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) \times L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d)),$$
$$W^\ell_{b,\phi} = L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) \times L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d))$$

and define respective seminorms similarly as in (2.4) for the parabolic spaces.

LEMMA 3.1. *There exists $\mu_0 > 0$ such that for all weight functions of growth $\mu \in [0,\mu_0)$ and all $\ell > 0$ the operators $L : \Phi^\ell_{b,\phi} \to W^\ell_{b,\phi}$ and $e : \Phi^\ell_{b,\phi} \to W_{b,\phi}$ are Lipschitz continuous on $\mathcal{X}$.*

In the next section we will use a weaker version of Lemma 3.1, more precisely the Lipschitz continuities $L : W^\ell_{b,\phi} \to W^\ell_{b,\phi}$ and $e : W^\ell_{b,\phi} \to W_{b,\phi}$, both of which follow from the proof by adding $\|\nabla w_t(s)\|^2_{\bar{x},\varepsilon}$ to the right-hand side of (3.1). A similar remark also applies to Lemma 4.1.

Proof. Let χ_1, $\chi_2 \in \mathcal{X}$, let u_1 and u_2 be the respective solutions and denote $w = u_1 - u_2$. By Lemma [15, Lemma 9.2] the semigroup $S(t) : \Phi_{\bar{x},\varepsilon} \to W_{\bar{x},\varepsilon}$ is Lipschitz continuous on $\mathcal{B}$ uniformly w.r.t. $t \in [0,T]$, i.e.

$$\begin{aligned}\|w(t)\|^2_{\bar{x},\varepsilon} + \|\nabla w(t)\|^2_{\bar{x},\varepsilon} + \|w_t(t)\|^2_{\bar{x},\varepsilon} + \|\nabla w_t(t)\|^2_{\bar{x},\varepsilon} \\ \leq C_{t,s}\left(\|w(s)\|^2_{\bar{x},\varepsilon} + \|\nabla w(s)\|^2_{\bar{x},\varepsilon} + \|w_t(s)\|^2_{\bar{x},\varepsilon}\right)\end{aligned} \tag{3.1}$$

for $0 < s < t$ and $\varepsilon > 0$ sufficiently small. The Lipschitz continuity of L then follows by integration over $s \in (0, \ell)$, $t \in (\ell, 2\ell)$, multiplication by $\phi(\bar{x})$, applying supremum over $\bar{x} \in \mathbb{R}^d$ to both sides of the estimate and using the equivalence of norms from Lemma 2.3. The Lipschitz continuity of e follows in a similar manner. □

DEFINITION 3.2. *The mapping $L : \mathcal{X} \to \mathcal{X}$ has a* squeezing property *for weight function ϕ if there exists $\varepsilon > 0$ such that for every $\gamma > 0$ we may find ℓ, κ, $R > 0$ so that for every χ_1, $\chi_2 \in \mathcal{X}$ and their respective solutions u_1 and u_2 we have*

$$\begin{aligned}\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_\ell^{2\ell} \int_{\mathbb{R}^d} \left(E[w] + |\nabla w_t|^2\right) \phi_{\bar{x},\varepsilon}\, dx\, dt \le \gamma \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\, dx\, dt \\ + \kappa \left(\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{B(\bar{x},R)} |w|^2\, dx\, dt + \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{B(\bar{x},R)} |w_t|^2\, dx\, dt \right) \quad (3.2) \\ + \kappa \left(\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_\ell^{2\ell} \int_{B(\bar{x},R)} |w|^2\, dx\, dt + \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_\ell^{2\ell} \int_{B(\bar{x},R)} |w_t|^2\, dx\, dt \right),\end{aligned}$$

where $w = u_1 - u_2$.

LEMMA 3.3. *Let the nonlinear term f be subcritical, i.e. let $0 \le q < 4/(d-2)$. Then for every weight function ϕ of sufficiently small growth the operator L has the squeezing property.*

Proof. The proof is similar to [12, Lemma 3.1]. Let χ_1, $\chi_2 \in \mathcal{X}$ and let u_1, u_2 be the respective solutions. Let $0 < \tau < \ell$ and denote $w = u_1 - u_2$. We test both the equations for u_1 and u_2 by $w_t + w/2$ to get

$$\begin{aligned}\frac{1}{2}\left(\|w_t(2\ell) + \frac{1}{2}w(2\ell)\|^2_{\bar{x},\varepsilon} + \frac{1}{8}\|w(2\ell)\|^2_{\bar{x},\varepsilon} + \frac{3}{4}\|\nabla w(2\ell)\|^2_{\bar{x},\varepsilon} \right) + \frac{1}{2}\int_\tau^{2\ell} \|w_t\|^2_{\bar{x},\varepsilon}\, dt \\ + \int_\tau^{2\ell} \|\nabla w_t\|^2_{\bar{x},\varepsilon} + \frac{1}{2}\|\nabla w\|^2_{\bar{x},\varepsilon}\, dt + \int_\tau^{2\ell} \left(f(u_1) - f(u_2), w_t + \frac{1}{2}w \right)_{\bar{x},\varepsilon} dt \\ + \int_\tau^{2\ell} \left(\nabla w_t, (w_t + \frac{1}{2}w)\nabla\phi_{\bar{x},\varepsilon} \right) + \left(\nabla w, (w_t + \frac{1}{2}w)\nabla\phi_{\bar{x},\varepsilon} \right) dt \\ = \frac{1}{2}\left(\|w_t(\tau) + \frac{1}{2}w(\tau)\|^2_{\bar{x},\varepsilon} + \frac{1}{8}\|w(\tau)\|^2_{\bar{x},\varepsilon} + \frac{3}{4}\|\nabla w(\tau)\|^2_{\bar{x},\varepsilon} \right). \quad (3.3)\end{aligned}$$

Relying on a standard but a rather tedious argument comprised of using Lemma 2.2, Hölder's and Young's inequalities, subcritical growth estimates (1.2) on the nonlinearity f and compact Sobolev embedding on bounded domains together with dissipation estimates (1.3) we obtain

$$\left| \int_{\mathbb{R}^d} (f(u_1) - f(u_2))(w_t + w)\phi_{\bar{x},\varepsilon}\, dx \right| \le \eta(\|w\|^2_{\bar{x},\varepsilon} + \|\nabla w\|^2_{\bar{x},\varepsilon}) + C\|w_t\|^2_{L^p_{\bar{x},\varepsilon}}$$

for $\eta > 0$ determined later and $1 \le p < 2d/(d-2)$. Putting the previous estimates into (3.3) and employing (2.1) and Young's inequality we get

$$\begin{aligned}C\left(\|w_t(2\ell) + \frac{1}{2}w(2\ell)\|^2_{\bar{x},\varepsilon} + \|w(2\ell)\|^2_{\bar{x},\varepsilon} + \|\nabla w(2\ell)\|^2_{\bar{x},\varepsilon} \right) \\ + \zeta \int_\ell^{2\ell} \|w_t\|^2_{\bar{x},\varepsilon} + \|\nabla w_t\|^2_{\bar{x},\varepsilon} + \|\nabla w\|^2_{\bar{x},\varepsilon} + \|w\|^2_{\bar{x},\varepsilon}\, dt \\ \le \int_{\mathbb{R}^d} E[w](\tau)\phi_{\bar{x},\varepsilon}\, dx + C\int_0^{2\ell} \|w\|^2_{\bar{x},\varepsilon} + \|w_t\|^2_{L^p_{\bar{x},\varepsilon}} + \|w\|^2_{L^p_{\bar{x},\varepsilon}}\, dt\end{aligned}$$

for some $\zeta > 0$. We note that from now on the value of ε will not change. We integrate over $\tau \in (0,\ell)$ and apply the weighted version of Ehrling's lemma (Lemma 2.4) to the functions $w(t)$ and $w_t(t)$ both on the time intervals $(0,\ell)$ and $(\ell,2\ell)$ to obtain

$$\begin{aligned}\zeta\ell \int_\ell^{2\ell}\int_{\mathbb{R}^d} \left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\,dx\,dt &\le \int_0^\ell \int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\,dx\,dt + C\ell\int_0^{2\ell}\|w\|^2_{\bar{x},\varepsilon}\,dt\\ &+ C\ell\theta\left(\int_0^\ell \|w\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\,dt + \int_0^\ell \|w_t\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\,dt + \int_\ell^{2\ell} \|w\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\,dt + \int_\ell^{2\ell} \|w_t\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\,dt\right)\\ &+ C\ell\left(\int_0^\ell\int_{B(\bar{x},R)} |w|^2 + |w_t|^2\,dx\,dt + \int_\ell^{2\ell}\int_{B(\bar{x},R)} |w|^2 + |w_t|^2\,dx\,dt\right)\end{aligned}$$

for some $R > 0$ fixed, $\theta > 0$ determined later and some $\tilde{\varepsilon} > 0$ such that $W^{1,2}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$, i.e. $2\varepsilon/q > \tilde{\varepsilon}$. If we restrict ourselves to weight functions ϕ of growth $\mu \in [0, \min\{\varepsilon,\tilde{\varepsilon}\})$, multiply by $\phi(\bar{x})$ and apply supremum over $\bar{x} \in \mathbb{R}^d$, then by Lemma 2.3 and by choosing θ sufficiently small we obtain

$$\begin{aligned}\tilde{\zeta}\ell \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{\mathbb{R}^d}\left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\,dx\,dt &\le C\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\,dx\,dt\\ &+ C\ell\left(\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{B(\bar{x},R)}|w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{B(\bar{x},R)}|w_t|^2\,dx\,dt\right)\\ &+ C\ell\left(\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w_t|^2\,dx\,dt\right).\end{aligned}$$

for some $0 < \tilde{\zeta} < \zeta$. The conclusion follows by dividing by $\tilde{\zeta}\ell$ and choosing ℓ sufficiently large. □

4. Entropy estimate. Let X be a metric space and let $K \subseteq X$ be precompact. We define the *Kolmogorov's ε-entropy* by

$$H_\varepsilon(K,X) = \ln N_\varepsilon(K,X),$$

where $N_\varepsilon(K,X)$ is the smallest number of ε-balls in X with centres in K that cover the set K.

LEMMA 4.1. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be bounded and let*

$$\mathbb{I}(\mathcal{O}) \le C_0 \operatorname{vol}(\mathcal{O}) \tag{4.1}$$

for some $C_0 > 0$. Let $\varepsilon > 0$ and $\theta \in (0,1)$. Let $(u_0,u_1) \in \mathcal{B}$ and let $(\chi_0,(\chi_0)_t)$ be the trajectory starting from (u_0,u_1). Let ϕ be a weight function such that the operator L has the squeezing property for ϕ and denote $B = B_\varepsilon((\chi_0,(\chi_0)_t);\Phi^\ell_{b,\phi}) \cap \mathcal{X}$. Then there exist C_1, $\ell > 0$ such that

$$H_{\theta\varepsilon}\left((LB)|_{\mathcal{O}}, W^\ell_{b,\phi}(\mathcal{O})\right) \le C_1 \operatorname{vol}(\mathcal{O}),$$

where the constant C_1 depends only on C_0 and θ and is independent of (u_0,u_1), ε, ϕ and $\mathcal{O}$ as long as (4.1) holds and the constants in (2.1) remain the same.

Proof. The proof combines the technique of [12, Lemma 4.1] and [7, Lemma 2.6] and adapts these to the squeezing property at hand. We will prove the assertion for

$\phi \equiv 1$. The general case then follows by the same argument as in [7, Lemma 2.6], namely by showing that $\|\chi\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\mathcal{O}))} \approx \|F\chi\|_{L^2_{b,1}(0,\ell;W^{1,2}(\mathcal{O}))}$ with $F : \chi \to \phi^{1/2}\chi$.

First fix $0 < \gamma < \theta^2$ and using Lemma 3.3 find κ, $\ell > 0$ such that L has the squeezing property for the weight function ϕ and γ. Let $\delta > 0$ be such that $\gamma + 4\kappa\delta^2 < \theta^2$. For $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$ fixed we denote

$$P_{x_1,x_2,x_3,x_4}\left((\chi, \partial_t \chi)\right) = \left(\chi|_{B(x_1,R)}, \partial_t\chi|_{B(x_2,R)}, L\chi|_{B(x_3,R)}, \partial_t L\chi|_{B(x_4,R)}\right),$$

where $R > 0$ comes from the squeezing property (3.2). Employing the standard Aubin-Lions lemma and the Lipschitz continuity of L we observe that the set

$$X(x_1, x_2, x_3, x_4) = \{P_{x_1,x_2,x_3,x_4}\left((\chi, \partial_t\chi)\right); (\chi, \partial_t\chi) \in B\}$$

equipped with the product topology $\prod_{i=1}^4 L^2(0, \ell; L^2(B(x_i, R)))$ can be covered by N balls of diameter $\delta\varepsilon$ with N independent of ε and x_i.

Let now $\chi_1, \chi_2 \in B$, let u_1, u_2 be their respective solutions and set $w = u_1 - u_2$. Then we find $x_i^M \in \mathbb{R}^d$ such that

$$\begin{aligned}
\sup_{\bar{x}\in\mathbb{R}^d} \int_0^\ell \int_{B(\bar{x},R)} |w|^2 \, dx \, dt + \sup_{\bar{x}\in\mathbb{R}^d} \int_0^\ell \int_{B(\bar{x},R)} |w_t|^2 \, dx \, dt \\
+ \sup_{\bar{x}\in\mathbb{R}^d} \int_\ell^{2\ell} \int_{B(\bar{x},R)} |w|^2 \, dx \, dt + \sup_{\bar{x}\in\mathbb{R}^d} \int_\ell^{2\ell} \int_{B(\bar{x},R)} |w_t|^2 \, dx \, dt \\
\le \int_0^\ell \int_{B(x_1^M,R)} |w|^2 \, dx \, dt + \int_0^\ell \int_{B(x_2^M,R)} |w_t|^2 \, dx \, dt \\
+ \int_\ell^{2\ell} \int_{B(x_3^M,R)} |w|^2 \, dx \, dt + \int_\ell^{2\ell} \int_{B(x_4^M,R)} |w_t|^2 \, dx \, dt + \frac{1}{M}
\end{aligned}$$

with $M \in \mathbb{N}$ large enough to have $\gamma\varepsilon^2 + 4\kappa\delta^2\varepsilon^2 + \kappa/M \le \theta^2\varepsilon^2$. By the previous observation we may cover the set $X(x_1^M, x_2^M, x_3^M, x_4^M)$ by $\delta\varepsilon$-balls centered at $P_{x_1^M,x_2^M,x_3^M,x_4^M}\left((\chi^i, \partial_t\chi^i)\right)$ for some $(\chi^i, \partial_t\chi^i) \in B$, $i = 1, \ldots, N$. For arbitrary $(\chi, \partial_t\chi) \in B$ we may now find $(\chi^i, \partial_t\chi^i) \in B$ such that

$$\|P_{x^M}\left((\chi, \partial_t\chi)\right) - P_{x^M}\left((\chi^i, \partial_t\chi^i)\right)\|_{X(x_1^M,x_2^M,x_3^M,x_4^M)} < \delta\varepsilon.$$

The squeezing property now leads to the estimate

$$\sup_{\bar{x}\in\mathbb{R}^d} \int_\ell^{2\ell} \int_{\mathbb{R}^d} \left(E[w] + |\nabla w_t|^2\right) \, dx \, dt \le \gamma\varepsilon^2 + 4\kappa\delta^2\varepsilon^2 + \frac{\kappa}{M} \le \theta^2\varepsilon^2,$$

which finishes the proof. ∎

We will use the following auxiliary function in the spirit of [17]: let $\bar{x} \in \mathbb{R}^d$, $R > 0$ and $\nu > 0$. Define

$$\psi(\bar{x}, R) = \psi(\bar{x}, R)(x) = \begin{cases} 1, & |x - \bar{x}| \le R + \sqrt{d}, \\ \exp\left(\nu\left(R + \sqrt{d} - |x - \bar{x}|\right)\right), & \text{otherwise.} \end{cases}$$

The function $\psi(\bar{x}, R)$ is clearly a weight function of growth ν with, in the notation of (2.1), $C_{\psi(\bar{x},R)} = 1$ for every $\bar{x} \in \mathbb{R}^d$ and $R > 0$. Also we have

$$H_\varepsilon\left(B|_{B(\bar{x},R)}, W_b(B(\bar{x}, R))\right) \le H_\varepsilon\left(B, W_{b,\psi(\bar{x},R)}\right), \tag{4.2}$$

where $W_b(B(\bar{x}, R))$ is a seminorm defined similarly as in (2.4) and $B \subseteq W_b^\ell$.

Lemma 4.2 ([7, Lemma 5.4]). *For every $\varepsilon_0 > 0$ we there exists $R' > 0$ such that for every $\bar{x} \in \mathbb{R}^d$, $R \geq 1$, $\varepsilon \in (0, \varepsilon_0)$ and $\chi_1, \chi_2 \in W^\ell_{b,\psi(\bar{x},R)}$ one has*

$$\|\chi_1 - \chi_2\|_{W^\ell_{b,\psi(\bar{x},R)}} \leq \max\left\{\varepsilon, \|\chi_1 - \chi_2\|_{W^\ell_{b,\psi(\bar{x},R)}(B(\bar{x},R+R'\ln(\varepsilon_0/\varepsilon)))}\right\}.$$

Recall that $\mathcal{A} \subseteq W_b^{2,2}(\mathbb{R}^d) \times W_b^{1,2}(\mathbb{R}^d)$ is the locally compact attractor of the set (1.1) defined in Section 1.

Theorem 4.3. *There exist constants C_0, C_1, $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $\bar{x} \in \mathbb{R}^d$ and $R \geq 1$ one has the estimate*

$$H_\varepsilon\left(\mathcal{A}|_{B(\bar{x},R)}, W_b(B(\bar{x}, R))\right) \leq C_0 \left(R + C_1 \ln \frac{\varepsilon_0}{\varepsilon}\right)^d \ln \frac{\varepsilon_0}{\varepsilon}.$$

Proof. The proof is standard and runs in almost the same way as in [10, Theorem 6.5] and [7, Theorem 5.1] with only minor differences.

Let $\bar{x} \in \mathbb{R}^d$, $R \geq 1$ and let $\psi(\bar{x}, R)$ be of sufficiently small growth such that L has the squeezing property for $\psi(\bar{x}, R)$ and let $\ell > 0$ be such that Lemma 4.1 holds with $\theta = 1/2 \operatorname{Lip}(L) < 1$, where $\operatorname{Lip}(L)$ denotes the Lipschitz constant of L from Lemma 3.1. The smallness of growth of $\psi(\bar{x}, R)$ can always be achieved by choosing ν small in the definition of $\psi(\bar{x}, R)$. By the Lipschitz continuity of e and the property of the weight function $\psi(\bar{x}, R)$ (4.2) we get

$$H_\varepsilon\left(\mathcal{A}|_{B(\bar{x},R)}, W_b(B(\bar{x}, R))\right) \leq H_\varepsilon\left(\mathcal{A}, W_{b,\psi(\bar{x},R)}\right) \leq H_{\varepsilon/\operatorname{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{b,\psi(\bar{x},R)}\right),$$

where $\mathcal{A}_\ell = \{(\chi, \chi_t) \in \Phi_b^\ell; (\chi(0), \chi_t(0) \in \mathcal{A}\}$. By the dissipation estimates (1.3) and the invariance of $\mathcal{A}$ we observe that actually $\mathcal{A}_\ell \subseteq W_b^\ell$ and $\mathcal{A}_\ell$ is invariant w.r.t. $L(t)$. Also the dissipation estimates (1.3) imply that for some $\chi \in \mathcal{A}_\ell$ and $\varepsilon_0 > 0$ sufficiently large we have

$$H_{\varepsilon_0/\operatorname{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{b,\psi(\bar{x},R)}\right) = 0.$$

The key part of the proof is to show that for $k \in \mathbb{N} \cup \{0\}$ one has

$$H_{\varepsilon_0 2^{-k}/\operatorname{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{b,\psi(\bar{x},R)}\right) \leq C\left(R + C' \ln 2^k\right)^d k \tag{4.3}$$

for some $C' > 0$. Indeed, once we have established (4.3) for given $\varepsilon \in (0, \varepsilon_0)$ we may find $k \in \mathbb{N}$ such that $2^{-k}\varepsilon_0 \leq \varepsilon < 2^{-k+1}\varepsilon_0$ and the desired entropy bound follows.

The estimate (4.3) clearly holds for $k = 0$. Assume that (4.3) holds for $k \geq 0$, i.e.

$$\mathcal{A}_\ell \subseteq \bigcup_{i=1}^{N_k} B_{\varepsilon_0 2^{-k}/\operatorname{Lip}(e)}\left((\chi^i, \chi^i_t); W^\ell_{b,\psi(\bar{x},R)}\right) \tag{4.4}$$

for some $N_k \in \mathbb{N}$ such that $\ln N_k \leq C(R + C' \ln 2^k)^d k$ and $(\chi^i, \chi^i_t) \in \mathcal{A}_\ell$ for $1 \leq i \leq N_k$. Applying L to (4.4) and recalling the invariance of $\mathcal{A}_\ell$ under L and the Lipschitz continuity of L, we get

$$\mathcal{A}_\ell = L(\mathcal{A}_\ell) \subseteq \bigcup_{i=1}^{N} B_{\operatorname{Lip}(L)\varepsilon_0 2^{-k}/\operatorname{Lip}(e)}\left((L\chi^i, \partial_t L\chi^i); W^\ell_{b,\psi(\bar{x},R)}\right) \tag{4.5}$$

By Lemma 4.1 with $\theta = 1/2\,\mathrm{Lip}(L)$ each of the balls on the right-hand side of (4.5) localized to the spatial domain $B(\bar{x}, R + R' \ln 2^{k+1})$ can be covered by $\varepsilon_0 2^{-(k+1)}$-balls in the space $W^{\ell}_{b,\psi(\bar{x},R)}$ in such a way that

$$\begin{aligned} H_{\varepsilon_0 2^{-(k+1)}/\,\mathrm{Lip}(e)} &\Big(\mathcal{A}_\ell|_{B(\bar{x},R+R'\ln 2^{k+1})}, W^{\ell}_{b,\psi(\bar{x},R)}(B(\bar{x}, R + R'\ln 2^{k+1}))\Big) \\ &\le H_{\varepsilon_0 2^{-k}/\,\mathrm{Lip}(e)} \Big(\mathcal{A}_\ell, W^{\ell}_{\bar{x},\psi(\bar{x},\varepsilon)}\Big) + C\left(R + R'\ln 2^{k+1}\right)^d \\ &\le C\left(R + R'\ln 2^{k+1}\right)^d (k+1). \end{aligned}$$

The proof is finished since by Lemma 4.2 every $\varepsilon_0 2^{-(k+1)}/\,\mathrm{Lip}(e)$-covering in the space $W^{\ell}_{b,\psi(\bar{x},R)}(B(\bar{x}, R(\varepsilon_0 2^{-(k+1)})))$ is also an $\varepsilon_0 2^{-(k+1)}/\,\mathrm{Lip}(e)$-covering in $W^{\ell}_{b,\psi(\bar{x},R)}$. $\square$

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