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POSITIVE SOLUTIONS FOR A SYSTEM OF FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. We investigate the existence and multiplicity of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with nonnegative nonlinearities which can be nonsingular or singular functions, subject to multi-point boundary conditions that contain fractional derivatives.

Key words. Riemann-Liouville fractional differential equations, multi-point boundary conditions, positive solutions, existence

AMS subject classifications. 34A08, 34B15, 45G15

1. Introduction. We consider the system of nonlinear ordinary fractional differential equations

$$(S)\qquad \begin{cases} D_{0+}^{\alpha}u(t)+f(t,v(t))=0, & t\in(0,1),\\ D_{0+}^{\beta}v(t)+g(t,u(t))=0, & t\in(0,1), \end{cases}$$

with the multi-point boundary conditions

$$(BC)\qquad \begin{cases} u^{(j)}(0)=0,\ j=0,\dots,n-2;\ \ D_{0+}^{p_1}u(t)|_{t=1}=\displaystyle\sum_{i=1}^{N}a_iD_{0+}^{q_1}u(t)|_{t=\xi_i},\\ v^{(j)}(0)=0,\ j=0,\dots,m-2;\ \ D_{0+}^{p_2}v(t)|_{t=1}=\displaystyle\sum_{i=1}^{M}b_iD_{0+}^{q_2}v(t)|_{t=\eta_i}, \end{cases}$$

where $\alpha,\ \beta\in\mathbf{R}$, $\alpha\in(n-1,n]$, $\beta\in(m-1,m]$, $n,\ m\in\mathbf{N}$, $n,\ m\geq 3$, $p_1,\ p_2,\ q_1,\ q_2\in\mathbf{R}$, $p_1\in[1,n-2]$, $p_2\in[1,m-2]$, $q_1\in[0,p_1]$, $q_2\in[0,p_2]$, $\xi_i,\ a_i\in\mathbf{R}$ for all $i=1,\dots,N$ ($N\in\mathbf{N}$), $0<\xi_1<\cdots<\xi_N\leq 1$, $\eta_i,\ b_i\in\mathbf{R}$ for all $i=1,\dots,M$ ($M\in\mathbf{N}$), $0<\eta_1<\cdots<\eta_M\leq 1$, and D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k=\alpha,\ \beta,\ p_1,\ p_2,\ q_1,\ q_2$).

Under sufficient conditions on functions f and g, which can be nonsingular or singular in the points $t=0$ and/or $t=1$, we study the existence and multiplicity of positive solutions of problem $(S)-(BC)$. We use some theorems from the fixed point index theory (from [1] and [27]) and the Guo-Krasnosel'skii fixed point theorem (see [9]). By a positive solution of problem $(S)-(BC)$ we mean a pair of functions $(u,v)\in C([0,1];\mathbf{R}_+)\times C([0,1];\mathbf{R}_+)$ ($\mathbf{R}_+=[0,\infty)$) satisfying (S) and (BC) with $u(t)>0$ and $v(t)>0$ for all $t\in(0,1]$. The system (S) with the boundary conditions

$$(\widetilde{BC})\qquad \begin{cases} u^{(j)}(0)=0,\ j=0,\dots,n-2;\ \ u(1)=\int_0^1 u(s)\,dH(s),\\ v^{(j)}(0)=0,\ j=0,\dots,m-2;\ \ v(1)=\int_0^1 v(s)\,dK(s), \end{cases}$$

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where the integrals from $(\widetilde{BC})$ are Riemann-Stieltjes integrals, has been investigated in [10]. The existence, multiplicity and nonexistence of positive solutions for the system (S) and the corresponding one with some positive parameters, namely the system

$$(S') \qquad \begin{cases} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t), v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{\beta} v(t) + \mu g(t, u(t), v(t)) = 0, \ \ t \in (0,1), \end{cases}$$

subject to coupled boundary conditions

$$(BC') \qquad \begin{cases} u^{(j)}(0) = 0, \ j = 0, \dots, n-2; \ \ u(1) = \int_0^1 v(s)\, dH(s), \\ v^{(j)}(0) = 0, \ j = 0, \dots, m-2; \ \ v(1) = \int_0^1 u(s)\, dK(s), \end{cases}$$

were studied in [11], [12], [13], [14], [16], [19], where the nonlinearities f and g are nonnegative or sign-changing functions. Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics and rheology (see [2], [3], [4], [5], [6], [7], [8], [17], [18], [20], [21], [22], [23], [24], [25], [26]). Fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations, and give the properties of the Green functions associated to our problem. Section 3 contains the existence and multiplicity results for the positive solutions of problem $(S) - (BC)$ in the nonsingular case, and Section 4 presents the existence results in the singular case. Finally, in Section 5 we give two examples which support our main results.

2. Auxiliary results. We present here the definitions of Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative, and some auxiliary results from [15] that will be used to prove our main results.

Definition 2.1 *The (left-sided) fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbf{R}$ is given by*

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ \ t > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^{\infty} t^{\alpha-1} e^{-t}\, dt$, $\alpha > 0$.

Definition 2.2 *The Riemann-Liouville fractional derivative of order $\alpha \ge 0$ for a function $f : (0, \infty) \to \mathbf{R}$ is given by*

$$(D_{0+}^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n \left(I_{0+}^{n-\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds, \ \ t > 0,$$

where $n = \lfloor \alpha \rfloor + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

The notation $\lfloor \alpha \rfloor$ stands for the largest integer not greater than α. If $\alpha = m \in \mathbf{N}$ then $D_{0+}^m f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$ then $D_{0+}^0 f(t) = f(t)$ for $t > 0$.

We consider now the fractional differential equation

$$D_{0+}^{\alpha}u(t)+x(t)=0,\ \ 0<t<1, \tag{2.1}$$

with the multi-point boundary conditions

$$u^{(j)}(0)=0,\ \ j=0,\ldots,n-2;\ \ D_{0+}^{p_1}u(t)\big|_{t=1}=\sum_{i=1}^{N}a_iD_{0+}^{q_1}u(t)\big|_{t=\xi_i}, \tag{2.2}$$

where $\alpha\in(n-1,n]$, $n\in\mathbf{N}$, $n\geq 3$, a_i, $\xi_i\in\mathbf{R}$, $i=1,\ldots,N$ $(N\in\mathbf{N})$, $0<\xi_1<\cdots<\xi_N\leq 1$, p_1, $q_1\in\mathbf{R}$, $p_1\in[1,n-2]$, $q_1\in[0,p_1]$, and $x\in C(0,1)\cap L^1(0,1)$. We denote by $\Delta_1=\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)}-\frac{\Gamma(\alpha)}{\Gamma(\alpha-q_1)}\sum_{i=1}^{N}a_i\xi_i^{\alpha-q_1-1}$.

Lemma 2.1. *([15]) If $\Delta_1\neq 0$, then the function $u\in C[0,1]$ given by*

$$u(t)=\int_0^1 G_1(t,s)x(s)\,ds,\ \ t\in[0,1], \tag{2.3}$$

is solution of problem (2.1)-(2.2), where

$$G_1(t,s)=g_1(t,s)+\frac{t^{\alpha-1}}{\Delta_1}\sum_{i=1}^{N}a_ig_2(\xi_i,s),\ \ \forall\,(t,s)\in[0,1]\times[0,1], \tag{2.4}$$

and

$$\begin{aligned} g_1(t,s)&=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-p_1-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1,\\ t^{\alpha-1}(1-s)^{\alpha-p_1-1}, & 0\leq t\leq s\leq 1,\end{cases}\\ g_2(t,s)&=\frac{1}{\Gamma(\alpha-q_1)}\begin{cases} t^{\alpha-q_1-1}(1-s)^{\alpha-p_1-1}-(t-s)^{\alpha-q_1-1}, \\ \qquad 0\leq s\leq t\leq 1,\\ t^{\alpha-q_1-1}(1-s)^{\alpha-p_1-1}, \ \ 0\leq t\leq s\leq 1.\end{cases}\end{aligned} \tag{2.5}$$

Lemma 2.2. *([15]) The functions g_1 and g_2 given by (2.5) have the properties:*

a) $g_1(t,s)\leq h_1(s)$ for all t, $s\in[0,1]$, where
$h_1(s)=\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-p_1-1}(1-(1-s)^{p_1})$, *$s\in[0,1]$;*

b) $g_1(t,s)\geq t^{\alpha-1}h_1(s)$ for all t, $s\in[0,1]$;

c) $g_1(t,s)\leq\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, for all t, $s\in[0,1]$;

d) $g_2(t,s)\geq t^{\alpha-q_1-1}h_2(s)$ for all t, $s\in[0,1]$, where
$h_2(s)=\frac{1}{\Gamma(\alpha-q_1)}(1-s)^{\alpha-p_1-1}(1-(1-s)^{p_1-q_1})$, *$s\in[0,1]$;*

e) $g_2(t,s)\leq\frac{1}{\Gamma(\alpha-q_1)}t^{\alpha-q_1-1}$ for all t, $s\in[0,1]$;

f) The functions g_1 and g_2 are continuous on $[0,1]\times[0,1]$; $g_1(t,s)\geq 0$, $g_2(t,s)\geq 0$ for all t, $s\in[0,1]$; $g_1(t,s)>0$, $g_2(t,s)>0$ for all t, $s\in(0,1)$.

Lemma 2.3. *([15]) Assume that $a_i\geq 0$ for all $i=1,\ldots,N$ and $\Delta_1>0$. Then the function G_1 given by (2.4) is a nonnegative continuous function on $[0,1]\times[0,1]$ and satisfies the inequalities:*

a) $G_1(t,s)\leq J_1(s)$ for all t, $s\in[0,1]$, where $J_1(s)=h_1(s)+\frac{1}{\Delta_1}\sum_{i=1}^{N}a_ig_2(\xi_i,s)$, $s\in[0,1]$;

b) $G_1(t,s)\geq t^{\alpha-1}J_1(s)$ for all t, $s\in[0,1]$;

c) $G_1(t,s)\leq\sigma_1t^{\alpha-1}$, for all t, $s\in[0,1]$, where $\sigma_1=\frac{1}{\Gamma(\alpha)}+\frac{1}{\Delta_1\Gamma(\alpha-q_1)}\times\sum_{i=1}^{N}a_i\xi_i^{\alpha-q_1-1}$.

LEMMA 2.4. *([15]) Assume that $a_i \geq 0$ for all $i = 1, \ldots, N$, $\Delta_1 > 0$, $x \in C(0,1) \cap L^1(0,1)$ and $x(t) \geq 0$ for all $t \in (0,1)$. Then the solution u of problem (2.1)-(2.2) given by (2.3) satisfies the inequality $u(t) \geq t^{\alpha-1} u(t')$ for all t, $t' \in [0,1]$.*

We can also formulate similar results as Lemmas 2.1-2.4 for the fractional boundary value problem

$$D_{0+}^{\beta} v(t) + y(t) = 0, \;\; 0 < t < 1, \tag{2.6}$$

$$v^{(j)}(0) = 0, \;\; j = 0, \ldots, m-2; \;\; D_{0+}^{p_2} v(t)\big|_{t=1} = \sum_{i=1}^{M} b_i D_{0+}^{q_2} v(t)\big|_{t=\eta_i}, \tag{2.7}$$

where $\beta \in (m-1, m]$, $m \in \mathbf{N}$, $m \geq 3$, b_i, $\eta_i \in \mathbf{R}$, $i = 1, \ldots, M$ ($M \in \mathbf{N}$), $0 < \eta_1 < \cdots < \eta_M \leq 1$, p_2, $q_2 \in \mathbf{R}$, $p_2 \in [1, m-2]$, $q_2 \in [0, p_2]$, and $y \in C(0,1) \cap L^1(0,1)$.

We denote by Δ_2, g_3, g_4, G_2, h_3, h_4, J_2 and σ_2 the corresponding constants and functions for problem (2.6)-(2.7) defined in a similar manner as Δ_1, g_1, g_2, G_1, h_1, h_2, J_1 and σ_1, respectively. More precisely, we have

$$\begin{aligned}
&\Delta_2 = \tfrac{\Gamma(\beta)}{\Gamma(\beta-p_2)} - \tfrac{\Gamma(\beta)}{\Gamma(\beta-q_2)} \textstyle\sum_{i=1}^{M} b_i \eta_i^{\beta-q_2-1},\\
&g_3(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-p_2-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1,\\ t^{\beta-1}(1-s)^{\beta-p_2-1}, & 0 \leq t \leq s \leq 1, \end{cases}\\
&g_4(t,s) = \frac{1}{\Gamma(\beta-q_2)} \begin{cases} t^{\beta-q_2-1}(1-s)^{\beta-p_2-1} - (t-s)^{\beta-q_2-1}, & 0 \leq s \leq t \leq 1,\\ t^{\beta-q_2-1}(1-s)^{\beta-p_2-1}, & 0 \leq t \leq s \leq 1, \end{cases}\\
&G_2(t,s) = g_3(t,s) + \tfrac{t^{\beta-1}}{\Delta_2} \textstyle\sum_{i=1}^{M} b_i g_4(\eta_i, s), \;\; \forall\, (t,s) \in [0,1] \times [0,1],\\
&h_3(s) = \tfrac{1}{\Gamma(\beta)} (1-s)^{\beta-p_2-1}(1-(1-s)^{p_2}), \;\; s \in [0,1],\\
&h_4(s) = \tfrac{1}{\Gamma(\beta-q_2)} (1-s)^{\beta-p_2-1}(1-(1-s)^{p_2-q_2}), \;\; s \in [0,1],\\
&J_2(s) = h_3(s) + \tfrac{1}{\Delta_2} \sum_{i=1}^{M} b_i g_4(\eta_i, s), \;\; s \in [0,1],\\
&\sigma_2 = \tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Delta_2 \Gamma(\beta-q_2)} \textstyle\sum_{i=1}^{M} b_i \eta_i^{\beta-q_2-1}.
\end{aligned}$$

The inequalities from Lemmas 2.3 and 2.4 for the functions G_2 and v are the following $G_2(t,s) \leq J_2(s)$, $G_2(t,s) \geq t^{\beta-1} J_2(s)$, $G_2(t,s) \leq \sigma_2 t^{\beta-1}$, for all t, $s \in [0,1]$, and $v(t) \geq t^{\beta-1} v(t')$ for all t, $t' \in [0,1]$.

The proofs of our results in the nonsingular case are based on the following fixed point index theorems. Let E be a real Banach space, $P \subset E$ a cone, "$\leq$" the partial ordering defined by P and θ the zero element in E. For $\varrho > 0$, let $B_\varrho = \{u \in E, \; \|u\| < \varrho\}$ be the open ball of radius ϱ centered at 0, and its boundary $\partial B_\varrho = \{u \in E, \; \|u\| = \varrho\}$.

THEOREM 2.5. *([1]) Let $A : \bar{B}_\varrho \cap P \to P$ be a completely continuous operator which has no fixed point on $\partial B_\varrho \cap P$. If $\|Au\| \leq \|u\|$ for all $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 1$.*

THEOREM 2.6. *([1]) Let $A : \bar{B}_\varrho \cap P \to P$ be a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that $u - Au \neq \lambda u_0$, for all $\lambda \geq 0$ and $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 0$.*

THEOREM 2.7. *([27]) Let $A : \bar{B}_\varrho \cap P \to P$ be a completely continuous operator which has no fixed point on $\partial B_\varrho \cap P$. If there exists a linear operator $L : P \to P$ and $u_0 \in P \setminus \{\theta\}$ such that*

$$i)\; u_0 \leq L u_0, \;\; ii)\; Lu \leq Au, \;\; \forall\, u \in \partial B_\varrho \cap P,$$

then $i(A, B_\varrho \cap P, P) = 0$.

We also present the Guo-Krasnosel'skii fixed point theorem (see [9]) that we will use in the proofs of our main results in the singular case.

THEOREM 2.8. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) $\|\mathcal{A}u\| \le \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$, *or*

ii) $\|\mathcal{A}u\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. The nonsingular case. In this section, we investigate the existence and multiplicity of positive solutions for problem $(S) - (BC)$ under various assumptions on nonsingular functions f and g.

We present the assumptions that we shall use in the sequel.

$(H1)$ α, $\beta \in \mathbf{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, n, $m \in \mathbf{N}$, n, $m \ge 3$, p_1, p_2, q_1, $q_2 \in \mathbf{R}$, $p_1 \in [1, n-2]$, $p_2 \in [1, m-2]$, $q_1 \in [0, p_1]$, $q_2 \in [0, p_2]$, $\xi_i \in \mathbf{R}$, $a_i \ge 0$ for all $i = 1, \dots, N$ ($N \in \mathbf{N}$), $0 < \xi_1 < \cdots < \xi_N \le 1$, $\eta_i \in \mathbf{R}$, $b_i \ge 0$ for all $i = 1, \dots, M$ ($M \in \mathbf{N}$), $0 < \eta_1 < \cdots < \eta_M \le 1$, $\Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_1)} \sum_{i=1}^N a_i \xi_i^{\alpha-q_1-1} > 0$, $\Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta-p_2)} - \frac{\Gamma(\beta)}{\Gamma(\beta-q_2)} \sum_{i=1}^M b_i \eta_i^{\beta-q_2-1} > 0$.

$(H2)$ The functions f, $g : [0,1] \times \mathbf{R}_+ \to \mathbf{R}_+$ are continuous and $f(t, 0) = g(t, 0) = 0$ for all $t \in [0,1]$.

If the pair of functions $(u, v) \in C[0,1] \times C[0,1]$ is a solution of the nonlinear integral system

$$
(3.1) \qquad \begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds, \ \ t \in [0,1], \\ v(t) = \displaystyle\int_0^1 G_2(t,s) g(s, u(s))\, ds, \ \ t \in [0,1], \end{cases}
$$

then it is a solution of problem $(S) - (BC)$.

We consider the Banach space $X = C[0,1]$ with supremum norm $\|\cdot\|$ and define the cone $P \subset X$ by $P = \{u \in X, \ u(t) \ge 0, \ \forall\, t \in [0,1]\}$.

We also define the operators $\mathcal{A} : P \to X$ by

$$
(\mathcal{A}u)(t) = \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds, \ t \in [0,1], \ u \in P,
$$

and $\mathcal{B} : P \to X$, $\mathcal{C} : P \to X$ by

$$
(\mathcal{B}u)(t) = \int_0^1 G_1(t,s) u(s)\, ds, \ \ (\mathcal{C}u)(t) = \int_0^1 G_2(t,s) u(s)\, ds, \ \ t \in [0,1], \ u \in P.
$$

Under the assumptions $(H1)$ and $(H2)$ it is easy to see that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are completely continuous from P to P. Thus we will investigate the existence and multiplicity of fixed points u of operator $\mathcal{A}$, which together with v given in (3.1) will be solutions of problem $(S) - (BC)$.

Using Theorems 2.5-2.6 and some similar arguments as those used in the proofs of Theorems 3.1-3.3 from [10], we obtain for our problem $(S) - (BC)$ the following results.

THEOREM 3.1. *Assume that $(H1) - (H2)$ hold. If the functions f and g also satisfy the conditions*

(*H*3) *There exist positive constants* $p \in (0,1]$ *and* $c \in (0,1)$ *such that*

$$i)\ f_\infty^i = \liminf_{u\to\infty} \inf_{t\in[c,1]} \frac{f(t,u)}{u^p} \in (0,\infty];\quad ii)\ g_\infty^i = \lim_{u\to\infty} \inf_{t\in[c,1]} \frac{g(t,u)}{u^{1/p}} = \infty,$$

(*H*4) *There exists positive constants* $\beta_1,\ \beta_2 > 0$ *with* $\beta_1\beta_2 \ge 1$ *such that*

$$i)\ f_0^s = \limsup_{u\to0^+} \sup_{t\in[0,1]} \frac{f(t,u)}{u^{\beta_1}} \in [0,\infty);\quad ii)\ g_0^s = \lim_{u\to0^+} \sup_{t\in[0,1]} \frac{g(t,u)}{u^{\beta_2}} = 0,$$

then the problem $(S)-(BC)$ *has at least one positive solution* $(u(t),v(t)),\ t\in[0,1]$.

THEOREM 3.2. *Assume that* $(H1)-(H2)$ *hold. If the functions* f *and* g *also satisfy the conditions*

(*H*5) *There exist positive constants* $\alpha_1,\ \alpha_2 > 0$ *with* $\alpha_1\alpha_2 \le 1$ *such that*

$$i)\ f_\infty^s = \limsup_{u\to\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u^{\alpha_1}} \in [0,\infty);\quad ii)\ g_\infty^s = \lim_{u\to\infty} \sup_{t\in[0,1]} \frac{g(t,u)}{u^{\alpha_2}} = 0,$$

(*H*6) *There exists* $c \in (0,1)$ *such that*

$$i)\ f_0^i = \liminf_{u\to0^+} \inf_{t\in[c,1]} \frac{f(t,u)}{u} \in (0,\infty];\quad ii)\ g_0^i = \lim_{u\to0^+} \inf_{t\in[c,1]} \frac{g(t,u)}{u} = \infty,$$

then the problem $(S)-(BC)$ *has at least one positive solution* $(u(t),v(t)),\ t\in[0,1]$.

THEOREM 3.3. *Assume that* $(H1)-(H3)$ *and* $(H6)$ *hold. If the functions* f *and* g *also satisfy the condition*

(*H*7) *For each* $t\in[0,1]$, $f(t,u)$ *and* $g(t,u)$ *are nondecreasing with respect to* u, *and there exists a constant* $N_0 > 0$ *such that*

$$f\left(t, m_0\int_0^1 g(s,N_0)\,ds\right) < \frac{N_0}{m_0},\ \ \forall\, t\in[0,1],$$

where $m_0 = \max\{K_1,K_2\}$, $K_1 = \max_{s\in[0,1]} J_1(s)$, $K_2 = \max_{s\in[0,1]} J_2(s)$ *and* $J_1,\ J_2$ *are defined in Section 2,*

then the problem $(S)-(BC)$ *has at least two positive solutions* $(u_1(t),v_1(t))$, $(u_2(t),v_2(t)),\ t\in[0,1]$.

4. The singular case. In this section we study the existence of positive solutions for our problem $(S)-(BC)$ under various assumptions on functions f and g which may be singular at $t=0$ and/or $t=1$.

The basic assumptions used here are the following.

(*A*1) $\equiv (H1)$,

(*A*2) The functions $f,\ g \in C((0,1)\times\mathbf{R}_+,\mathbf{R}_+)$ and there exist $\widetilde{p}_i \in C((0,1),\mathbf{R}_+)$, $\widetilde{q}_i \in C(\mathbf{R}_+,\mathbf{R}_+)$, $i=1,2$, with $0 < \int_0^1 \widetilde{p}_i(t)\,dt < \infty$, $i=1,2$, $\widetilde{q}_1(0)=0$, $\widetilde{q}_2(0)=0$ such that

$$f(t,x) \le \widetilde{p}_1(t)\widetilde{q}_1(x),\ \ g(t,x) \le \widetilde{p}_2(t)\widetilde{q}_2(x),\ \ \forall\, t\in(0,1),\ \ x\in\mathbf{R}_+.$$

We consider the Banach space $X = C([0,1])$ with supremum norm and define the cone $P \subset X$ by $P = \{u\in X,\ u(t)\ge 0,\ \forall\, t\in[0,1]\}$. We also define the operator $\widetilde{\mathcal{A}} : P \to X$ by

$$(\widetilde{\mathcal{A}}u)(t) = \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau,u(\tau))\,d\tau\right) ds,\ \ t\in[0,1],\ \ u\in P.$$

Using Theorem 2.8 and similar arguments as those used in the proofs of Lemma 4.1 and Theorems 4.1-4.2 from [10], we obtain for our problem $(S)-(BC)$ the following results.

LEMMA 4.1. *Assume that* $(A1)-(A2)$ *hold. Then* $\widetilde{\mathcal{A}} : P \to P$ *is completely continuous.*

THEOREM 4.2. *Assume that* $(A1)-(A2)$ *hold. If the functions* f *and* g *also satisfy the conditions*

$(A3)$ *There exist* $\alpha_1,\, \alpha_2 \in (0,\infty)$ *with* $\alpha_1\alpha_2 \le 1$ *such that*

$$i)\ q_{1\infty}^s = \limsup_{x\to\infty} \frac{\widetilde{q}_1(x)}{x^{\alpha_1}} \in [0,\infty);\quad ii)\ q_{2\infty}^s = \lim_{x\to\infty} \frac{\widetilde{q}_2(x)}{x^{\alpha_2}} = 0,$$

$(A4)$ *There exist* $\beta_1,\, \beta_2 \in (0,\infty)$ *with* $\beta_1\beta_2 \le 1$ *and* $c \in (0,1/2)$ *such that*

$$i)\ \widetilde{f}_0^i = \liminf_{x\to 0^+} \inf_{t\in[c,1-c]} \frac{f(t,x)}{x^{\beta_1}} \in (0,\infty];\quad ii)\ \widetilde{g}_0^i = \lim_{x\to 0^+} \inf_{t\in[c,1-c]} \frac{g(t,x)}{x^{\beta_2}} = \infty,$$

then the problem $(S)-(BC)$ *has at least one positive solution* $(u(t), v(t)),\ t \in [0,1]$.

THEOREM 4.3. *Assume that* $(A1)-(A2)$ *hold. If the functions* f *and* g *also satisfy the conditions*

$(A5)$ *There exist* $r_1,\, r_2 \in (0,\infty)$ *with* $r_1 r_2 \ge 1$ *such that*

$$i)\ q_{10}^s = \limsup_{x\to 0^+} \frac{\widetilde{q}_1(x)}{x^{r_1}} \in [0,\infty);\quad ii)\ q_{20}^s = \lim_{x\to 0^+} \frac{\widetilde{q}_2(x)}{x^{r_2}} = 0,$$

$(A6)$ *There exist* $l_1,\, l_2 \in (0,\infty)$ *with* $l_1 l_2 \ge 1$ *and* $c \in (0,1/2)$ *such that*

$$i)\ \widetilde{f}_\infty^i = \liminf_{x\to \infty} \inf_{t\in[c,1-c]} \frac{f(t,x)}{x^{l_1}} \in (0,\infty];\quad ii)\ \widetilde{g}_\infty^i = \lim_{x\to \infty} \inf_{t\in[c,1-c]} \frac{g(t,x)}{x^{l_2}} = \infty,$$

then the problem $(S)-(BC)$ *has at least one positive solution* $(u(t), v(t)),\ t \in [0,1]$.

As against to the positive solutions obtained in the paper [10], in this paper, by using Lemma 2.4, we deduce that the fixed points u of operators $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ together with v given in (3.1) satisfy the conditions $u(t) > 0$ and $v(t) > 0$ for all $t \in (0,1]$, that is the pairs (u,v) are positive solutions of problem $(S)-(BC)$ in the sense of definition from Section 1.

5. Examples. Let $n=3$, $m=5$, $\alpha=\frac{5}{2}$, $\beta=\frac{17}{4}$, $p_1=1$, $q_1=\frac{1}{2}$, $p_2=\frac{7}{3}$, $q_2=\frac{3}{2}$, $N=2$, $M=1$, $\xi_1=\frac{1}{3}$, $\xi_2=\frac{2}{3}$, $a_1=2$, $a_2=\frac{1}{2}$, $\eta_1=\frac{1}{2}$, $b_1=4$.

We consider the system of fractional differential equations

$$(S_0)\qquad \begin{cases} D_{0+}^{5/2}u(t) + f(t,v(t)) = 0,\ \ t\in(0,1),\\ D_{0+}^{17/4}v(t) + g(t,u(t)) = 0,\ \ t\in(0,1),\end{cases}$$

with the multi-point boundary conditions

$$(BC_0)\qquad \begin{cases} u(0)=u'(0)=0,\ u'(1) = 2D_{0+}^{1/2}u(t)|_{t=\frac{1}{3}} + \frac{1}{2}D_{0+}^{1/2}u(t)|_{t=\frac{2}{3}},\\ v(0)=v'(0)=v''(0)=v'''(0)=0,\ D_{0+}^{7/3}v(t)|_{t=1} = 4D_{0+}^{3/2}v(t)|_{t=\frac{1}{2}}.\end{cases}$$

We have $\Delta_1 = \frac{6-3\sqrt{\pi}}{4} \approx 0.17065961 > 0$, $\Delta_2 = \frac{\Gamma(17/4)}{\Gamma(23/12)} - \frac{2^{1/4}\Gamma(17/4)}{\Gamma(11/4)} \approx 2.43672831 > 0$. So assumptions $(H1)$ and $(A1)$ are satisfied.

Besides we deduce

$$
\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(5/2)} \begin{cases} t^{3/2}(1-s)^{1/2} - (t-s)^{3/2}, & 0 \le s \le t \le 1, \\ t^{3/2}(1-s)^{1/2}, & 0 \le t \le s \le 1, \end{cases} \\
g_2(t,s) &= \begin{cases} t(1-s)^{1/2} - (t-s), & 0 \le s \le t \le 1, \\ t(1-s)^{1/2}, & 0 \le t \le s \le 1, \end{cases} \\
g_3(t,s) &= \frac{1}{\Gamma(17/4)} \begin{cases} t^{13/4}(1-s)^{11/12} - (t-s)^{13/4}, & 0 \le s \le t \le 1, \\ t^{13/4}(1-s)^{11/12}, & 0 \le t \le s \le 1, \end{cases} \\
g_4(t,s) &= \frac{1}{\Gamma(11/4)} \begin{cases} t^{7/4}(1-s)^{11/12} - (t-s)^{7/4}, & 0 \le s \le t \le 1, \\ t^{7/4}(1-s)^{11/12}, & 0 \le t \le s \le 1. \end{cases}
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
G_1(t,s) &= g_1(t,s) + \frac{t^{3/2}}{\Delta_1}\left(2g_2\left(\frac{1}{3},s\right) + \frac{1}{2}g_2\left(\frac{2}{3},s\right)\right), \\
G_2(t,s) &= g_3(t,s) + \frac{4t^{13/4}}{\Delta_2} g_4\left(\frac{1}{2},s\right), \\
h_1(s) &= \frac{4}{3\sqrt{\pi}} s(1-s)^{1/2}, \ h_3(s) = \frac{1}{\Gamma(17/4)}(1-s)^{11/12}(1-(1-s)^{7/3}), \\
J_1(s) &= \frac{4}{3\sqrt{\pi}} s(1-s)^{1/2} + \frac{1}{\Delta_1}\left(2g_2\left(\frac{1}{3},s\right) + \frac{1}{2}g_2\left(\frac{2}{3},s\right)\right) \\
&= \begin{cases} \frac{4}{3\sqrt{\pi}} s(1-s)^{1/2} + \frac{1}{2\Delta_1}\left[2(1-s)^{1/2} + 5s - 2\right], & 0 \le s < \frac{1}{3}, \\ \frac{4}{3\sqrt{\pi}} s(1-s)^{1/2} + \frac{1}{6\Delta_1}\left[6(1-s)^{1/2} + 3s - 2\right], & \frac{1}{3} \le s < \frac{2}{3}, \\ \frac{4}{3\sqrt{\pi}} s(1-s)^{1/2} + \frac{1}{\Delta_1}(1-s)^{1/2}, & \frac{2}{3} \le s \le 1. \end{cases} \\
J_2(s) &= \frac{1}{\Gamma(17/4)}(1-s)^{11/12}(1-(1-s)^{7/3}) + \frac{4}{\Delta_2} g_4\left(\frac{1}{2},s\right) \\
&= \begin{cases} \frac{1}{\Gamma(17/4)}(1-s)^{11/12}(1-(1-s)^{7/3}) + \frac{2^{1/4}}{\Delta_2\Gamma(11/4)}[(1-s)^{11/12} - (1-2s)^{7/4}], \\ \qquad 0 \le s < \frac{1}{2}, \\ \frac{1}{\Gamma(17/4)}(1-s)^{11/12}(1-(1-s)^{7/3}) + \frac{2^{1/4}}{\Delta_2\Gamma(11/4)}(1-s)^{11/12}, \quad \frac{1}{2} \le s \le 1. \end{cases}
\end{aligned}
$$

Example 1. We consider the functions

$$
f(t,u) = a(u^{\alpha_0} + u^{\beta_0}), \ g(t,u) = b(u^{\gamma_0} + u^{\delta_0}), \ t \in [0,1], \ u \ge 0,
$$

where $\alpha_0 > 1$, $0 < \beta_0 < 1$, $\gamma_0 > 2$, $0 < \delta_0 < 1$, $a, b > 0$. We have $K_1 = \max_{s\in[0,1]} J_1(s) \approx 4.01249183$, $K_2 = \max_{s\in[0,1]} J_2(s) \approx 0.22467674$. Then $m_0 = \max\{K_1, K_2\} = K_1$. The functions $f(t,u)$ and $g(t,u)$ satisfy the assumption $(H2)$. Besides, they are nondecreasing with respect to u, for any $t \in [0,1]$, and for $p = 1/2$ and $c \in (0,1)$ the assumptions $(H3)$ and $(H6)$ are satisfied; indeed we obtain

$$
\begin{aligned}
f_\infty^i &= \lim_{u\to\infty} \frac{a(u^{\alpha_0} + u^{\beta_0})}{u^{1/2}} = \infty, \ g_\infty^i = \lim_{u\to\infty} \frac{b(u^{\gamma_0} + u^{\delta_0})}{u^2} = \infty, \\
f_0^i &= \lim_{u\to 0^+} \frac{a(u^{\alpha_0} + u^{\beta_0})}{u} = \infty, \ g_0^i = \lim_{u\to 0^+} \frac{b(u^{\gamma_0} + u^{\delta_0})}{u} = \infty.
\end{aligned}
$$

We take $N_0 = 1$ and then $\int_0^1 g(s,1)\,ds = 2b$ and $f(t, 2bm_0) = a[(2bm_0)^{\alpha_0} + (2bm_0)^{\beta_0}]$. If $a[(2bm_0)^{\alpha_0} + (2bm_0)^{\beta_0}] < \frac{1}{m_0} \Leftrightarrow a\left[m_0^{\alpha_0+1}(2b)^{\alpha_0} + m_0^{\beta_0+1}(2b)^{\beta_0}\right] < 1$, then the assumption $(H7)$ is satisfied. For example, if $\alpha_0 = 3/2$, $\beta_0 = 1/3$, $b = 1/2$ and $a < \dfrac{1}{m_0^{5/2} + m_0^{4/3}}$ (e.g. $a \le 0.0258$), then the above inequality is satisfied. By Theorem 3.3, we deduce that the problem $(S_0) - (BC_0)$ has at least two positive solutions.

Example 2. We consider the functions

$$f(t,x) = \frac{x^a}{t^{\zeta_1}(1-t)^{\rho_1}}, \quad g(t,x) = \frac{x^b}{t^{\zeta_2}(1-t)^{\rho_2}},$$

with $a, b > 1$ and $\zeta_1, \rho_1, \zeta_2, \rho_2 \in (0,1)$. Here $f(t,x) = \widetilde{p}_1(t)\widetilde{q}_1(x)$ and $g(t,x) = \widetilde{p}_2(t)\widetilde{q}_2(x)$, where

$$\widetilde{p}_1(t) = \frac{1}{t^{\zeta_1}(1-t)^{\rho_1}}, \ \widetilde{p}_2(t) = \frac{1}{t^{\zeta_2}(1-t)^{\rho_2}}, \ \widetilde{q}_1(x) = x^a, \ \widetilde{q}_2(x) = x^b.$$

We have $0 < \int_0^1 \widetilde{p}_1(s)\,ds < \infty$, $0 < \int_0^1 \widetilde{p}_2(s)\,ds < \infty$, so the functions f and g satisfy the assumption $(A2)$.

In $(A5)$, for $r_1 < a,\ r_2 < b$ and $r_1 r_2 \ge 1$, we obtain

$$\lim_{x\to 0^+} \frac{\widetilde{q}_1(x)}{x^{r_1}} = 0, \quad \lim_{x\to 0^+} \frac{\widetilde{q}_2(x)}{x^{r_2}} = 0.$$

In $(A6)$, for $l_1 < a,\ l_2 < b$, $l_1 l_2 \ge 1$ and $c \in (0, 1/2)$, we have

$$\lim_{x\to\infty} \inf_{t\in[c,1-c]} \frac{f(t,x)}{x^{l_1}} = \infty, \quad \lim_{x\to\infty} \inf_{t\in[c,1-c]} \frac{g(t,x)}{x^{l_2}} = \infty.$$

For example, if $a = 3/2$, $b = 2$, $r_1 = 1$, $r_2 = 3/2$, $l_1 = 1$, $l_2 = 3/2$, the above conditions are satisfied. Then, by Theorem 4.3, we deduce that the problem $(S_0) - (BC_0)$ has at least one positive solution.

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