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BOUNDEDNESS AND STABILIZATION IN A THREE-DIMENSIONAL TWO-SPECIES CHEMOTAXIS-NAVIER-STOKES SYSTEM WITH COMPETITIVE KINETICS*

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Abstract. This paper is concerned with the two-species chemotaxis-Navier–Stokes system with Lotka–Volterra competitive kinetics

$$\begin{cases} (n_1)_t + u\cdot\nabla n_1 = \Delta n_1 - \chi_1\nabla\cdot(n_1\nabla c) + \mu_1 n_1(1-n_1-a_1n_2) & \text{in } \Omega\times(0,\infty), \\ (n_2)_t + u\cdot\nabla n_2 = \Delta n_2 - \chi_2\nabla\cdot(n_2\nabla c) + \mu_2 n_2(1-a_2n_1-n_2) & \text{in } \Omega\times(0,\infty), \\ c_t + u\cdot\nabla c = \Delta c - (\alpha n_1+\beta n_2)c & \text{in } \Omega\times(0,\infty), \\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + (\gamma n_1+\delta n_2)\nabla\Phi, \quad \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty) \end{cases}$$

under homogeneous Neumann boundary conditions and initial conditions, where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary. Recently, in the 2-dimensional setting, global existence and stabilization of classical solutions to the above system were first established. However, the 3-dimensional case has not been studied: Because of difficulties in the Navier–Stokes system, we can not expect existence of classical solutions to the above system. The purpose of this paper is to obtain global existence of weak solutions to the above system, and their eventual smoothness and stabilization.

Key words. chemotaxis, Navier–Stokes, Lotka–Volterra, large-time behaviour

AMS subject classifications. 35B40, 35K55, 35Q30, 92C17

1. Introduction. This paper deals with the following two-species chemotaxis-Navier–Stokes system with Lotka–Volterra competitive kinetics:

$$\begin{cases} (n_1)_t + u\cdot\nabla n_1 = \Delta n_1 - \chi_1\nabla\cdot(n_1\nabla c) + \mu_1 n_1(1-n_1-a_1n_2) & \text{in } \Omega\times(0,\infty), \\ (n_2)_t + u\cdot\nabla n_2 = \Delta n_2 - \chi_2\nabla\cdot(n_2\nabla c) + \mu_2 n_2(1-a_2n_1-n_2) & \text{in } \Omega\times(0,\infty), \\ c_t + u\cdot\nabla c = \Delta c - (\alpha n_1+\beta n_2)c & \text{in } \Omega\times(0,\infty), \\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + (\gamma n_1+\delta n_2)\nabla\Phi, \quad \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty), \\ \partial_\nu n_1 = \partial_\nu n_2 = \partial_\nu c = 0, \quad u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ n_1(\cdot,0)=n_{1,0}, \quad n_2(\cdot,0)=n_{2,0}, \quad c(\cdot,0)=c_0, \quad u(\cdot,0)=u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and ∂_ν denotes differentiation with respect to the outward normal of $\partial\Omega$; $\kappa = 1$, $\chi_1, \chi_2, a_1, a_2 \ge 0$ and $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$ are constants; $n_{1,0}, n_{2,0}, c_0, u_0, \Phi$ are known functions satisfying

$$0 < n_{1,0}, n_{2,0} \in C(\overline{\Omega}), \quad 0 < c_0 \in W^{1,q}(\Omega), \quad u_0 \in D(A^\theta), \tag{1.2}$$

$$\Phi \in C^{1+\lambda}(\overline{\Omega}) \tag{1.3}$$

for some $q > 3$, $\theta \in (\frac{3}{4}, 1)$, $\lambda \in (0,1)$ and A denotes the realization of the Stokes operator under homogeneous Dirichlet boundary conditions in the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega)$.

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In the mathematical point of view, difficulties of this problem are mainly caused by the chemotaxis terms $-\chi_1\nabla\cdot(n_1\nabla c)$, $-\chi_2\nabla\cdot(n_2\nabla c)$, the competitive kinetics $\mu_1 n_1(1-n_1-a_1n_2)$, $\mu_2n_2(1-a_2n_1-n_2)$ and the Navier–Stokes equation which is the fourth equation in (1.1). In the case that $n_2 = 0$, global existence of weak solutions, and their eventual smoothness and stabilization were shown in [5]. On the other hand, in the case that $n_2 \neq 0$ and $\Omega\subset\mathbb{R}^2$, global existence and boundedness of classical solutions to (1.1) have been attained ([4]). Moreover, in the case that $\kappa = 0$ in (1.1), which namely means that the fourth equation in (1.1) is the *Stokes* equation, global existence and stabilization can be found in [2]; in the case that $\kappa = 0$ in (1.1) and that $-(\alpha n_1+\beta n_2)c$ is replaced with $+\alpha n_1+\beta n_2 - c$, global existence and boundedness of classical solutions to the Keller–Segel-Stokes system and their asymptotic behaviour are found in [3].

As we mentioned above, global classical solutions are found in (1.1) in the 2-dimensional setting and the case that $\kappa = 0$. However, global existence of solutions in 3-dimensional setting has not been attained. Thus the main purposes of this paper is to obtain global existence of solutions to (1.1) in the case that $\Omega\subset\mathbb{R}^3$. Nevertheless, because of the difficulties of the Navier–Stokes equation, we can not expect global existence of *classical solutions* to (1.1) in the 3-dimensional case. Therefore our goal is to obtain global existence of *weak solutions* to (1.1) in the following sense.

Definition 1.1. *A quadruple (n_1,n_2,c,u) is called a (global) weak solution of (1.1) if*

$$\begin{aligned}
n_1,n_2 &\in L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))\cap L^{\frac43}_{\mathrm{loc}}([0,\infty);W^{1,\frac43}(\Omega)),\\
c &\in L^2_{\mathrm{loc}}([0,\infty);W^{1,2}(\Omega)),\\
u &\in L^2_{\mathrm{loc}}([0,\infty);W^{1,2}_{0,\sigma}(\Omega))
\end{aligned}$$

and for all $T>0$ the identities

$$\begin{aligned}
&-\int_0^\infty\!\!\int_\Omega n_1\varphi_t - \int_\Omega n_{1,0}\varphi(\cdot,0) - \int_0^\infty\!\!\int_\Omega n_1 u\cdot\nabla\varphi\\
&\quad= -\int_0^\infty\!\!\int_\Omega \nabla n_1\cdot\nabla\varphi + \chi_1\int_0^\infty\!\!\int_\Omega n_1\nabla c\cdot\nabla\varphi + \mu_1\int_0^\infty\!\!\int_\Omega n_1(1-n_1-a_1n_2)\varphi,\\
&-\int_0^\infty\!\!\int_\Omega n_2\varphi_t - \int_\Omega n_{2,0}\varphi(\cdot,0) - \int_0^\infty\!\!\int_\Omega n_2 u\cdot\nabla\varphi\\
&\quad= -\int_0^\infty\!\!\int_\Omega \nabla n_2\cdot\nabla\varphi + \chi_2\int_0^\infty\!\!\int_\Omega n_2\nabla c\cdot\nabla\varphi + \mu_2\int_0^\infty\!\!\int_\Omega n_2(1-a_2n_1-n_2)\varphi,\\
&-\int_0^\infty\!\!\int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) - \int_0^\infty\!\!\int_\Omega cu\cdot\nabla\varphi\\
&\quad= -\int_0^\infty\!\!\int_\Omega \nabla c\cdot\nabla\varphi - \int_0^\infty\!\!\int_\Omega(\alpha n_1+\beta n_2)c\varphi,\\
&-\int_0^\infty\!\!\int_\Omega u\cdot\psi_t - \int_\Omega u_0\cdot\psi(\cdot,0) - \int_0^\infty\!\!\int_\Omega u\otimes u\cdot\nabla\psi\\
&\quad= -\int_0^\infty\!\!\int_\Omega \nabla u\cdot\nabla\psi + \int_0^\infty\!\!\int_\Omega(\gamma n_1+\delta n_2)\nabla\psi\cdot\nabla\Phi
\end{aligned}$$

hold for all $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ and all $\psi\in C^\infty_{0,\sigma}(\Omega\times[0,\infty))$, respectively.

Now the main results read as follows. The first theorem is concerned with global existence of weak solutions to (1.1).

THEOREM 1.2. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded smooth domain and let* $\chi_1, \chi_2, a_1, a_2 \geq 0$ *and* $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$. *Assume that* $n_{1,0}, n_{2,0}, c_0, u_0$ *satisfy* (1.2) *with some* $q > 3$ *and* $\theta \in (\frac{3}{4}, 1)$ *and* $\Phi \in C^{1+\lambda}(\overline{\Omega})$ *for some* $\lambda \in (0,1)$. *Then there is a weak solution of* (1.1)*, which can be approximated by a sequence of solutions* $(n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon, u_\varepsilon)$ *of* (2.1) (*see Section* 2) *in a pointwise manner.*

The second theorem gives eventual smoothness and stabilization.

THEOREM 1.3. *Let the assumption of Theorem* 1.2 *be satisfied. Then there are* $T > 0$ *and* $\alpha' \in (0,1)$ *such that the solution* (n_1, n_2, c, u) *given by Theorem* 1.2 *satisfies*

$$n_1, n_2, c \in C^{2+\alpha', 1+\frac{\alpha'}{2}}(\overline{\Omega} \times [T, \infty)), \quad u \in C^{2+\alpha', 1+\frac{\alpha'}{2}}(\overline{\Omega} \times [T, \infty)).$$

Moreover, the solution of (1.1) *has the following properties*:

(i) *Assume that* $a_1, a_2 \in (0,1)$. *Then*

$$n_1(\cdot, t) \to N_1, \quad n_2(\cdot, t) \to N_2, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega)$$

as $t \to \infty$, *where*

$$N_1 := \frac{1 - a_1}{1 - a_1 a_2}, \quad N_2 := \frac{1 - a_2}{1 - a_1 a_2}.$$

(ii) *Assume that* $a_1 \geq 1 > a_2$. *Then*

$$n_1(\cdot, t) \to 0, \quad n_2(\cdot, t) \to 1, \quad c(\cdot, t) \to 0, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega)$$

as $t \to \infty$.

The proofs of the main theorems are based on the arguments in [5]. The strategies for the proofs is to construct energy estimates for the solution $(n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon, u_\varepsilon)$ of (2.1). In Section 2 we consider the energy function $\mathcal{F}_\varepsilon$ defined as

$$\mathcal{F}_\varepsilon := \int_\Omega n_{1,\varepsilon} \log n_{1,\varepsilon} + \int_\Omega n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{\chi}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + k_4 \chi \int_\Omega |u_\varepsilon|^2$$

with some constant $\chi > 0$. Noting that for all $\rho, \xi_i > 0$ there exists $C > 0$ such that

$$\begin{aligned} &\int_\Omega \nabla c_\varepsilon \cdot \nabla n_{i,\varepsilon} \left(\frac{\chi_i}{1 + \varepsilon n_{i,\varepsilon}} - \frac{\chi\alpha \text{ (or } \chi\beta)}{1 + \varepsilon(\alpha n_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right) \\ &\leq \rho \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \xi_i \int_\Omega \frac{|\nabla n_{i,\varepsilon}|^2}{n_{i,\varepsilon}} + C \int_\Omega n_{i,\varepsilon}^2 \quad (i = 1, 2), \end{aligned}$$

which did not appear in the previous work [5], from the estimate for the energy function $\mathcal{F}_\varepsilon$ we obtain global-in-time solvability of approximate solutions. Then we moreover see convergence as $\varepsilon \searrow 0$. Furthermore, in Section 3, according to an argument similar to [4], by putting

$$\mathcal{G}_{\varepsilon,B} := \int_\Omega \left(n_{1,\varepsilon} - N_1 \log \frac{n_{1,\varepsilon}}{N_1} \right) + \int_\Omega \left(n_{2,\varepsilon} - N_2 \log \frac{n_{2,\varepsilon}}{N_2} \right) + \frac{B}{2} \int_\Omega c_\varepsilon^2$$

with suitable constant $B > 0$ and establishing the Hölder estimates for the solution of (1.1) through the estimate for the energy function $\mathcal{G}_{\varepsilon,B}$, we can discuss convergence of $\big(n_1(\cdot, t), n_2(\cdot, t), c(\cdot, t), u(\cdot, t)\big)$ as $t \to \infty$.

2. Proof of Theorem 1.2 (Global existence). We will start by considering an approximate problem with parameter $\varepsilon > 0$, namely:

$$\begin{cases} (n_{1,\varepsilon})_t + u_\varepsilon \cdot \nabla n_{1,\varepsilon} = \Delta n_{1,\varepsilon} - \chi_1 \nabla \cdot \left(\frac{n_{1,\varepsilon}}{1+\varepsilon n_{1,\varepsilon}} \nabla c_\varepsilon \right) + \mu_1 n_{1,\varepsilon}(1 - n_{1,\varepsilon} - a_1 n_{2,\varepsilon}), \\ (n_{2,\varepsilon})_t + u_\varepsilon \cdot \nabla n_{2,\varepsilon} = \Delta n_{2,\varepsilon} - \chi_2 \nabla \cdot \left(\frac{n_{2,\varepsilon}}{1+\varepsilon n_{2,\varepsilon}} \nabla c_\varepsilon \right) + \mu_2 n_{2,\varepsilon}(1 - a_2 n_{1,\varepsilon} - n_{2,\varepsilon}), \\ (c_\varepsilon)_t + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon \frac{1}{\varepsilon} \log\left(1 + \varepsilon(\alpha n_{1,\varepsilon} + \beta n_{2,\varepsilon})\right), \\ (u_\varepsilon)_t + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + (\gamma n_{1,\varepsilon} + \delta n_{2,\varepsilon}) \nabla \Phi, \quad \nabla \cdot u_\varepsilon = 0, \\ \partial_\nu n_{1,\varepsilon}|_{\partial\Omega} = \partial_\nu n_{2,\varepsilon}|_{\partial\Omega} = \partial_\nu c_\varepsilon|_{\partial\Omega} = 0, \quad u_\varepsilon|_{\partial\Omega} = 0, \\ n_{1,\varepsilon}(\cdot,0) = n_{1,0}, \quad n_{2,\varepsilon}(\cdot,0) = n_{2,0}, \quad c_\varepsilon(\cdot,0) = c_0, \quad u_\varepsilon(\cdot,0) = u_0, \end{cases} \tag{2.1}$$

where $Y_\varepsilon = (1 + \varepsilon A)^{-1}$, and provide estimates for its solutions. We first give the following result which states local existence in (1.1).

Lemma 2.1. *Let $\chi_1, \chi_2, a_1, a_2 \geq 0$, $\mu_1, \mu_2, \alpha, \beta, \gamma, \delta > 0$, and $\Phi \in C^{1+\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$ and assume that $n_{1,0}, n_{2,0}, c_0, u_0$ satisfy (1.2) with some $q > 3, \theta \in (\frac{3}{4}, 1)$. Then for all $\varepsilon > 0$ there are $T_{\max,\varepsilon}$ and uniquely determined functions:*

$$\begin{aligned} n_{1,\varepsilon}, n_{2,\varepsilon} &\in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ c_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \cap L^\infty_{\mathrm{loc}}([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)), \\ u_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \end{aligned}$$

which together with some $P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0, T_{\max,\varepsilon}))$ solve (2.1) classically. Moreover, $n_{1,\varepsilon}$, $n_{2,\varepsilon}$ and c_ε are positive and the following alternative holds: $T_{\max,\varepsilon} = \infty$ or

$$\|n_{1,\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|n_{2,\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\theta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \tag{2.2}$$

as $t \nearrow T_{\max,\varepsilon}$.

We next show the following lemma which holds a key for the proof of Theorem 1.2. This lemma derives the estimate for the energy function.

Lemma 2.2. *For all $\xi_1, \xi_2 \in (0,1)$ and $\chi > 0$ there are $C, \overline{C}, \widetilde{C}, k, \overline{k} > 0$ such that*

$$\mathcal{F}_\varepsilon := \int_\Omega n_{1,\varepsilon} \log n_{1,\varepsilon} + \int_\Omega n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{\chi}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \overline{k}\chi \int_\Omega |u_\varepsilon|^2$$

satisfies

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_\varepsilon \leq & -\frac{\mu_1}{4} \int_\Omega n_{1,\varepsilon}^2 \log n_{1,\varepsilon} - \frac{\mu_2}{4} \int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon} \\ & - (1-\xi_1) \int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} - (1-\xi_2) \int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{2,\varepsilon}} + C \int_\Omega n_{1,\varepsilon}^2 + \overline{C} \int_\Omega n_{2,\varepsilon}^2 + \widetilde{C} \\ & - k \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 - k \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} - k \int_\Omega |\nabla u_\varepsilon|^2 \end{aligned}$$

on $(0, T_{\max,\varepsilon})$ for all $\varepsilon > 0$.

Proof. Noting, the boundedness of $s(1-s)$ and $s(1-\frac{s}{2})\log s$, we have that there exists $C_1 > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_{1,\varepsilon}\log n_{1,\varepsilon} \\
&= -\int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} + \chi_1\int_\Omega \frac{\nabla c_\varepsilon\cdot\nabla n_{1,\varepsilon}}{1+\varepsilon n_{1,\varepsilon}} \\
&\quad + \mu_1\int_\Omega n_{1,\varepsilon}(1-n_{1,\varepsilon}-a_1n_{2,\varepsilon})\log n_{1,\varepsilon} + \mu_1\int_\Omega n_{1,\varepsilon}(1-n_{1,\varepsilon}-a_1n_{2,\varepsilon}) \\
&\le -\int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} + \chi_1\int_\Omega \frac{\nabla c_\varepsilon\cdot\nabla n_{1,\varepsilon}}{1+\varepsilon n_{1,\varepsilon}} - \frac{\mu_1}{2}\int_\Omega n_{1,\varepsilon}^2\log n_{1,\varepsilon} \\
&\quad - \mu_1a_1\int_\Omega n_{1,\varepsilon}n_{2,\varepsilon}\log n_{1,\varepsilon} - \mu_1a_1\int_\Omega n_{1,\varepsilon}n_{2,\varepsilon} + C_1. \qquad (2.3)
\end{aligned}$$

Similarly, there is $C_2 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n_{2,\varepsilon}\log n_{2,\varepsilon} \le &-\int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{2,\varepsilon}} + \chi_2\int_\Omega \frac{\nabla c_\varepsilon\cdot\nabla n_{2,\varepsilon}}{1+\varepsilon n_{2,\varepsilon}} - \frac{\mu_2}{2}\int_\Omega n_{2,\varepsilon}^2\log n_{2,\varepsilon} \\
&- \mu_2a_2\int_\Omega n_{1,\varepsilon}n_{2,\varepsilon}\log n_{2,\varepsilon} - \mu_2a_2\int_\Omega n_{1,\varepsilon}n_{2,\varepsilon} + C_2. \qquad (2.4)
\end{aligned}$$

According to an argument similar to that in the proof of [5, Lemma 2.8], there exist $k_1, C_3, C_4 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \le &-k_1\int_\Omega c_\varepsilon|D^2\log c_\varepsilon|^2 - k_1\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\
&+ C_3 + C_4\int_\Omega |\nabla u_\varepsilon|^2 - 2\int_\Omega \frac{\alpha\nabla c_\varepsilon\cdot\nabla n_{1,\varepsilon} + \beta\nabla c_\varepsilon\cdot\nabla n_{2,\varepsilon}}{1+\varepsilon(\alpha n_{1,\varepsilon}+\beta n_{2,\varepsilon})}. \qquad (2.5)
\end{aligned}$$

Now we let $\overline{k}, \eta_1, \eta_2, k$ be constants satisfying $\frac{C_4}{2} - \overline{k} = -\frac{k_1}{4}$, $\eta_1 = \frac{\mu_1}{4\overline{k}\chi}$, $\eta_2 = \frac{b\mu_2}{4\overline{k}\chi}$ and $k = \frac{\chi k_1}{4}$. Then we have

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 = -2\int_\Omega |\nabla u_\varepsilon|^2 - 2\int_\Omega u_\varepsilon\cdot(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon + 2\int_\Omega u_\varepsilon\cdot(\gamma n_{1,\varepsilon}+\delta n_{2,\varepsilon})\nabla\Phi.$$

From the Schwarz inequality, the Poincaré inequality, the Young inequality and the fact that $\int_\Omega \varphi^2 \le a\int_\Omega \varphi^2\log\varphi + |\Omega|e^{\frac{1}{a}}$ holds for any positive function φ and any $a > 0$, there exist $C_5, C_{\eta_1}, C_{\eta_2} > 0$ such that

$$\begin{aligned}
\gamma\int_\Omega |n_{1,\varepsilon}\nabla\Phi\cdot u_\varepsilon| &\le \gamma\|\nabla\Phi\|_{L^\infty}\left(\int_\Omega n_{1,\varepsilon}^2\right)^{\frac{1}{2}}\left(\int_\Omega |u_\varepsilon|^2\right)^{\frac{1}{2}} \\
&\le \gamma\|\nabla\Phi\|_{L^\infty}\left(\int_\Omega n_{1,\varepsilon}^2\right)^{\frac{1}{2}}\left(C_5\int_\Omega |\nabla u_\varepsilon|^2\right)^{\frac{1}{2}} \\
&\le \gamma^2C_5\|\nabla\Phi\|_{L^\infty}^2\int_\Omega n_{1,\varepsilon}^2 + \frac{1}{4}\int_\Omega |\nabla u_\varepsilon|^2 \\
&\le \frac{\eta_1}{2}\int_\Omega n_{1,\varepsilon}^2\log n_{1,\varepsilon} + \frac{C_{\eta_1}}{2} + \frac{1}{4}\int_\Omega |\nabla u_\varepsilon|^2
\end{aligned}$$

and

$$\delta \int_\Omega |n_{2,\varepsilon} \nabla \Phi \cdot u_\varepsilon| \le \frac{\eta_2}{2} \int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon} + \frac{C_{\eta_2}}{2} + \frac{1}{4} \int_\Omega |\nabla u_\varepsilon|^2$$

hold. Therefore we have

$$\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 \le - \int_\Omega |\nabla u_\varepsilon|^2 + \eta_1 \int_\Omega n_{1,\varepsilon}^2 \log n_{1,\varepsilon} + \eta_2 \int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon} + C_{\eta_1} + C_{\eta_2}. \tag{2.6}$$

Thus a combination of (2.3), (2.4), (2.5) and (2.6) leads to

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega n_{1,\varepsilon} \log n_{1,\varepsilon} + \int_\Omega n_{2,\varepsilon} \log n_{2,\varepsilon} + \frac{\chi}{2} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \overline{k}\chi \int_\Omega |u_\varepsilon|^2 \right] \\
&\le \left(\overline{k}\chi\eta_1 - \frac{\mu_1}{2} \right) \int_\Omega n_{1,\varepsilon}^2 \log n_{1,\varepsilon} + \left(\overline{k}\chi\eta_2 - \frac{\mu_2}{2} \right) \int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon} \\
&\quad - \left(\int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} + \int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{2,\varepsilon}} \right) + \left(\frac{\chi}{2} C_4 - \overline{k}\chi \right) \int_\Omega |\nabla u_\varepsilon|^2 \\
&\quad + \int_\Omega \nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon} \left(\frac{\chi_1}{1+\varepsilon n_{1,\varepsilon}} - \frac{\chi\alpha}{1+\varepsilon(\alpha n_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right) \\
&\quad + \int_\Omega \nabla c_\varepsilon \cdot \nabla n_{2,\varepsilon} \left(\frac{\chi_2}{1+\varepsilon n_{2,\varepsilon}} - \frac{\chi\beta}{1+\varepsilon(\alpha n_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right) \\
&\quad - \frac{\chi}{2} k_1 \int_\Omega c_\varepsilon |D^2 \log c_\varepsilon|^2 - \frac{\chi}{2} k_1 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_1 + C_2 + \frac{\chi}{2} C_3 + \overline{k}\chi (C_{\eta_1} + C_{\eta_2}) \\
&\quad - \mu_1 a_1 \int_\Omega n_{1,\varepsilon} n_{2,\varepsilon} (\log n_{1,\varepsilon} + 1) - \mu_2 a_2 \int_\Omega n_{1,\varepsilon} n_{2,\varepsilon} (\log n_{2,\varepsilon} + 1).
\end{aligned}$$

Here, since $n_{1,\varepsilon}, n_{2,\varepsilon}$ are nonnegative, we can find $C_6, C_7 > 0$ such that

$$\begin{aligned}
&\int_\Omega \nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon} \left(\frac{\chi_1}{1+\varepsilon n_{1,\varepsilon}} - \frac{\chi\alpha}{1+\varepsilon(\alpha n_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right) \\
&\le (\chi_1 + \chi\alpha) \int_\Omega |\nabla c_\varepsilon \cdot \nabla n_{1,\varepsilon}| \\
&\le \frac{\chi k_1}{8\|c_0\|_{L^\infty}^3} \int_\Omega |\nabla c_\varepsilon|^4 + C_6 \int_\Omega |\nabla n_{1,\varepsilon}|^{\frac{4}{3}} \\
&\le \frac{\chi k_1}{8} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \xi_1 \int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} + C_7 \int_\Omega n_{1,\varepsilon}^2
\end{aligned}$$

and there is $C_8 > 0$ such that

$$\begin{aligned}
&\int_\Omega \nabla c_\varepsilon \cdot \nabla n_{2,\varepsilon} \left(\frac{\chi_2}{1+\varepsilon n_{2,\varepsilon}} - \frac{\chi\beta}{1+\varepsilon(\alpha n_{1,\varepsilon} + \beta n_{2,\varepsilon})} \right) \\
&\le \frac{\chi k_1}{8} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \xi_2 \int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{2,\varepsilon}} + C_8 \int_\Omega n_{2,\varepsilon}^2,
\end{aligned}$$

which with the fact that $s\log s \geq -\frac{1}{e}$ $(s>0)$ enables us to obtain

$$\begin{aligned}
&\Big(\overline{k}\chi\eta_1 - \frac{\mu_1}{2}\Big)\int_\Omega n_{1,\varepsilon}^2 \log n_{1,\varepsilon} + \Big(\overline{k}\chi\eta_2 - \frac{\mu_2}{2}\Big)\int_\Omega n_{2,\varepsilon}^2 \log n_{2,\varepsilon}\\
&-\bigg(\int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} + \int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{2,\varepsilon}}\bigg) + \Big(\frac{\chi}{2}C_4 - \overline{k}\chi\Big)\int_\Omega |\nabla u_\varepsilon|^2\\
&+\int_\Omega \nabla c_\varepsilon\cdot\nabla n_{1,\varepsilon}\bigg(\frac{\chi_1}{1+\varepsilon n_{1,\varepsilon}} - \frac{\chi\alpha}{1+\varepsilon(\alpha n_{1,\varepsilon}+\beta n_{2,\varepsilon})}\bigg)\\
&+\int_\Omega \nabla c_\varepsilon\cdot\nabla n_{2,\varepsilon}\bigg(\frac{\chi_2}{1+\varepsilon n_{2,\varepsilon}} - \frac{\chi\beta}{1+\varepsilon(\alpha n_{1,\varepsilon}+\beta n_{2,\varepsilon})}\bigg)\\
&-\frac{\chi}{2}k_1\int_\Omega c_\varepsilon |D^2\log c_\varepsilon|^2 - \frac{\chi}{2}k_1\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_1 + C_2 + \frac{\chi}{2}C_3 + \overline{k}\chi(C_{\eta_1}+C_{\eta_2})\\
&-\mu_1 a_1\int_\Omega n_{1,\varepsilon}n_{2,\varepsilon}(\log n_{1,\varepsilon}+1) - \mu_2 a_2\int_\Omega n_{1,\varepsilon}n_{2,\varepsilon}(\log n_{2,\varepsilon}+1)\\
&\leq -\frac{\mu_1}{4}\int_\Omega n_{1,\varepsilon}^2\log n_{1,\varepsilon} - \frac{\mu_2}{4}\int_\Omega n_{2,\varepsilon}^2\log n_{2,\varepsilon}\\
&\quad -(1-\xi_1)\int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} - (1-\xi_2)\int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{1,\varepsilon}}\\
&\quad -k\int_\Omega |\nabla u_\varepsilon|^2 - k\int_\Omega c_\varepsilon|D^2\log c_\varepsilon| - k\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_7\int_\Omega n_{1,\varepsilon}^2 + C_8\int_\Omega n_{2,\varepsilon}^2 + C_9.
\end{aligned}$$

Therefore we obtain this lemma. □

Proof of Theorem 1.2. Let $\tau = \min\{1, \frac{1}{2}T_{\max,\varepsilon}\}$, $\xi_1,\xi_2\in(0,1)$ and $\chi>0$. Lemma 2.2, the facts that $s^2\log s \geq s\log s - \frac{1}{2e}$ $(s>0)$ and $n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon > 0$ imply

$$\frac{d}{dt}\mathcal{F}_\varepsilon + \mathcal{F}_\varepsilon \leq C\int_\Omega n_{1,\varepsilon}^2 + \overline{C}\int_\Omega n_{2,\varepsilon}^2 + \widetilde{C}'$$

for some $C, \overline{C}, \widetilde{C}' > 0$. According to [5, Lemma 2.5], there exists $C_1>0$ such that

$$\int_t^{t+\tau}\!\!\int_\Omega n_{i,\varepsilon}^2 \leq C_1$$

for all $t\in(0, T_{\max,\varepsilon}-\tau)$ and each $i=1,2$. From the uniform Gronwall type lemma (see e.g., [6, Lemma 3.2]) we can find $C_2>0$ such that

$$\int_\Omega n_{1,\varepsilon}\log n_{1,\varepsilon} + \int_\Omega n_{2,\varepsilon}\log n_{2,\varepsilon} + \frac{\chi}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \overline{k}\chi\int_\Omega |u_\varepsilon|^2 \leq C_2 \tag{2.7}$$

for all $t\in(0,T_{\max,\varepsilon})$. Moreover, we have from integration of the differential inequality in Lemma 2.2 over $(t,t+\tau)$ that for all $\xi_1,\xi_2\in(0,1)$ there is $C_3>0$ such that

$$\begin{aligned}
&\frac{\mu_1}{4}\int_t^{t+\tau}\!\!\int_\Omega n_{1,\varepsilon}^2\log n_{1,\varepsilon} + \frac{\mu_2}{4}\int_t^{t+\tau}\!\!\int_\Omega n_{2,\varepsilon}^2\log n_{2,\varepsilon} + k\int_t^{t+\tau}\!\!\int_\Omega c_\varepsilon|D^2\log c_\varepsilon|^2\\
&\quad + (1-\xi_1)\int_t^{t+\tau}\!\!\int_\Omega \frac{|\nabla n_{1,\varepsilon}|^2}{n_{1,\varepsilon}} + (1-\xi_2)\int_t^{t+\tau}\!\!\int_\Omega \frac{|\nabla n_{2,\varepsilon}|^2}{n_{2,\varepsilon}} \leq C_3
\end{aligned}\tag{2.8}$$

and

$$\int_t^{t+\tau}\!\!\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_t^{t+\tau}\!\!\int_\Omega |\nabla u_\varepsilon|^2 \leq C_3 \tag{2.9}$$

as well as

$$\int_t^{t+\tau}\int_\Omega |\nabla n_{1,\varepsilon}|^{\frac{4}{3}} + \int_t^{t+\tau}\int_\Omega |\nabla n_{2,\varepsilon}|^{\frac{4}{3}} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_t^{t+\tau}\int_\Omega |\nabla c_\varepsilon|^4 + \int_t^{t+\tau}\int_\Omega n_{1,\varepsilon}^2 + \int_t^{t+\tau}\int_\Omega n_{2,\varepsilon}^2 \le C_3 \tag{2.10}$$

for all $t \in [0, T_{\max,\varepsilon} - \tau)$. Now we assume $T_{\max,\varepsilon} < \infty$ for some $\varepsilon > 0$. From (2.7), (2.8), (2.9) and (2.10), we can see that there exists $C_4 > 0$ such that

$$\begin{aligned} \|n_{1,\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} &\le C_4, & \|n_{2,\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} &\le C_4, \\ \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} &\le C_4, & \|A^\sigma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le C_4 \end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, which is inconsistent with (2.2). Therefore we obtain $T_{\max,\varepsilon} = \infty$ for all $\varepsilon > 0$, which means global existence and boundedness of $(n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon, u_\varepsilon)$. We next verify convergence of the solution $(n_{1,\varepsilon}, n_{2,\varepsilon}, c_\varepsilon, u_\varepsilon)$. Due to Lemma 2.2 and arguments similar to those in [5], we establish that for all $T > 0$ there is $C_5 > 0$ such that

$$\begin{aligned} \|(n_{1,\varepsilon})_t\|_{L^1((0,T);(W_0^{2,4}(\Omega))^*)} &\le C_5, & \|(n_{2,\varepsilon})_t\|_{L^1((0,T);(W_0^{2,4}(\Omega))^*)} &\le C_5, \\ \|(c_\varepsilon)_t\|_{L^2((0,T);(W_0^{1,2}(\Omega))^*)} &\le C_5, & \|(u_\varepsilon)_t\|_{L^2((0,T);(W^{1,3}(\Omega))^*)} &\le C_5 \end{aligned} \tag{2.11}$$

for all $\varepsilon > 0$, which together with arguments in [5] implies that there exist a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and functions n_1, n_2, c, u such that

$$\begin{aligned} n_1, n_2 &\in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \cap L^{\frac{4}{3}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{4}{3}}(\Omega)), \\ c &\in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)), \\ u &\in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_{0,\sigma}(\Omega)) \end{aligned}$$

and that

$$\begin{aligned} n_{1,\varepsilon} &\to n_1 && \text{in } L^{\frac{4}{3}}_{\mathrm{loc}}([0,\infty); L^p(\Omega)) \quad \text{for all } p \in \left[1, \tfrac{12}{5}\right) \quad \text{and a.e. in } \Omega \times (0,\infty), \\ n_{2,\varepsilon} &\to n_2 && \text{in } L^{\frac{4}{3}}_{\mathrm{loc}}([0,\infty); L^p(\Omega)) \quad \text{for all } p \in \left[1, \tfrac{12}{5}\right) \quad \text{and a.e. in } \Omega \times (0,\infty), \\ c_\varepsilon &\to c && \text{in } C^0_{\mathrm{loc}}([0,\infty); L^p(\Omega)) \quad \text{for all } p \in [1,6) \quad \text{and a.e. in } \Omega \times (0,\infty), \\ u_\varepsilon &\to u && \text{in } L^2_{\mathrm{loc}}([0,\infty); L^p(\Omega)) \quad \text{for all } p \in [1,6) \quad \text{and a.e. in } \Omega \times (0,\infty), \\ c_\varepsilon &\to c && \text{weakly}^* \text{ in } L^\infty(\Omega \times (t,t+1)) \quad \text{for all } t \ge 0, \\ \nabla n_{1,\varepsilon} &\to \nabla n_1 && \text{weakly in } L^{\frac{4}{3}}_{\mathrm{loc}}([0,\infty); L^{\frac{4}{3}}(\Omega)), \\ \nabla n_{2,\varepsilon} &\to \nabla n_2 && \text{weakly in } L^{\frac{4}{3}}_{\mathrm{loc}}([0,\infty); L^{\frac{4}{3}}(\Omega)), \\ \nabla c_\varepsilon &\to \nabla c && \text{weakly}^* \text{ in } L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega)), \\ \nabla u_\varepsilon &\to \nabla u && \text{weakly in } L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)), \\ Y_\varepsilon u_\varepsilon &\to u && \text{in } L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)), \\ n_{1,\varepsilon} &\to n_1 && \text{in } L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)), \\ n_{2,\varepsilon} &\to n_2 && \text{in } L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \end{aligned} \tag{2.12}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Thus we see that (n_1, n_2, c, u) is a weak solution to (1.1) in the sense of Definition 1.1, which means the end of the proof. $\square$

3. Proof of Theorem 1.3 (Eventual smoothness and stabilization). In this section we will prove Theorem 1.3. The following lemma plays an important role in the proof of Theorem 1.3.

LEMMA 3.1.

(i) *Assume that $a_1, a_2 \in (0,1)$. Then there exists $C > 0$ such that for all $\varepsilon > 0$,*

$$\int_0^\infty\int_\Omega (n_{1,\varepsilon} - N_1)^2 \le C, \qquad \int_0^\infty\int_\Omega (n_{2,\varepsilon} - N_2)^2 \le C,$$

where $N_1 = \frac{1-a_1}{1-a_1a_2}$, $N_2 = \frac{1-a_2}{1-a_1a_2}$.

(ii) *Assume $a_1 \ge a_2 > 0$. Then there exists $C > 0$ such that for all $\varepsilon > 0$,*

$$\int_0^\infty\int_\Omega n_{1,\varepsilon}^2 \le C, \qquad \int_0^\infty\int_\Omega (n_{2,\varepsilon} - 1)^2 \le C.$$

Proof. Due to arguments similar to those in [4, Lemmas 4.1–4.4], by using the energy functions

$$\mathcal{G}_{\varepsilon,B} := \int_\Omega \left(n_{1,\varepsilon} - N_1 \log \frac{n_{1,\varepsilon}}{N_1} \right) + \int_\Omega \left(n_{2,\varepsilon} - N_2 \log \frac{n_{2,\varepsilon}}{N_2} \right) + \frac{B}{2}\int_\Omega c_\varepsilon^2$$

in the case that $a_1, a_2 \in (0,1)$, and

$$\mathcal{G}_{\varepsilon,B} := \int_\Omega n_{1,\varepsilon} + \int_\Omega \left(n_{2,\varepsilon} - \log n_{2,\varepsilon} \right) + \frac{B}{2}\int_\Omega c_\varepsilon^2$$

in the case that $a_1 \ge 1 > a_2 > 0$, we can see this lemma. $\square$

Proof of Theorem 1.3. According to an argument similar to that in the proof of [5, Lemmas 3.4 and 3.5], for all $\eta > 0$ and $p \in (1,\infty)$ there are $T > 0$, $\varepsilon_0 > 0$ and $C_1 > 0$ such that for all $t > T$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < \eta, \quad \|n_{1,\varepsilon}^p(\cdot,t)\|_{L^p(\Omega)} \le C_1, \quad \|n_{2,\varepsilon}^p(\cdot,t)\|_{L^p(\Omega)} \le C_1.$$

We next consider the estimate for u_ε. Since $\nabla \cdot u_\varepsilon = 0$, it follows from the Young inequality, the Poincaré inequality, boundedness of $\nabla\Phi$ and (2.1) that there exists $C_2 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 &= -2\int_\Omega |\nabla u_\varepsilon|^2 - 2\int_\Omega u_\varepsilon \cdot (Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + 2\int_\Omega u_\varepsilon \cdot (\gamma n_{1,\varepsilon} + \delta n_{2,\varepsilon})\nabla\Phi \\
&= -2\int_\Omega |\nabla u_\varepsilon|^2 - 2\int_\Omega u_\varepsilon \cdot (Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon \\
&\quad + 2\gamma\int_\Omega u_\varepsilon \cdot (n_{1,\varepsilon} - n_{1,\infty})\nabla\Phi + 2\delta\int_\Omega u_\varepsilon \cdot (n_{2,\varepsilon} - n_{2,\infty})\nabla\Phi \\
&\le -\int_\Omega |\nabla u_\varepsilon|^2 - 2\int_\Omega u_\varepsilon \cdot (Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon \\
&\quad + C_2\int_\Omega (n_{1,\varepsilon} - n_{1,\infty})^2 + C_2\int_\Omega (n_{2,\varepsilon} - n_{2,\infty})^2,
\end{aligned}$$

where $(n_{1,\infty}, n_{2,\infty}) = (N_1, N_2)$ in the case that $a_1, a_2 \in (0,1)$ and $(n_{1,\infty}, n_{2,\infty}) = (0,1)$ in the case that $a_1 \ge 1 > a_2 > 0$. Then, noticing from straightforward calculations that $\int_\Omega u_\varepsilon \cdot (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \frac{1}{2}\int_\Omega \nabla \cdot (Y_\varepsilon u_\varepsilon)|u_\varepsilon|^2 = 0$, thanks to Lemma 3.1, we obtain from integration of the above inequality over $(0,\infty)$ that there exists $C_3 > 0$ such that

$$\int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \le C_3.$$

According to an argument similar to that in the proof of [5, Lemmas 3.7–3.11], there exist $\alpha' > 0$, $T^* > T$, $C_4 > 0$ such that for all $t > T^*$ there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1)$,

$$\begin{aligned} \|n_{1,\varepsilon}\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4, & \|n_{2,\varepsilon}\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4,\\ \|c_\varepsilon\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4, & \|u_\varepsilon\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4. \end{aligned}$$

Then aided by arguments similar to those in the proofs of [5, Corollary 3.3–Lemma 3.13], from (2.11) there are $\alpha' \in (0,1)$ and $T_0 > 0$ as well as a subsequence $\varepsilon_j \searrow 0$ such that for all $t > T_0$

$$n_{1,\varepsilon} \to n_1, \quad n_{2,\varepsilon} \to n_2, \quad c_\varepsilon \to c, \quad u_\varepsilon \to u \quad \text{in } C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])$$

as $\varepsilon = \varepsilon_j \searrow 0$, and then

$$\begin{aligned} \|n_1\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4, & \|n_2\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4,\\ \|c\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4, & \|u\|_{C^{1+\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[t,t+1])} &\le C_4. \end{aligned} \tag{3.1}$$

Then we obtain

$$n_1, n_2, c, u \in C^{2+\alpha', 1+\frac{\alpha'}{2}}(\overline{\Omega}\times[T_0,\infty)).$$

Finally, from (3.1) the solution (n_1,n_2,c,u) of (2.1) constructed in (2.12) fulfills

$$n_1(\cdot,t) \to N_1, \quad n_2(\cdot,t) \to N_2, \quad c(\cdot,t) \to 0, \quad u(\cdot,t) \to 0 \quad \text{in } C^1(\overline{\Omega}) \quad (t\to\infty)$$

in the case that $a_1, a_2 \in (0,1)$, and

$$n_1(\cdot,t) \to 0, \quad n_2(\cdot,t) \to 1, \quad c(\cdot,t) \to 0, \quad u(\cdot,t) \to 0, \quad \text{in } C^1(\overline{\Omega}) \quad (t\to\infty)$$

in the case that $a_1 \ge 1 > a_2 > 0$, which enable us to see Theorem 1.3. □

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