Proceedings of EQUADIFF 2017
pp. 107–116

ON LYAPUNOV STABILITY IN HYPOPLASTICITY*

VICTOR A. KOVTUNENKO†, PAVEL KREJČÍ‡, ERICH BAUER§, LENKA SIVÁKOVÁ¶, AND ANNA V. ZUBKOVA‖

Abstract. We investigate the Lyapunov stability implying asymptotic behavior of a nonlinear ODE system describing stress paths for a particular hypoplastic constitutive model of the Kolymbas type under proportional, arbitrarily large monotonic coaxial deformations. The attractive stress path is found analytically, and the asymptotic convergence to the attractor depending on the direction of proportional strain paths and material parameters of the model is proved rigorously with the help of a Lyapunov function.

Key words. Nonlinear ODE, rate-independent problem, asymptotic behavior, attractor, Lyapunov function, proportional loading, hypoplasticity, granular media

AMS subject classifications. 37B25, 34D20, 74C15

1. Introduction. A rate-independent nonlinear ODE system describing the constitutive stress–strain relation for hypoplastic granular materials like cohesionless soil or broken rock is investigated here. The hypoplastic constitutive equation is of the rate type, incrementally non-linear and based on the hypoplastic concept proposed by Kolymbas [8]. Various physical aspects of hypoplastic models are discussed in engineering literature, e. g., [3, 5, 6, 11, 12, 13]. For mathematical approaches to granular and multiphase media within the variational theory, we refer to [1, 7, 9]. An important feature of the hypoplastic concept is the asymptotic behavior under monotonic proportional loading paths accompanied with a sweeping out of the memory on the initial state. This is a general property also observed in experiments with granular materials. Although for particular monotonic strain paths some numerical simulations and analytical investigations indicate the existence of asymptotic states pointed out, e. g., in [10, Chapter 3.4], a rigorous mathematical proof is missing so far. The main difficulty of developing proper mathematical tools suitable for hypoplastic models is a strongly nonlinear behavior of the corresponding ODE.

For a particular simplified version of a hypoplastic model by Bauer [2] we identify the domain of physical parameters of the model which guarantee that proportional strain paths are stable in the sense of Lyapunov. Our proof of asymptotic stability for unrestricted monotonic deformations is inspired by the rate-independent technique developed in [4].

*This work was supported by the OeAD Scientific & Technological Cooperation (WTZ CZ 01/2016) financed by the Austrian Federal Ministry of Science, Research and Economy (BMWFW) and by the Czech Ministry of Education, Youth and Sports (MŠMT).

†Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstr.36, 8010 Graz, Austria, and Lavrent'ev Institute of Hydrodynamics, Siberian Division of the Russian Academy of Sciences, 630090 Novosibirsk, Russia (victor.kovtunenko@uni-graz.at).

‡Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic (krejci@math.cas.cz).

§Institute of Applied Mechanics, Graz University of Technology, Technikerstr.4, 8010 Graz, Austria (erich.bauer@tugraz.at).

¶Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7, 166 29 Praha 6, Czech Republic, (lenka.sivak@gmail.com).

‖Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstr.36, 8010 Graz, Austria (anna.zubkova@uni-graz.at).

2. Problem of Lyapunov stability. Consider the general form of a hypoplastic constitutive equation of the Kolymbas type [8] in which the objective stress rate can be stated as follows:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{L}(\boldsymbol{\sigma}) : \dot{\boldsymbol{\varepsilon}} + \mathbf{N}(\boldsymbol{\sigma})\|\dot{\boldsymbol{\varepsilon}}\|, \tag{2.1}$$

where $\mathbf{L}(\boldsymbol{\sigma})$ is a fourth order tensor and $\mathbf{N}(\boldsymbol{\sigma})$ is a second order tensor depending on the stress $\boldsymbol{\sigma}$. The current Cauchy stress tensor $\boldsymbol{\sigma}$ and the strain rate tensor $\dot{\boldsymbol{\varepsilon}}$ are assumed to be symmetric and of second order. The right-hand side of (2.1) is positively homogeneous of degree one in $\dot{\boldsymbol{\varepsilon}}$. With respect to the Frobenius norm $\|\dot{\boldsymbol{\varepsilon}}\| = \sqrt{\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}}$ the constitutive equation is incrementally nonlinear.

Here we consider the particular version of (2.1) by Bauer [2] in a simplified manner:

$$\overset{\circ}{\boldsymbol{\sigma}} = c\big\{a^2 \mathrm{tr}(\boldsymbol{\sigma})\dot{\boldsymbol{\varepsilon}} + \frac{1}{\mathrm{tr}(\boldsymbol{\sigma})}(\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}})\boldsymbol{\sigma} + a(2\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})I)\|\dot{\boldsymbol{\varepsilon}}\|\big\}, \tag{2.2}$$

with the constitutive constants $c < 0$ and $a > 0$. We emphasize that the second term in the right-hand side of (2.2) is nonlinear in $\boldsymbol{\sigma}$.

For the following investigations we consider cartesian coordinates and we assume coaxial deformations such that $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ and $\dot{\varepsilon}_{12} = \dot{\varepsilon}_{13} = \dot{\varepsilon}_{23} = 0$. Then the objective stress rate $\overset{\circ}{\boldsymbol{\sigma}}$ equals to the material time derivative, i.e. the rate $\dot{\boldsymbol{\sigma}}$. For the constitutive equation (2.2) only negative principal stresses are relevant. In this case, using the Voigt notation of the time-dependent 3-vector-valued functions

$$t \mapsto \sigma : \ \mathbb{R}_+ \mapsto \mathbb{R}^3_-, \quad t \mapsto \dot{\varepsilon} : \ \mathbb{R}_+ \mapsto \mathbb{R}^3,$$

the stress components σ_i and strain rate components $\dot{\varepsilon}_i$ can be combined in

$$\begin{aligned}\sigma &= (\sigma_1, \sigma_2, \sigma_3)^\top := (\sigma_{11}, \sigma_{22}, \sigma_{33})^\top,\\ \dot{\varepsilon} &= (\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3)^\top := (\dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, \dot{\varepsilon}_{33})^\top.\end{aligned}$$

Here $^\top$ swaps between rows and columns. We use respective vector notation for the inner product and the associated Euclidean norm:

$$\sigma \cdot \dot{\varepsilon} := \sum_{i=1}^{3} \sigma_i \dot{\varepsilon}_i, \quad \|\dot{\varepsilon}\| := \sqrt{\dot{\varepsilon} \cdot \dot{\varepsilon}}, \quad \mathrm{tr}(\sigma) := \sigma_1 + \sigma_2 + \sigma_3.$$

In this case, $\mathrm{tr}(\boldsymbol{\sigma}) = \mathrm{tr}(\sigma)$ and $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \sigma \cdot \dot{\varepsilon}$, hence from (2.2) we derive the corresponding matrix equation

$$\dot{\sigma} = c\big(L(\sigma)\dot{\varepsilon} + N(\sigma)\|\dot{\varepsilon}\|\big), \tag{2.3a}$$

with the corresponding 3-by-3 symmetric matrix L depending on σ:

$$L(\sigma) = a^2 \mathrm{tr}(\sigma) I + \frac{1}{\mathrm{tr}(\sigma)} \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 & \sigma_1\sigma_3 \\ \sigma_1\sigma_2 & \sigma_2^2 & \sigma_2\sigma_3 \\ \sigma_1\sigma_3 & \sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3b}$$

and the 3-vector

$$N(\sigma) = 2a\sigma - \frac{a}{3}\mathrm{tr}(\sigma)\mathbf{1}, \quad \mathbf{1} := (1,1,1)^\top, \tag{2.3c}$$

where we have employed the usual matrix product rule, e.g.:

$$\{L(\sigma)\dot{\varepsilon}\}_{i=1}^{3} = \Big\{\sum_{j=1}^{3} L(\sigma)_{ij}\dot{\varepsilon}_j\Big\}_{i=1}^{3}.$$

We consider here strain paths pointing in one fixed direction. Since the dynamical system (2.3) is rate-independent, without loss of generality we can assume that the loading speed is constant and consider the strain in the form

$$\varepsilon(t) = tU, \quad \|U\| = 1, \quad t \geq 0, \tag{2.4a}$$

along a prescribed unit vector $U = (U_1, U_2, U_3)^\top \in \mathbb{R}^3$. Here t is to be interpreted as a dimensionless monotonically increasing time-like loading parameter. Physically, $\mathrm{tr}(U) < 0$ corresponds to proportional compression and $\mathrm{tr}(U) > 0$ to extension. After inserting (2.4a) in (2.3a), due to $d\,\varepsilon/dt = U$ we get the equivalent system

$$\frac{d}{dt}\sigma = c\{L(\sigma)U + N(\sigma)\}. \tag{2.4b}$$

The ODE (2.4b) for the unknown vector $\sigma(t)$ is considered for $t > 0$, with a prescribed initial condition

$$\sigma(0) = \sigma^0, \tag{2.4c}$$

where $\sigma^0 = (\sigma_1^0, \sigma_2^0, \sigma_3^0)^\top \in \mathbb{R}^3_-$ is a fixed vector. In the next sections we study the asymptotic behavior as $t \nearrow \infty$ of solutions to the Cauchy problem (2.4).

3. Isotropic proportional loading. The strain path (2.4a) is said to be *isotropic* if its direction is parallel to the vector $\mathbf{1}$. In the following we consider two proportional strain paths, i.e. isotropic compression and isotropic extension.

3.1. Isotropic compression. According to (2.4a), the case of the monotonic isotropic compression $\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \dot{\varepsilon}_3 < 0$ implies that

$$\varepsilon(t) = tU, \quad U = -\frac{1}{\sqrt{3}}\mathbf{1}. \tag{3.1a}$$

In this particular case, due to $\sigma \cdot \dot{\varepsilon} = \sigma \cdot U = -\frac{1}{\sqrt{3}}\mathrm{tr}(\sigma)$, we have

$$L(\sigma)\dot{\varepsilon} = -\frac{1}{\sqrt{3}}\big(a^2\mathrm{tr}(\sigma)\mathbf{1} + \sigma\big),$$

and the system (2.4b) turns out to be linear

$$\frac{d}{dt}\sigma = A\sigma, \quad t > 0, \tag{3.1b}$$

with the 3-by-3 system matrix

$$A = b\mathbb{1} + dI, \quad b = -c\big(\frac{a^2}{\sqrt{3}} + \frac{a}{3}\big), \quad d = c\big(2a - \frac{1}{\sqrt{3}}\big), \tag{3.1c}$$

where $\mathbb{1}$ stands for the 3-by-3 matrix of ones: $\mathbb{1} := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

The characteristic equation for (3.1) is calculated as

$$\det(A-\lambda I)=(d-\lambda)^2(d+3b-\lambda)=0, \tag{3.2a}$$

it has one double and one single roots:

$$\lambda_1=\lambda_2=d,\quad \lambda_3=d+3b=-c\big(\sqrt{3}a^2-a+\frac{1}{\sqrt{3}}\big)=-c\frac{3a^3+\frac{1}{\sqrt{3}}}{\sqrt{3}a+1}. \tag{3.2b}$$

Recalling that $c<0$, we have

$$\lambda_3>0,\quad \lambda_1=\lambda_2<0\quad \text{for } a>\frac{1}{2\sqrt{3}}. \tag{3.2c}$$

From a physical point of view, the lower bound for the constitutive parameter a in (3.2c) implies a restriction of the granular friction angle as discussed in Section 5. For the isotropic case, this condition is necessary and sufficient for the Lyapunov stability as stated in Theorem 3.1.

Let vectors $V^1,V^2,V^3\in\mathbb{R}^3$ form an orthonormal eigenbasis for the eigenvalues from (3.2b) such that $(A-\lambda_i I)V^i=0$, i.e.

$$\big(b\mathbb{1}+(d-\lambda_i)I\big)V^i=0,\quad i=1,2,3. \tag{3.2d}$$

We note that V^1 and V^2 with the corresponding negative eigenvalues λ_1 and λ_2 lie in the deviatoric stress plane due to $\mathrm{tr}(V^i)\mathbf{1}=\mathbb{1}V^i=0$ for $i=1,2$ in (3.2d), thus

$$V^1=\frac{(p,q,-p-q)^\top}{\sqrt{2(p^2+q^2+pq)}},\quad V^2=\frac{(2p+q,-p-2q,-p+q)^\top}{\sqrt{6(p^2+q^2+pq)}},\quad p,q\in\mathbb{R},$$

for example, $V^1=\frac{1}{\sqrt{6}}(1,1,-2)^\top$ and $V^2=\frac{1}{\sqrt{2}}(1,-1,0)^\top$. For the positive eigenvalue λ_3, we normalize the eigenvector perpendicular to the deviatoric stress plan as follows

$$V^3=-\frac{1}{\sqrt{3}}\mathbf{1}, \tag{3.2e}$$

which coincides with U in the isotropic case.

The following exponential stability theorem is a straightforward consequence of the formulas (3.2).

Theorem 3.1. *(Isotropic compression)*

The solution of the linear problem (3.1) *with initial condition* (2.4c) *for given* $\sigma^0\in\mathbb{R}^3_-$ *is expressed by the explicit formula*

$$\sigma(t)=\sum_{i=1}^{3}(\sigma^0\cdot V^i)V^i e^{\lambda_i t} \tag{3.3a}$$

in terms of the orthonormal eigenbasis (V^1,V^2,V^3) *corresponding to the eigenvalues* $\lambda_1=\lambda_2$ *and* λ_3 *from* (3.2).

If $a>a_\star=\frac{1}{2\sqrt{3}}$ *and* $c<0$*, then the dynamic system* (3.1) *is exponentially stable as* $t\nearrow\infty$ *in the sense of Lyapunov:*

$$\sigma(t)-\sigma_{V^3}(t)=\big(\sigma(0)-\sigma_{V^3}(0)\big)e^{2c\big(a-\frac{1}{2\sqrt{3}}\big)t} \tag{3.3b}$$

with respect to the attractive trajectory along the V^3-axis:

$$\sigma_{V^3}(t) = (\sigma^0 \cdot V^3)V^3 e^{\lambda_3 t}. \tag{3.3c}$$

Conversely, if $a < a_\star$, then $\sigma(t) - \sigma_{V^3}(t)$ diverges according to (3.3b).

A typical configuration is illustrated in Figure 3.1. In the left plot (a), the strain

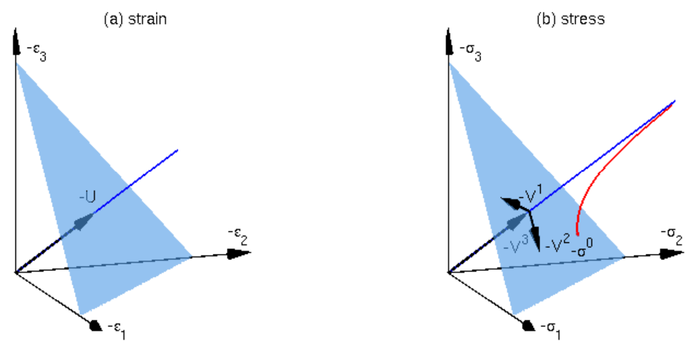

FIG. 3.1. *(a) strain space ; (b) stress space*

path in the direction of -U is depicted in the first octant of the $(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3)$-coordinates. In the right plot (b), in the first octant of the $(-\sigma_1, -\sigma_2, -\sigma_3)$-coordinate system there are presented the stress path attracting the axis along $-V^3$ vector, and the eigenbasis vectors $-V^1$ and $-V^2$ lying in the deviatoric stress plane.

If the initial stress in (2.4c) is isotropic such that $\sigma^0 = sV^3$ with some $s \in \mathbb{R}_+$, then $\sigma(t) = sV^3 e^{\lambda_3 t}$ uniquely solves the system (3.1) under the initial condition (2.4c). This case is the direct consequence of the formula of the solution (3.3a) given in Theorem 3.1. Such $\sigma(t)$ remains isotropic and propagates along the V^3-axis as $t \nearrow \infty$. In the general case when $\sigma^0 \neq sV^3$, an asymptotic stress path attracting the V^3-axis is illustrated in plot (b) of Figure 3.1.

3.2. Isotropic extension. In the case of monotonic isotropic extension, we have $U = \frac{1}{\sqrt{3}}\mathbf{1}$ in (3.1a). It follows that $b = c(\frac{a^2}{\sqrt{3}} - \frac{a}{3})$ and $d = c(2a + \frac{1}{\sqrt{3}})$ in (3.1c). Calculated from (3.2b), the corresponding eigenvalues $\lambda_1 = \lambda_2 = c(2a + \frac{1}{\sqrt{3}})$ and $\lambda_3 = c(\sqrt{3}a^2 + a + \frac{1}{\sqrt{3}})$ are negative since $c < 0$. Therefore, due to the representation formula (3.3a), starting at arbitrary initial stress $\sigma^0 \in \mathbb{R}^3_-$, the stress $\sigma(t)$ decays exponentially to zero as $t \nearrow \infty$ under isotropic extension.

In the next section we investigate the stress path under a non-isotropic loading.

4. Non-isotropic proportional strain paths. For the case of non-isotropic proportional loading, the strain is expressed by formula (2.4a) with an arbitrary unit vector $U \in \mathbb{R}^3$. As mentioned above, this can describe both loading, i.e. compression, and unloading, i.e. extension, tests according to the sign of the trace of U.

The constitutive equation (2.4b) with L and N from (2.3b) and (2.3c) takes the specific form depending on U as a parameter:

$$\frac{d}{dt}\sigma = c\big\{a\big(aU - \frac{1}{3}\mathbf{1}\big)\mathrm{tr}(\sigma) + \big(2a + \frac{\sigma\cdot U}{\mathrm{tr}(\sigma)}\big)\sigma\big\}. \tag{4.1a}$$

The right-hand side of (4.1a) is a nonlinear vector function of σ and represents the principal difficulty in the analysis.

We start with the following two consequences of formula (4.1a) which will be used in the sequel. First, after scalar multiplication of (4.1a) with $\mathbf{1}$ using the fact that $\sigma\cdot\mathbf{1} = \mathrm{tr}(\sigma)$, it follows that

$$\frac{d}{dt}\mathrm{tr}(\sigma) = c\big\{a(a\mathrm{tr}(U) + 1)\mathrm{tr}(\sigma) + (\sigma\cdot U)\big\}. \tag{4.1b}$$

Second, multiplying (4.1a) with $-U$ we get

$$\frac{d}{dt}(-\sigma\cdot U) = c\big\{a\big(-a + \frac{1}{3}\mathrm{tr}(U)\big)\mathrm{tr}(\sigma) - \big(2a + \frac{\sigma\cdot U}{\mathrm{tr}(\sigma)}\big)(\sigma\cdot U)\big\}. \tag{4.1c}$$

Analogously with (3.3c) we look for a linear attractive trajectory of (4.1a) such that $\frac{d}{dt}\sigma = \lambda_3\sigma$ which can be expressed in the form

$$\sigma(t) = (\sigma^0\cdot V^3)V^3 e^{\lambda_3 t} \tag{4.2a}$$

with unknown parameters $\lambda_3 \in \mathbb{R}$ and nonzero $V^3 \in \mathbb{R}^3$ such that $\mathrm{tr}(V^3) \neq 0$. When $\sigma^0\cdot V^3 = 0$, this special case describes the attractive point 0.

Inserting (4.2a) in (4.1b), since $\frac{d}{dt}\mathrm{tr}(\sigma) = \lambda_3\mathrm{tr}(\sigma)$ and $\frac{\sigma\cdot U}{\mathrm{tr}(\sigma)} = \frac{V^3\cdot U}{\mathrm{tr}(V^3)}$ we get

$$\lambda_3 = c\big(a^2\mathrm{tr}(U) + a + \frac{V^3\cdot U}{\mathrm{tr}(V^3)}\big).$$

Substituting this expression together with (4.2a) in (4.1a) such that

$$(a^2\mathrm{tr}(U) - a)V^3 = (a^2U - \frac{1}{3}a\mathbf{1})\mathrm{tr}(V^3),$$

we find a vector $V^3 = a^2U - \frac{1}{3}a\mathbf{1}$ with the trace $\mathrm{tr}(V^3) = a^2\mathrm{tr}(U) - a$ satisfying this equality, then $\frac{V^3\cdot U}{\mathrm{tr}(V^3)} = \frac{-a + \frac{1}{3}\mathrm{tr}(U)}{-a\,\mathrm{tr}(U) + 1}$, and, consequently, after normalization we arrive at

$$\lambda_3 = c\,\frac{\mathrm{tr}(U)\big(-a^3\mathrm{tr}(U) + \frac{1}{3}\big)}{-a\,\mathrm{tr}(U) + 1}, \quad V^3 = \frac{aU - \frac{1}{3}\mathbf{1}}{\sqrt{a^2 - \frac{2}{3}a\,\mathrm{tr}(U) + \frac{1}{3}}}. \tag{4.2b}$$

The above formula is meaningless if $U = \frac{1}{3a}\mathbf{1}$, that is, $a = \frac{1}{\sqrt{3}}$ and $U = \frac{1}{\sqrt{3}}\mathbf{1}$. According to (4.1a), this corresponds to the special case of fully isotropic extension along every stress direction. As well the case $a\,\mathrm{tr}(U) - 1 = 0$ implying $\mathrm{tr}(V^3) = 0$ should be excluded from the consideration.

If $\lambda_3 > 0$ in (4.2b), then $\sigma(t)$ from (4.2a) propagates as $t \nearrow \infty$ exponentially along the V^3-direction. This behavior corresponds to the sketch in Figure 3.1. Otherwise, if $\lambda_3 < 0$, then $\sigma(t) \searrow 0$ which implies unloading.

We note that $-V^3$ in (4.2b) will be directed strictly inside the first octant $\mathbb{R}^3_+$, if a and the direction U of loading in (2.4a) are such that $-aU + \frac{1}{3}\mathbf{1} > 0$, hence $-a\,\mathrm{tr}(U) + 1 > 0$. In this case, for $\sigma(t) \in \mathbb{R}^3_-$ it holds $\sigma^0 \cdot V^3 > 0$.

In particular, for the isotropic compression with $U = -\frac{1}{\sqrt{3}}\mathbf{1}$, from (4.2b) it follows formulas (3.2b) of λ_3 and (3.2e) of V^3.

Next we look for the orthogonal projection of any solution σ of (4.1a) on the V^3-axis, that is

$$\sigma_{V^3}(t) := (\sigma(t) \cdot V^3)V^3. \tag{4.3a}$$

The equivalent form of (4.1a) reads

$$\frac{d}{dt}\sigma = c\big\{a\sqrt{(a^2 - \frac{2}{3}a\,\mathrm{tr}(U) + \frac{1}{3})}\; V^3\mathrm{tr}(\sigma) + \big(2a + \frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big)\sigma\big\}, \tag{4.3b}$$

after multiplication (4.3b) with V^3 we derive the following equation

$$\frac{d}{dt}(\sigma_{V^3}) = c\big\{a\sqrt{(a^2 - \frac{2}{3}a\,\mathrm{tr}(U) + \frac{1}{3})}\; V^3\mathrm{tr}(\sigma) + \big(2a + \frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big)\sigma_{V^3}\big\} \tag{4.3c}$$

for σ_{V^3} from (4.3a). The subtraction of (4.3c) from (4.3b) provides formula for the difference

$$\frac{d}{dt}(\sigma - \sigma_{V^3}) = c\big(2a + \frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big)(\sigma - \sigma_{V^3}). \tag{4.3d}$$

Now we introduce the Lyapunov function $\Lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by

$$\Lambda(t) := \frac{1}{2}\|\sigma(t) - \sigma_{V^3}(t)\|^2, \tag{4.4a}$$

which expresses the distance between the trajectories $\sigma(t)$ and $\sigma_{V^3}(t)$. Differentiating (4.4a) with respect to time and using (4.3d) we get the differential equation for Λ:

$$\frac{d}{dt}\Lambda(t) = 2c\big(2a + \frac{\sigma(t) \cdot U}{\mathrm{tr}(\sigma(t))}\big)\Lambda(t), \quad t > 0. \tag{4.4b}$$

Either negative or positive sign of the factor $2c\big(2a + \frac{\sigma(t) \cdot U}{\mathrm{tr}(\sigma(t))}\big)$ in (4.4b) provides, respectively, either Lyapunov stability or instability of the system. This is the key issue of the following theorem.

THEOREM 4.1. *(Non-isotropic proportional loading) Let $\delta > 0$ be arbitrary fixed, and let $U \in \mathbb{R}^3$ and $a > 0$ be such that $U \neq \frac{1}{3a}\mathbf{1}$ and the following inequalities hold*

$$-a\,\mathrm{tr}(U) + 1 > 0, \quad \big(-2a^2 + \frac{1}{3}\big)\mathrm{tr}(U) + a > 0. \tag{4.5a}$$

For $c < 0$ and arbitrary initial data $\sigma^0 \in \mathcal{C}_\delta$ lying in the cone

$$\mathcal{C}_\delta := \{\sigma \in \mathbb{R}^3_- : \ (-\sigma) \cdot \big(U + (2a + \frac{\delta}{2c})\mathbf{1}\big) > 0\}, \tag{4.5b}$$

any solution $\sigma(t)$ of the nonlinear problem (4.1a) *endowed with the initial condition* (2.4c) *satisfies the inequality*

$$-2a - \frac{\sigma(t) \cdot U}{\mathrm{tr}(\sigma(t))} < \frac{\delta}{2c}, \quad t \geq 0. \tag{4.5c}$$

Moreover, if $\sigma(t) \in \mathbb{R}^3_-$, then $\sigma(t) \in \mathcal{C}_\delta$ for all $t \geq 0$.

In particular, by virtue of (4.5c)*, the dynamical system* (4.1a) *is exponentially stable as $t \nearrow \infty$ in the sense of Lyapunov:*

$$\|\sigma(t) - \sigma_{V^3}(t)\| \leq \|\sigma(0) - \sigma_{V^3}(0)\| e^{-\frac{1}{2}\delta t} \tag{4.5d}$$

with respect to the orthogonal projection $\sigma_{V^3}(t) = (\sigma(t) \cdot V^3)V^3$ on the V^3-axis, where V^3 is determined in formula (4.2b).

Proof. The main challenge is to prove the uniform bound in (4.5c). To do so, we subtract the equation (4.1b), multiplied with $-\frac{\sigma \cdot U}{\mathrm{tr}^2(\sigma)}$, from the equation (4.1c), divided by $\mathrm{tr}(\sigma)$, to calculate that

$$\frac{d}{dt}\big(-\frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big) = c\big\{a\big(-a + \frac{1}{3}\mathrm{tr}(U)\big) - a(-a\mathrm{tr}(U) + 1)\frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big\}.$$

By adding and subtracting the term $2a^2(-a\mathrm{tr}(U) + 1)$ here, this yields

$$\begin{aligned} \frac{d}{dt}\big(-2a - \frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big) &= c\big\{a\big[(-2a^2 + \frac{1}{3})\mathrm{tr}(U) + a\big] + a(-a\mathrm{tr}(U) + 1)\big(-2a - \frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big)\big\} \\ &< ca(-a\mathrm{tr}(U) + 1)\big(-2a - \frac{\sigma \cdot U}{\mathrm{tr}(\sigma)}\big), \end{aligned}$$

where we have used the second inequality in (4.5a) and $c < 0$ for the estimation. The integration of this inequality with respect to t and employing the initial condition (2.4c) results in the following upper bounds

$$-2a - \frac{\sigma(t) \cdot U}{\mathrm{tr}(\sigma(t))} < \big(-2a - \frac{\sigma^0 \cdot U}{\mathrm{tr}(\sigma^0)}\big)e^{ca(-a\mathrm{tr}(U)+1)t} \leq -2a - \frac{\sigma^0 \cdot U}{\mathrm{tr}(\sigma^0)},$$

when the first inequality in (4.5a) holds. This proves the inequality (4.5c) for the initial data σ^0 chosen such that

$$-2a - \frac{\sigma^0 \cdot U}{\mathrm{tr}(\sigma^0)} < \frac{\delta}{2c}.$$

Since $-\sigma^0$ is chosen in the first octant, multiplying the latter inequality with $\mathrm{tr}(\sigma^0) < 0$ we obtain the equivalent inequality

$$-\sigma^0 \cdot U - \mathrm{tr}(\sigma^0)\big(2a + \frac{\delta}{2c}\big) = (-\sigma^0) \cdot \big(U + (2a + \frac{\delta}{2c})\mathbf{1}\big) > 0,$$

which determines the cone in (4.5b).

If (4.5c) holds, then the integration of (4.4) leads immediately to the inequality (4.5d) and completes the proof. □

4.1. Analytic expression of the normalized stress. As a corollary, we consider the normalized stress $\hat{\sigma}$ defined as

$$\hat{\sigma} = \frac{\sigma}{\mathrm{tr}(\sigma)}. \tag{4.6a}$$

Similarly to (4.1) we derive the linear equation

$$\frac{d}{dt}\hat{\sigma} = \frac{1}{\mathrm{tr}(\sigma)}\frac{d}{dt}\sigma - \frac{\hat{\sigma}}{\mathrm{tr}(\sigma)}\frac{d}{dt}(\mathrm{tr}(\sigma)) = ca\big\{(-a\,\mathrm{tr}(U)+1)\hat{\sigma} + aU - \frac{1}{3}\mathbf{1}\big\}, \tag{4.6b}$$

which can be solved analytically:

$$\hat{\sigma}(t) = \frac{-aU + \frac{1}{3}\mathbf{1}}{-a\,\mathrm{tr}(U)+1} + \Big(\hat{\sigma}(0) - \frac{-aU + \frac{1}{3}\mathbf{1}}{-a\,\mathrm{tr}(U)+1}\Big)e^{ca(-a\,\mathrm{tr}(U)+1)t}. \tag{4.6c}$$

This analytical formula entails directly the next result.

Theorem 4.2. *(Normalized stress) If $U \in \mathbb{R}^3$ is such that $-a\,\mathrm{tr}(U)+1>0$, then $\hat{\sigma}(t) \to \dfrac{-aU + \frac{1}{3}\mathbf{1}}{-a\,\mathrm{tr}(U)+1}$ exponentially as $t \to \infty$ according to* (4.6).

From Theorem 4.2 we also conclude that no restriction is imposed on a for proportional loading with $\mathrm{tr}(U) < 0$.

5. Discussion. Let us make a few comments on Theorem 4.1.

Remark 1. According to (4.5d) we can establish that the maximal cone $\mathcal{C}_\delta$ is not less than $\mathcal{C}_0$ when passing $\delta \searrow 0^+$.

Remark 2. Conditions (4.5a) are sufficient for the Lyapunov stability.

Remark 3. If $\mathrm{tr}(U) \le 0$, in particular, when $U \in \mathbb{R}^3_-$ and the vector $-U$ lies in the first octant, then the first inequality in (4.5a) always holds.

Remark 4. If $\mathrm{tr}(U) \le 0$ and $a > \dfrac{1}{2\sqrt{3}}$, then we calculate

$$\big(-2a^2 + \frac{1}{3}\big)\mathrm{tr}(U) + a > \frac{1}{6}\big(\mathrm{tr}(U) + \sqrt{3}\big) \ge 0$$

since $|\mathrm{tr}(U)| \le \sqrt{3}\|U\| = \sqrt{3}$ in (2.4a). This suffices the second inequality in (4.5a).

Remark 5. In particular, for $U = -\dfrac{1}{\sqrt{3}}\mathbf{1}$ under isotropic compression, the second inequality in (4.5a) implies that $2\sqrt{3}(a + \dfrac{1}{\sqrt{3}})(a - \dfrac{1}{2\sqrt{3}}) > 0$ which holds for $a > \dfrac{1}{2\sqrt{3}}$. The inequality $\dfrac{1}{\sqrt{3}} - 2a < \dfrac{\delta}{2c}$ in (4.5c) determines the cone $\mathcal{C}_\delta$ for σ such that $-\sigma > 0$ and $-\mathrm{tr}(\sigma)(2a - \dfrac{1}{\sqrt{3}} + \dfrac{\delta}{2c}) > 0$ in (4.5b). In this particular case, the maximal cone $\mathcal{C}_0$ implies $\sigma < 0$ component-wisely and $-\mathrm{tr}(\sigma)(2a - \dfrac{1}{\sqrt{3}}) > 0$, that is the first octant when $a > \dfrac{1}{2\sqrt{3}}$. This fact is in accordance with Theorem 3.1.

Remark 6. For the granular friction angle ϕ such that $a = \dfrac{2\sqrt{2}\sin\phi}{\sqrt{3}(3-\sin\phi)}$, from $a > a_\star = \dfrac{1}{2\sqrt{3}} \approx 0.2887$, we have $\phi > \phi_\star$ and calculate the critical value $\sin\phi_\star = \dfrac{3}{1+4\sqrt{2}}$ and $\phi_\star \approx 26.78°$.

The analytical result for the minimum value of parameter a to achieve asymptotic behavior under isotropic straining can also be confirmed with numerical simulation, i.e. by numerical integration of the constitutive equation we could get the same results. For smaller values of a the stress path diverges.

Acknowledgments. V.A.K. and A.V.Z. are supported by the Austrian Science Fund (FWF) project P26147-N26 (PION), V.A.K. thanks the Austrian Academy of Sciences (OeAW) and A.V.Z. thanks IGDK1754 for the support.
P.K. has been supported by the GAČR Grant GA15-12227S and RVO: 67985840.
The authors thank N. Krenn and G.A. Rogacheva for the help in organizing the WTZ Austria - Czech Republic cooperation meetings.

REFERENCES

[1] B. D. Annin, V. A. Kovtunenko and V. M. Sadovskii, *Variational and hemivariational inequalities in mechanics of elastoplastic, granular media, and quasibrittle cracks*, in Analysis, Modelling, Optimization, and Numerical Techniques, G. O. Tost, O. Vasilieva, eds., Springer Proc. Math. Stat., 121 (2015), 49–56.

[2] E. Bauer, *Modelling limit states within the framework of hypoplasticity*, AIP Conf. Proc., 1227, J. Goddard, P. Giovine and J. T. Jenkin, eds., AIP, 2010, pp. 290–305.

[3] E. Bauer and W. Wu, *A hypoplastic model for granular soils under cyclic loading*, Proc. Int. Workshop Modern Approaches to Plasticity, D. Kolymbas, ed., Elsevier, 2010, pp. 247–258.

[4] M. Brokate and P. Krejčí, *Wellposedness of kinematic hardening models in elastoplasticity*, RAIRO Modél. Math. Anal. Numér., (1998), 177–209.

[5] G. Gudehus, *Physical Soil Mechanics*, Springer, Berlin, Heidelberg, 2011.

[6] W. Huang and E. Bauer, *Numerical investigations of shear localization in a micro-polar hypoplastic material*, Int. J. Numer. Anal. Meth. Geomech., 27 (2003), pp. 325–352.

[7] A.M. Khludnev and V.A. Kovtunenko, *Analysis of Cracks in Solids*, WIT-Press, Southampton, Boston, 2000.

[8] D. Kolymbas, *An outline of hypoplasticity*, Arch. Appl. Mech., 61 (1991), pp. 143–151.

[9] V.A. Kovtunenko and A.V. Zubkova, *Mathematical modeling of a discontinuous solution of the generalized Poisson–Nernst–Planck problem in a two-phase medium*, Kinet. Relat. Mod., 11 (2018), pp.119–135.

[10] A. Niemunis, *Extended Hypoplastic Models for Soils*, Habilitation thesis, Ruhr University, Bochum, 2002.

[11] A. Niemunis and I. Herle, *Hypoplastic model for cohesionless soils with elastic strain range*, Mech. Cohes.-Frict. Mat., 2 (1997), pp. 279–299.

[12] B. Svendsen, K. Hutter and L. Laloui, *Constitutive models for granular materials including quasi-static frictional behaviour: toward a thermodynamic theory of plasticity*, Continuum Mech. Therm., 4 (1999), pp. 263–275.

[13] W. Wu, E. Bauer and D. Kolymbas, *Hypoplastic constitutive model with critical state for granular materials*, Mech. Mater., 23 (1996), pp. 45–69.