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NONEXISTENCE OF SOLUTIONS OF SOME INEQUALITIES WITH GRADIENT NONLINEARITIES AND FRACTIONAL LAPLACIAN*

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Abstract. We obtain sufficient conditions for nonexistence of nontrivial solutions for some classes of nonlinear partial differential inequalities containing the fractional powers of the Laplace operator.

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1. Introduction. The necessary conditions of solvability of nonlinear partial differential equations and inequalities has been recently studied by many authors.

In particular, in [4, 1, 2] (see also references therein) such conditions were obtained for some classes of nonlinear elliptic and parabolic inequalities, in particular containing integer powers of the Laplacian, using the test function method developed by S. Pohozaev [5]. However, for similar inequalities with fractional powers of the Laplacian the problem remained open. For such inequalities with nonlinear terms of the form u^q it was considered in [6].

In the present paper we obtain sufficient conditions for nonexistence of solutions for a class of elliptic inequalities with fractional powers of the Laplacian and nonlinear terms of the form $|Du|^q$, as well as for elliptic systems of the same type.

The rest of the paper consists of three sections. In §2 we obtain some auxiliary estimates for the fractional Laplacian used further. In §3, we prove a nonexistence theorem for single elliptic inequalities with fractional powers of the Laplacian, and in §4, for systems of such inequalities.

2. Auxiliary estimates. We define the operator $(-\Delta)^s$ by the formula

$$(-\Delta)^s u(x) \stackrel{\text{def}}{=} c_{n,s} \cdot \text{p.v.} \int\limits_{\mathbb{R}^n} \frac{(-\Delta)^{[s]}u(y) - (-\Delta)^{[s]}u(x)}{|x-y|^{n+2\{s\}}}\, dy, \tag{2.1}$$

where

$$c_{n,s} \stackrel{\text{def}}{=} \frac{2^{\{s\}}\Gamma\left(\frac{n+\{s\}}{2}\right)}{\pi^{n/2}\left|\Gamma\left(-\frac{\{s\}}{2}\right)\right|}$$

(see, e.g., [3]).

We will use definition (2.1) for the proof of the following Lemmas.

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Lemma 2.1. *Let $s \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$ and $q, q' > 1$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Consider a function $\varphi_1 : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\varphi_1(x) \stackrel{\text{def}}{=} \begin{cases} 1 & (|x| \le 1), \\ (2-|x|)^\lambda & (1 < |x| < 2), \\ 0 & (|x| \ge 2) \end{cases} \tag{2.2}$$

with $\lambda > [s] + 2q'$. Then one has

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^s \varphi_1(x)|^{q'} (1+|x|)^{-\frac{\alpha q'}{q}} \varphi_1^{1-q'}(x)\, dx < \infty. \tag{2.3}$$

Remark. In the Mitidieri–Pohozaev approach such estimates were established by direct calculation of the iterated Laplacian of the test functions. This does not work for the fractional Laplacian, so we need to establish some additional estimates.

Proof. Let $\frac{3}{2} < |x| < 1$. Use (2.1) with notation $f(x,y) = \dfrac{\Delta^{[s]}\varphi_1(x) - \Delta^{[s]}\varphi_1(y)}{|x-y|^{n+2\{s\}}}$:

$$|(-\Delta)^s \varphi_1)(x)| = c_{n,s} \left| \int\limits_{\mathbb{R}^n} f(x,y)\, dy \right| = c_{n,s} \left| \sum_{i=1}^{2} \int\limits_{D_i} f(x,y)\, dy \right|, \tag{2.4}$$

where

$$D_1 \stackrel{\text{def}}{=} \{y \in \mathbb{R}^{\mathrm{n}} : |\mathrm{x}-\mathrm{y}| \ge (2-|\mathrm{x}|)/2\},$$
$$D_2 \stackrel{\text{def}}{=} \{y \in \mathbb{R}^{\mathrm{n}} : |\mathrm{x}-\mathrm{y}| < (2-|\mathrm{x}|)/2\}$$

(here and below the singular integrals are understood in the sense of the Cauchy principal value).

For any $\varepsilon \in (0, 2\{s\})$, since we have $|x-y| \ge (2-|x|)/2$ in D_1, we get

$$\begin{aligned} \int\limits_{D_1} f(x,y)\, dy &= \int\limits_{D_1} \frac{(-\Delta)^{[s]}\varphi_1(x) - (-\Delta)^{[s]}\varphi_1(y)}{|x-y|^{n+2\{s\}}}\, dy \le \\ &\le (-\Delta)^{[s]}\varphi_1(x) \int\limits_{D_1} \frac{dy}{|x-y|^{n+2\{s\}}} \le \\ &\le (-\Delta)^{[s]}\varphi_1(x) \cdot \left(\frac{2-|x|}{2}\right)^{\varepsilon - 2s} \int\limits_{D_1} \frac{dy}{|x-y|^{n+\varepsilon}} \le c_1 (2-|x|)^{\lambda+\varepsilon-2s} \end{aligned} \tag{2.5}$$

with some constant $c_1 > 0$.

Finally, the Lagrange Mean Value Theorem implies that

$$\begin{aligned} &\int\limits_{D_2} f(x,y)\, dy = \\ &= \frac{1}{2} \int\limits_{\tilde{D}_2} \frac{2(-\Delta)^{[s]}\varphi_1(x) - (-\Delta)^{[s]}\varphi_1(x+z) + (-\Delta)^{[s]}\varphi_1(x-z)}{|z|^{n+2s}}\, dz \le \\ &\le c_2 \cdot \max_{z \in \tilde{D}_2} |((2-|x+z|)^{\lambda-[s]})''| \int\limits_{\tilde{D}_2} \frac{|z|^2}{|z|^{n+2\{s\}}\, dy} = \\ &= c_3 \cdot \max_{z \in \tilde{D}_2} (2-|x+z|)^{\lambda-[s]-2} \cdot \int\limits_{\tilde{D}_2} \frac{dz}{|z|^{n+2\{s\}-2}}, \end{aligned}$$

where $\tilde{D}_2 = \{z \in \mathbb{R}^n : |z| < (2-|x|)/2\}$, with constants c_2, $c_3 > 0$.

For $z \in \tilde{D}_2$ we have

$$2-|x+z| = 2-|x|+|x|-|x+z| \le (2-|x|)+|z| \le \frac{3}{2}(2-|x|).$$

Hence

$$\int\limits_{D_2} f(x,y)\,dy \le c_4(2-|x|)^{\lambda-[s]-2} \tag{2.6}$$

with some constant $c_4 > 0$.

Combining (2.4)–(2.6), we obtain

$$|(-\Delta)^s\varphi_1(x)| \le c_5(2-|x|)^{\lambda-[s]-2} \tag{2.7}$$

and consequently

$$|(-\Delta)^s\varphi_1(x)|^{q'}(1+|x|)^{-\frac{\alpha q'}{q}}\varphi_1^{1-q'}(x) \le$$

$$\le c_6(2-|x|)^{(\lambda-[s]-2)q'-\lambda(1-q')} = c_6(2-|x|)^{\lambda-([s]+2)q'}$$

with some constants $c_5, c_6 > 0$ independent of x, which implies (2.3). □

LEMMA 2.2. *Let $s \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$ and $q, q' > 1, \frac{1}{q}+\frac{1}{q'} = 1$. For a family of functions $\varphi_R(x) = \varphi_1\left(\frac{x}{R}\right)$, where $R > 0$, one has*

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^s\varphi_R(x)|^{q'}(1+|x|)^{-\frac{\alpha q'}{q}}\varphi_R^{1-q'}(x)\,dx \le cR^{n-2q's-\frac{\alpha q'}{q}} \tag{2.8}$$

for any $R > 0$ and some $c > 0$ independent of R.

Proof. By (2.1) and a change of variables $\tilde{y} = \frac{y}{R}$, we have

$$(-\Delta)^s\varphi_R(x) = R^{-2s}(-\Delta)^s\varphi_1(x). \tag{2.9}$$

Substituting (2.9) into the left-hand side of (2.8) and applying Lemma 2.1, we obtain the claim. □

3. Single elliptic inequalities. Now consider the nonlinear elliptic inequality

$$(-\Delta)^s u \ge c|Du|^q(1+|x|)^\alpha \quad (x \in \mathbb{R}^n), \tag{3.1}$$

where $s > 1$, $c > 0$, $q > 1$ and α are real numbers.

DEFINITION 3.1. *A weak solution of inequality* (3.1) *is a function $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ such that for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^n)$ there holds the inequality*

$$-\int\limits_{\mathbb{R}^n}(Du, D((-\Delta)^{s-1}\varphi))\,dx \ge c\int\limits_{\mathbb{R}^n}|Du|^q(1+|x|)^\alpha\varphi\,dx. \tag{3.2}$$

THEOREM 3.2. *Inequality* (3.1) *has no nontrivial (i.e., distinct from a constant a.e.) weak solutions for* $\alpha > 1 - 2s$ *and*

$$1 < q \le \frac{n+\alpha}{n-2s+1}. \tag{3.3}$$

Proof. Introduce a test function $\varphi_R(x) = \varphi_1\left(\frac{x}{R}\right)$, where $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ is nonnegative and

$$\varphi_1(x) = \begin{cases} 1 & (|x| \le 1), \\ 0 & (|x| \ge 2). \end{cases} \tag{3.4}$$

Substituting $\varphi(x) = \varphi_R(x)$ into (3.1) and applying the Hölder inequality, we get

$$\begin{aligned} & c\int\limits_{\mathbb{R}^n} |Du|^q(1+|x|)^\alpha \varphi_R\,dx \le -\int\limits_{\mathbb{R}^n} (Du, D((-\Delta)^{s-1}\varphi))\varphi_R\,dx \le \\ & \le \int\limits_{\mathbb{R}^n} |Du|\cdot|D((-\Delta)^{s-1}\varphi_R)|\,dx \le \left(\int\limits_{\mathbb{R}^n} |Du|^q(1+|x|)^\alpha\varphi_R\,dx\right)^{\frac{1}{q}} \times \\ & \times \left(\int\limits_{\operatorname{supp}|D\varphi_R|} |(-\Delta)^s\varphi_R|^{q'}(1+|x|)^{\frac{\alpha q'}{q}}\varphi_R^{1-q'}\,dx\right)^{\frac{1}{q}}, \end{aligned} \tag{3.5}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence,

$$\int\limits_{\mathbb{R}^n} |Du|^q(1+|x|)^\alpha\varphi_R\,dx \le c\int\limits_{\mathbb{R}^n} |D((-\Delta)^{s-1}\varphi_R)|^{q'}(1+|x|)^{\frac{\alpha q'}{q}}\varphi_R^{1-q'}\,dx. \tag{3.6}$$

From Lemma 2.2 we have

$$\begin{aligned} & \int\limits_{\mathbb{R}^n} |(-\Delta)^s\varphi_R|^{q'}(1+|x|)^{\frac{\alpha q'}{q}}\varphi_R^{1-q'}\,dx \le \\ & cR^{n-q'(2s-1)-\frac{\alpha q'}{q}} \int\limits_{\mathbb{R}^n} |(-\Delta)^s\varphi_1(y)|^{q'}(1+|y|)^{\frac{\alpha q'}{q}}\varphi_1^{1-q'}(y)\,dy, \end{aligned} \tag{3.7}$$

where $y = \frac{x}{R}$. Combining (3.6) and (2.3), since the integral on the right-hand side of (3.7) converges for an appropriate choice of $\varphi_1(y)$, we obtain

$$\int\limits_{\mathbb{R}^n} |Du|^q(1+|x|)^\alpha\varphi_R\,dx \le cR^{n-q'(2s-1)-\frac{\alpha q'}{q}}.$$

Taking $R \to \infty$, in case of strict inequality in (3.3) we come to a contradiction, which proves the claim. In case of equality, we have

$$\int\limits_{\mathbb{R}^n} |Du|^q(1+|x|)^\alpha\,dx < \infty,$$

whence

$$\int\limits_{\mathrm{supp}|D\varphi_R|} |Du|^q(1+|x|)^\alpha \varphi_R\, dx \to 0 \text{ for } R\to\infty$$

and by (3.5)

$$\int\limits_{\mathbb{R}^n} |Du|^q(1+|x|)^\alpha\, dx = 0,$$

which completes the proof in this case as well. □

4. Systems of elliptic inequalities. Here we consider a system of nonlinear elliptic inequalities

$$\begin{cases} (-\Delta)^{s_1} u \ge c_1|Dv|^{q_1}(1+|x|)^{\alpha_1} & (x\in\mathbb{R}^n),\\ (-\Delta)^{s_2} u \ge c_2|Du|^{q_2}(1+|x|)^{\alpha_2} & (x\in\mathbb{R}^n),\end{cases} \tag{4.1}$$

where $s_1>1$, $s_2>1$, $q_1>1$, $q_2>1$, α_1 and α_2 are real numbers.

Definition 4.1. *A weak solution of system of inequalities (3.7) is a pair of functions $(u,v)\in W^{1,q_2}_{\mathrm{loc}}(\mathbb{R}^n)\times W^{1,q_1}_{\mathrm{loc}}(\mathbb{R}^n)$ such that for any nonnegative function $\varphi\in C_0^\infty(\mathbb{R}^n)$ there hold the inequalities*

$$\begin{aligned} \int\limits_{\mathbb{R}^n} (Du, D((-\Delta)^{s_1}\varphi))\, dx &\ge c_1\int\limits_{\mathbb{R}^n} |Dv|^{q_1}(1+|x|)^{\alpha_1}\varphi\, dx,\\ \int\limits_{\mathbb{R}^n} (Dv, D((-\Delta)^{s_2}\varphi))\, dx &\ge c_2\int\limits_{\mathbb{R}^n} |Du|^{q_2}(1+|x|)^{\alpha_2}\varphi\, dx.\end{aligned} \tag{4.2}$$

Denote

$$\begin{aligned}\beta_1 &= q_1((2s_2-1)q_2-(2s_1-1)-\alpha_2)-\alpha_1,\\ \beta_2 &= q_2((2s_1-1)q_1-(2s_2-1)-\alpha_2)-\alpha_2.\end{aligned}$$

We will prove the following

Theorem 4.2. *System* (4.1) *has no nontrivial (i.e., distinct from constants a.e.) weak solutions for*

$$n(q_1q_2-1)\le \max\{\beta_1,\beta_2\}. \tag{4.3}$$

Proof. Introduce a test function $\varphi_R(x)$ as in the proof of the previous theorem. Similarly to (3.5), we get

$$c_1\int\limits_{\mathbb{R}^n} v^{q_1}(1+|x|)^{\alpha_1}\varphi_R\, dx \le \left(\int\limits_{\mathbb{R}^n} |Du|^{q_2}(1+|x|)^{\alpha_2}\varphi_R\, dx\right)^{\frac{1}{q_2}}\times$$
$$\times\left(\int\limits_{\mathrm{supp}|D\varphi_R|} |D((-\Delta)^{s_2}\varphi_R)|^{q_2'}(1+|x|)^{\frac{\alpha_2 q_2'}{q_2}}\varphi_R^{1-q_2'}\, dx\right)^{\frac{1}{q_2'}},$$

$$c_2 \int\limits_{\mathbb{R}^n} u^{q_2}(1+|x|)^{\alpha_2}\varphi_R\,dx \le \left(\int\limits_{\mathbb{R}^n} |Dv|^{q_1}(1+|x|)^{\alpha_1}\varphi_R\,dx\right)^{\frac{1}{q_1}} \times$$
$$\times \left(\int\limits_{\mathrm{supp}|D\varphi_R|} |D((-\Delta)^{s_1}\varphi_R)|^{q_1'}(1+|x|)^{\frac{\alpha_1 q_1'}{q_1}}\varphi_R^{1-q_1'}\,dx\right)^{\frac{1}{q_1'}},$$

where $\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{q_2}+\frac{1}{q_2'}=1$. Estimating the second factors on the right-hand sides of the obtained inequalities similarly to (2.3), we get

$$\int\limits_{\mathbb{R}^n} |Dv|^{q_1}(1+|x|)^{\alpha_1}\varphi_R\,dx \le cR^{\frac{n}{q_2'}-(2s_2-1)-\frac{\alpha_2}{q_2}}\left(\int\limits_{\mathbb{R}^n} u^{q_2}(1+|x|)^{\alpha_2}\varphi_R\,dx\right)^{\frac{1}{q_2}}, \tag{4.4}$$

$$\int\limits_{\mathbb{R}^n} |Du|^{q_2}(1+|x|)^{\alpha_2}\varphi_R\,dx \le cR^{\frac{n}{q_1'}-(2s_1-1)-\frac{\alpha_1}{q_1}}\left(\int\limits_{\mathbb{R}^n} v^{q_1}(1+|x|)^{\alpha_1}\varphi_R\,dx\right)^{\frac{1}{q_1}} \tag{4.5}$$

and, substituting (4.5) into (4.4) and vice versa,

$$\int\limits_{\mathbb{R}^n} |Dv|^{q_1}(1+|x|)^{\alpha_1}\varphi_R\,dx \le cR^{n-\frac{q_1((2s_2-1)q_2-(2s_1-1)-\alpha_2)-\alpha_1}{q_1q_2-1}},$$

$$\int\limits_{\mathbb{R}^n} |Du|^{q_2}(1+|x|)^{\alpha_2}\varphi_R\,dx \le cR^{n-\frac{q_2((2s_1-1)q_1-(2s_2-1)-\alpha_1)-\alpha_2}{q_1q_2-1}}.$$

Passing to the limit as $R\to\infty$, we complete the proof of the theorem similarly to the previous one. ∎

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