Proceedings of EQUADIFF 2017
pp. 359–368

BEHAVIOUR OF THE SUPPORT OF THE SOLUTION APPEARING IN SOME NONLINEAR DIFFUSION EQUATION WITH ABSORPTION *

KENJI TOMOEDA†

Abstract. Numerical experiments suggest interesting properties in the several fields of fluid dynamics, plasma physics and population dynamics. Among such properties, we may observe the interesting phenomena; that is, the *repeated appearance and disappearance phenomena of the region penetrated by the fluid* in the flow through a porous media with absorption. The model equation in two dimensional space is written in the form of the initial-boundary value problem for a nonlinear diffusion equation with the effect of absorption. In this paper we show some numerical examples and prove such phenomena.

Key words. nonlinear diffusion, support dynamics, finite difference scheme

AMS subject classifications. 35K65, 35B99, 65M06

1. Introduction.

We are concerned with the dynamical behaviour of the region penetrated by the fluid in the filtration of the flow through an absorbing medium. The representative filtration is well known as the flow through porous media where the water evaporates. In particular, it is expected that such a seepage exhibits the *repeated appearance and disappearance phenomena of such a region*, which are caused by the interaction between the nonlinear diffusion and the penetration of the fluid from the boundary on which the flowing tide and the ebbing tide occur. To realize such phenomena we introduce the simplest model based on the following nonlinear diffusion equation with absorption in two dimensional space [7, 9] :

$$(1.1) \qquad \begin{cases} v_t(t,x,y) = \Delta(v^m) - cv^p & \text{in} \quad (0,\infty)\times\Omega, \\ v(t,x,y) = \psi(t,x,y) & \text{on} \quad (0,\infty)\times\partial\Omega, \\ v(0,x,y) = v^0(x,y) & \text{on} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with piecewisely smooth boundary $\partial\Omega$, and satisfies the exterior sphere condition. Moreover, $v(\geq 0)$ denotes the density of the fluid, $m > 1$, $0 < p < 1$, $c > 0$, $m + p \geq 2$, and $v^0(x,y)$ and $\psi(t,x,y)$ are nonnegative continuous functions. This equation is also used to describe the propagation of thermal waves in plasma physics [8].

From analytical points of view, the existence and uniqueness of a weak solution and the comparison theorem are proved by Bertsch [1].

We state some mathematical results in one dimensional case. For the initial value problem, Rosenau and Kamin [8] suggested the *support splitting phenomena* in several numerical examples, and we also constructed the initial function for which the *repeated support splitting and merging phenomena* appear [10]. Here the support means the region penetrated by the fluid; that is, the region where $v > 0$. For the

*This work was supported by Japan Society for the Promotion of Science through Grand-in-Aid for Scientific Research(C):No.16K05271.

†Graduate School of Informatics, Kyoto University, Sakyo-ward, Kyoto 606-8501, JAPAN (ktomoeda@acs.i.kyoto-u.ac.jp).

initial-boundary value problem Kersner proved the appearance of the *support splitting phenomena* [5], but he did not show that the *support merging phenomena* appear after the support splits. To investigate the occurrence of the *repeated support splitting and merging phenomena*, we construct two stationary solutions, the one is the support non-splitting solution and the other is the support splitting solution. We proved this occurrence by imposing the periodicity on the boundary value which takes the value greater than the former boundary value and less than the latter [11].

In two dimensional case, to the best of my knowledge, we are unable find any result concerned with the *repeated appearance and disappearance phenomena of the support*. By employing the argument used in the one dimensional case we try to justify the occurrence of such phenomena.

2. Stationary solutions and numerical examples. In this section we consider the profile of the stationary solution $w(x,y)(\geq 0)$ satisfying

$$\left\{\begin{array}{ll}\Delta(w^m)-cw^p=0 & \text{in} \quad \Omega,\\ w(t,x,y)=\varphi(x,y) & \text{on} \quad \partial\Omega,\end{array}\right. \tag{2.1}$$

where $m>1$, $0<p<1$, $c>0$, $m+p\geq 2$, and $\varphi(x,y)$ is a non-negative continuous function on $\partial\Omega$. Then the existence and uniqueness of the solution $w(x)$ follows (see Theorem 12.5 in [12]).

THEOREM 2.1. *The equation (2.1) has the unique solution $w(x,y)$ such that $w^m(x,y)\in C^{2,\frac{p}{m}}(\Omega)\cap C^0(\bar{\Omega})$.*

We introduce the radial solution of (2.1), which is used in the numerical computation. Put

$$\phi(x,y)=\left\{\left(\frac{m-p}{2m}\right)^2 c(x^2+y^2)\right\}^{\frac{1}{m-p}}. \tag{2.2}$$

It is obvious that (2.2) satisfies the first equation of (2.1) with $w=\phi$ and $w(x,y)>0$ for $(x,y)\neq(0,0)$.

To investigate the behaviour of the support of the solution (1.1) we tried numerical computation by our difference scheme, which approximates the following problem instead of (1.1):

$$\left\{\begin{array}{lll}u_t(t,x,y)=mu\Delta u+a(u_x^2+u_y^2)-(m-1)cu^q & \text{in} & (0,\infty)\times\Omega,\\ u(t,x,y)=\psi^{m-1}(t,x,y) & \text{on} & (0,\infty)\times\partial\Omega,\\ u(0,x,y)=(v^0)^{m-1}(x,y) & \text{on} & \Omega,\end{array}\right. \tag{2.3}$$

where $a=\frac{m}{m-1}$ and $q=\frac{m+p-2}{m-1}$, and this equation can be obtained by putting $u=v^{m-1}$ [6, 10].

We put $m=1.5$, $p=0.5$, $c=6$, $\Omega=(-1.5,\ 1.5)\times(-1.5,\ 1.5)$ and the space mesh width $h=\frac{1}{32}$, and show the numerical profiles in Cases I and II.

Case I. The boundary condition $\psi(t,x,y)$ is independent of t; that is,

Example 1. $\psi(t,x,y)=\left\{\phi^{m-1}(x,y)+0.5\right\}^{\frac{1}{m-1}}$ on $\partial\Omega$ (Left figures in Fig. 2.1),

where $v^0(x,y)=\left\{\phi^{m-1}(x,y)+0.5+1.25\cos\theta(x)\cos\theta(y)\right\}^{\frac{1}{m-1}}$ on Ω;

Example 2. $\psi(t,x,y)=\left\{\phi^{m-1}(x,y)-0.5\right\}^{\frac{1}{m-1}}$ on $\partial\Omega$ (Right figures in Fig. 2.1),

where $v^0(x,y)=\left\{[\phi^{m-1}(x,y)-0.5]_+ +1.5\cos^2\theta(x)\cos^2\theta(y)\right\}^{\frac{1}{m-1}}$ on Ω.

In both examples we put $\theta(\eta)=-\dfrac{\pi}{2}+\dfrac{\pi}{3}(\eta+1.5)$ $(-1.5\leq\eta\leq 1.5)$.

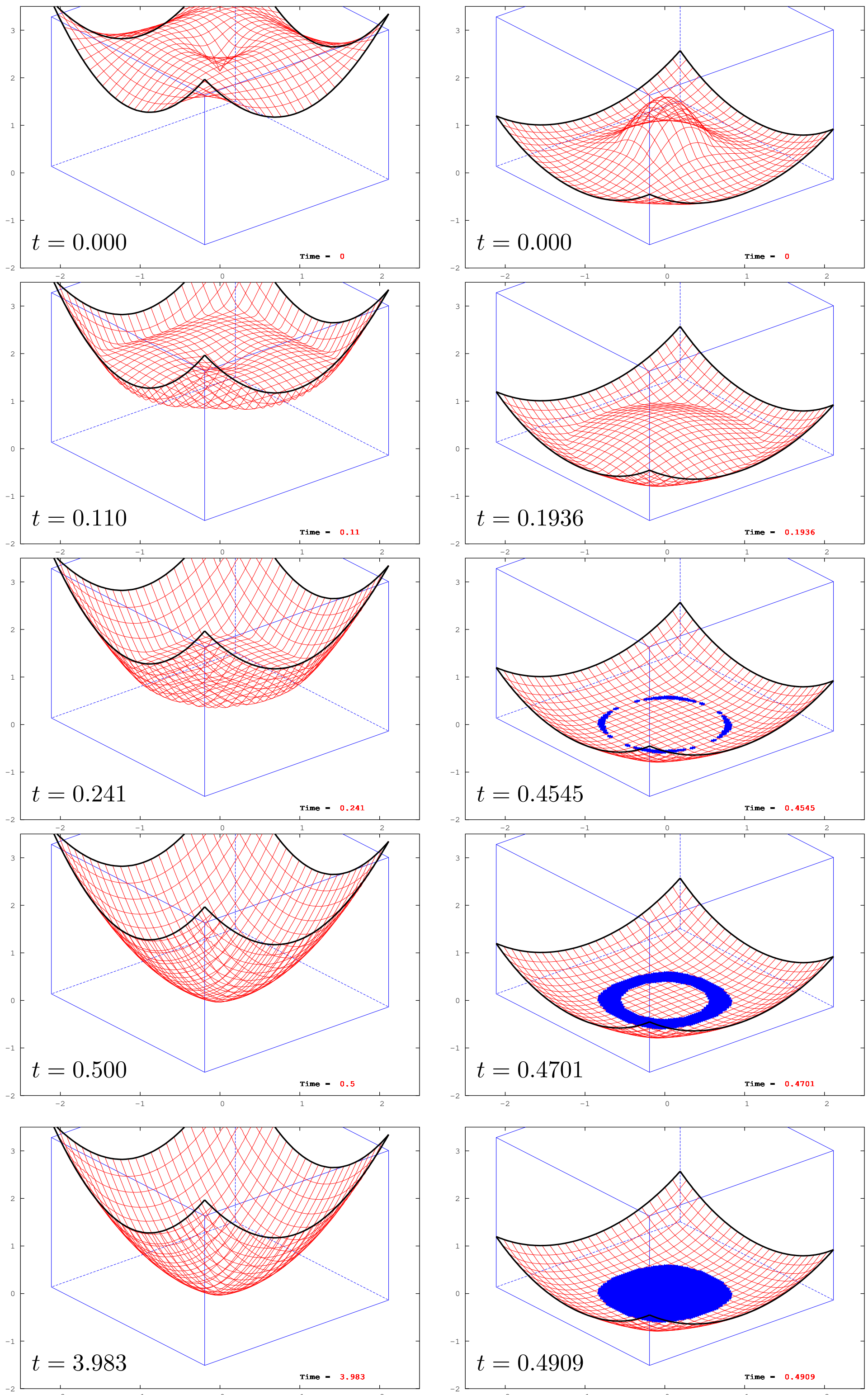

FIG. 2.1. *The non-appearance and appearance phenomena of the region where* $v = 0$

The region where $v = 0$, which is indicated in black, begins to appear in the right figure with $t = 0.4545$, but does not in the left figures. We note that such a region in the right figure with $t = 0.4909$ remains until $t = 3.972$ at which we stop computation. Thus numerical solutions converge to the stationary solutions as t increases, respectively.

Case II. We impose a period on $\psi(t, x, y)$ and put $v^0(x, y) = \phi(x, y)$ on Ω; that is,

Example 3. $\psi(t, x, y) = \left\{\phi^{m-1}(x, y) + 0.5\sin(2\pi t)\right\}^{\frac{1}{m-1}}$ on $\partial\Omega$ (Fig. 2.2);

Example 4. $\psi(t, x, y) = \left\{\phi^{m-1}(x, y) + 0.5\sin(8\pi t)\right\}^{\frac{1}{m-1}}$ on $\partial\Omega$ (Fig. 2.3).

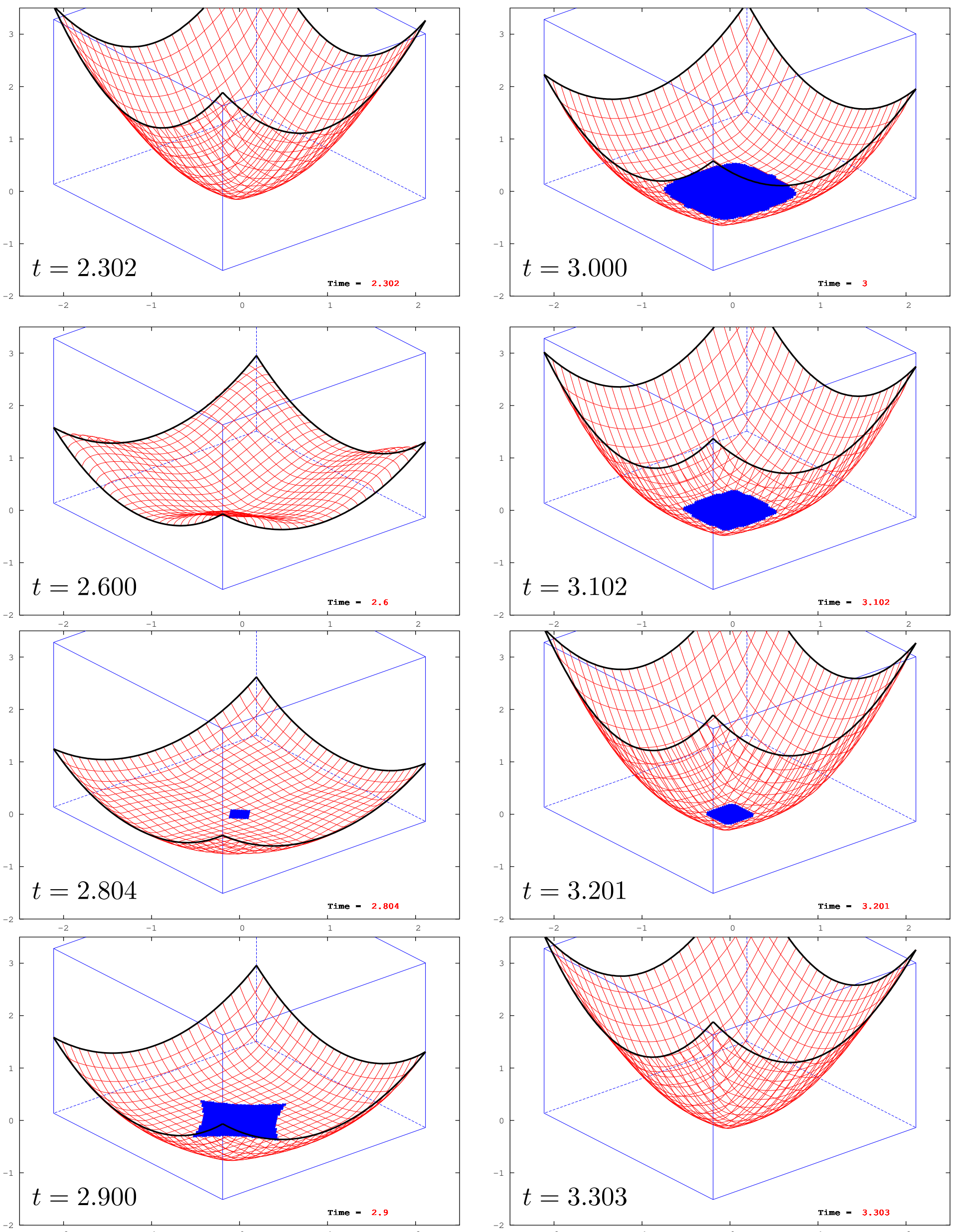

FIG. 2.2. *The repeated appearance and disappearance phenomena of the region where $v = 0$.*

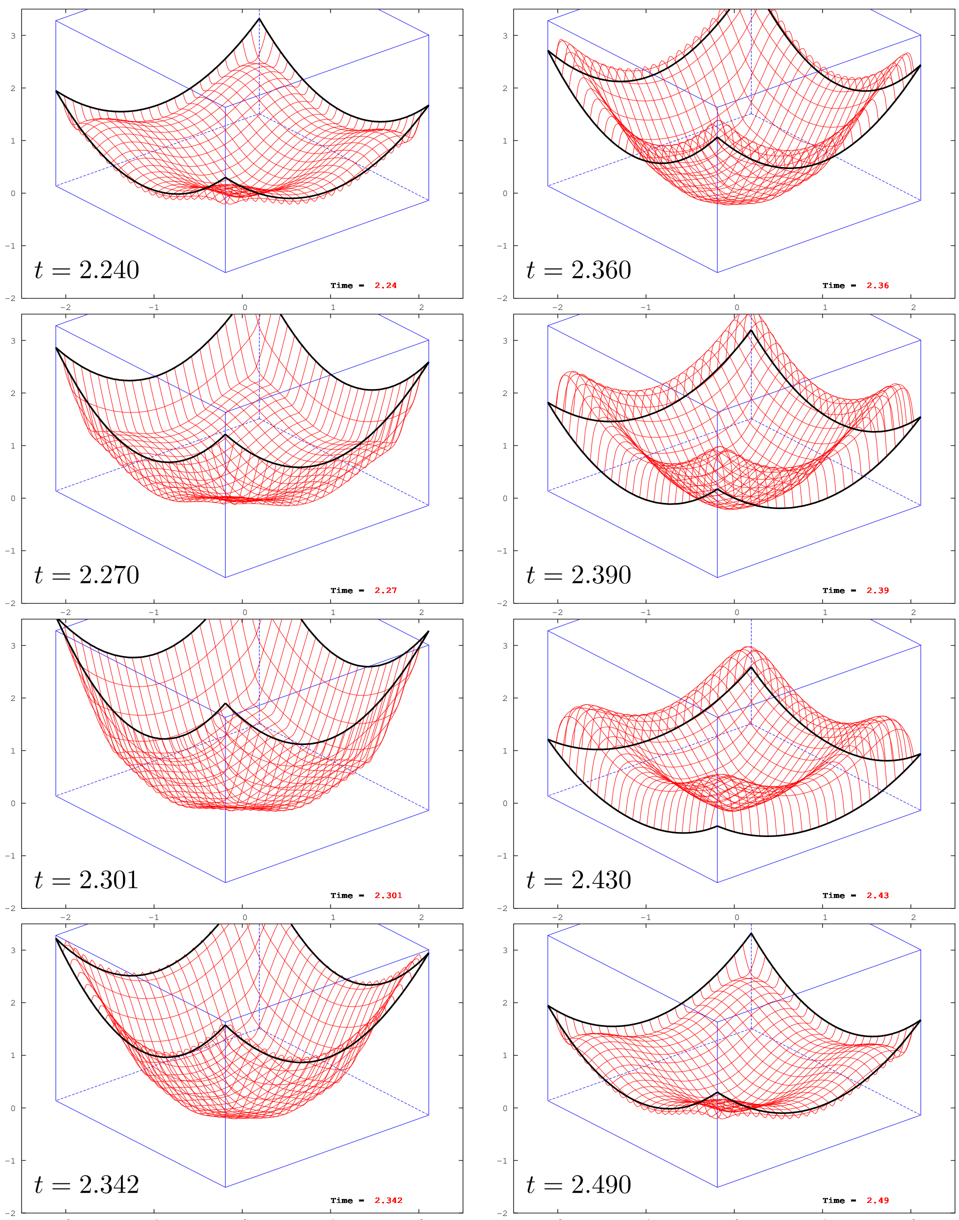

FIG. 2.3. *The appearance of the region where $v = 0$ is not observed.*

In Figs. 2.2 and 2.3 the initial and boundary profiles are located as equal to the same stationary solution $\phi(x, y)$. The increasing and decreasing profiles appear repeatedly as t increases in both figures. The region where $v = 0$ appears at $t = 2.804$ and disappears at $t = 3.303$ in Fig. 2.2. We may observe that the profile of the numerical solution at $t = 2.302$ approximately coincides with the one at $t = 3.303$. The numerical period $1.001 = 3.303 - 2.302$ corresponds to 1.00 of $\psi(t, x, y)$ in Example 3. Thus Fig. 2.2 shows the *repeated appearance and disappearance phenomena of the region where* $v = 0$. On the other hand, Fig. 2.3 shows the numerical period $0.250 =$

2.490 − 2.240, which coincides with $\frac{1}{4}$ of $\psi(t,x,y)$ in Example 4. The numerical solution is close to zero in the neighborhood of $(x,y)=(0,0)$, but not equal to zero. The region where $v=0$ never appears.

We mention our numerical method for (2.3), which is the following explicit finite difference scheme:

$$u_h^{n+1} = P_{k,h} D_{k,h} H_{k,h} u_h^n \quad (n = 0, 1, \cdots). \tag{2.4}$$

Here $u_h^n(x,y)$ is the numerical approximation to the solution $u(t_n,x,y)$. $P_{k,h}$, $D_{k,h}$ and $H_{k,h}$ approximate $u_t = mu\Delta u$, $u_t = -(m-1)cu^q$ and $u_t = a(u_x^2 + u_y^2)$, respectively, and $k \equiv k_{n+1} = t_{n+1} - t_n$ is a variable time step determined by

$$k_{n+1} = \frac{h}{2a \max(\|(u_h^n)_x\|_\infty, \|(u_h^n)_y\|_\infty)}. \tag{2.5}$$

Since $h = \frac{1}{32}$, it is observed that $k_{n+1} \approx 6.0 \times 10^{-4} \sim 5.2 \times 10^{-3}$ in Examples 1-4, which is very small. This may affect the shape of the region where $v=0$. In the right figures of Fig. 2.1 such a region looks like a square after $t = 0.4909$. Unfortunately, we are unable to analyze the appearance of such a figure. However, it is expected that these numerical solutions in Examples 1-4 qualitatively capture the *appearance and disappearance phenomena of the region where* $v=0$.

In the following section we prove the properties appearing in Figs. 2.1–2.2. At the present time it seems difficult for us to prove the occurrence of such phenomena in Fig. 2.3.

3. Stabilization.

THEOREM 3.1 (Stabilization). *Let $v(t,\cdot)$ be the solution of (1.1) with $\psi(t,x,y) = \varphi(x,y)$, where $\varphi(x,y)$ is a non-negative continuous function on $\partial\Omega$. Then $v(t,\cdot)$ converges to the unique stationary solution $w(\cdot)$ of (2.1) in $C(K)$ as $t \to \infty$, where $K \subset \Omega$ is an arbitrary fixed compact set.*

Proof. We state the proof briefly. For the solution $v(t,\cdot)$ we consider a continuous orbit $\gamma = \{v(t,\cdot) : t \geq 0\}$ in $C(K)$. By the result of DiBenedetto [2], γ is precompact in $C(K)$; that is,

$$^\exists\{t_n\}, ^\exists\,\hat{v}(\cdot) : \quad t_n \to \infty \ \text{ and } \ v(t_n,\cdot) \to \hat{v}(\cdot) \in \omega \ \text{ in } \ C(K) \ \text{ as } \ n \to \infty,$$

where ω is the ω-limit set of γ. On the other hand, the following inequality is proved for the solutions $v_1(t,\cdot)$ and $v_2(t,\cdot)$ of (1.1) by Bertsch [1]:

$$\|v_1(t,\cdot) - v_2(t,\cdot)\|_{L^1(\Omega)} \leq e^{Mt} \|v_1(0,\cdot) - v_2(0,\cdot)\|_{L^1(\Omega)} \text{ for } t \geq 0, \tag{3.1}$$

where M is the constant number satisfying

$$(-s^p) - (-r^p) \leq M(s-r) \text{ for any } (0 \leq r \leq s). \tag{3.2}$$

In general, $M = 0$. However, taking the boundedness of the solution v of (1.1) and the stationary solution w of (2.1) into consideration, we can take $|M(v,w)| << 1$ ($M(v,w) < 0$) depending on v and w, and obtain

$$\begin{aligned} \|v(t_n,\cdot) - w(\cdot)\|_{L^1(K)} &\leq \|v(t_n,\cdot) - w(\cdot)\|_{L^1(\Omega)} \\ &\leq e^{M(v,w)t_n} \|v(0,\cdot) - w(\cdot)\|_{L^1(\Omega)} \quad \text{for} \quad t \geq 0, \end{aligned} \tag{3.3}$$

which tends to 0 as $n \to \infty$. Thus $\hat{v}(x,y) = w(x,y)$ holds on K, and the theorem follows from the uniqueness of the stationary solution $w(x,y)$. □

Let $w_i(x,y)$ $(i = 1,2)$ be two non-negative solutions of (2.1) satisfying $w_2(x,y) > w_1(x,y)$ on $\bar{\Omega}$. Assume that $w_1(x,y)$ has the non-empty region where $w_1(x,y) = 0$. Then Theorem 3.1 predicts the following result:

If the time while we keep the boundary value $\psi(t,x,y)$ of $v(t,x,y)$ greater than $w_2(x,y)$ on $\partial\Omega$ is sufficiently long, then $v(t,x,y) > 0$ on Ω. Conversely, if the time while we keep $\psi(t,x,y)$ less than $w_1(x,y)$ on $\partial\Omega$ is sufficiently long, then the region where $v(t,x,y) = 0$ appears.

Thus we may expect the *repeated appearance and disappearance phenomena of the region where* $v = 0$ by imposing the period and magnitude on $\psi(t,x,y)$.

However, since Theorem 3.1 does not guarantee the appearance of the region where $v = 0$ in finite time, it is unclear in Fig. 2.1 and 2.2 whether or not such phenomena occur. So, we will prove it for the specific case in the next section.

4. Galaktionov and Vazquez's particular solution.

Let $m + p = 2$ and $0 < p < 1$. Then we can construct the Galaktionov-Vazquez's particular solution which satisfies the first equation of (1.1)[3, 4]. We briefly state its construction. In the first equation of (2.3) we find $q = 0$ and have

$$u_t = mu\Delta u + a(u_x^2 + u_y^2) - (m-1)c\chi_{\{u>0\}}, \quad a = \frac{m}{m-1}. \tag{4.1}$$

We assume that this explicit solution is written in the form $u(t,x) = f(t) + g(t)h(x,y)$. Then f, g and h satisfy

$$f' + g'h = m(f + gh)g(h_{xx} + h_{yy}) + ag^2(h_x^2 + h_y^2) - (m-1)c, \tag{4.2}$$

where $'$ denotes the derivative with respect to t. Let $h(x,y) = x^2 + y^2$ in (4.2). Then we have

$$\begin{cases} f' = 4mfg - (m-1)c, \\ g' = 4(m+a)g^2. \end{cases} \tag{4.3}$$

Solving (4.3), we obtain a solution for two parameters $\varepsilon > 0$ and $\hat{\sigma} > 0$:

$$\begin{aligned} u(t,x,y) &= \{E - 4(m+a)t\}^{-1} \\ &\quad \times \left[D\{E - 4(m+a)t\}^2 + G\{E - 4(m+a)t\}^{\frac{1}{m}} + x^2 + y^2 \right]_+, \end{aligned} \tag{4.4}$$

$$u(0,x,y) = \varepsilon(x^2 + y^2) + \hat{\sigma}, \tag{4.5}$$

where
$G \equiv G(m,c,\hat{\sigma},\varepsilon) = (\hat{\sigma} - DE)E^{\frac{m-1}{m}}$, $D \equiv D(m,c) = \dfrac{(m-1)c}{4(2m+a)}$, $E \equiv E(\varepsilon) = \varepsilon^{-1}$.

Using the same argument as used in [10], we have

LEMMA 4.1. *Let $\hat{\sigma} < DE$. Then*

$$\frac{\hat{\sigma}}{(m-1)c} < \hat{t}(m,c,\hat{\sigma},\varepsilon) < \hat{T}(m,\varepsilon) \tag{4.6}$$

holds and u satisfies

(4.7) $u(t,x,y) > 0$ *for* $(t,x,y) \in [0,\ \hat{T}(m,\varepsilon)) \times \mathbf{R}^2 \setminus S$,

(4.8) $u(t,x,y) = 0$ *for* $(t,x,y) \in S$,

(4.9) $\lim_{t \nearrow \hat{T}(m,\varepsilon)} u(t,0,0) = 0, \quad \lim_{t \nearrow \hat{T}(m,\varepsilon)} u(t,x,y) = \infty$ *for* $(x,y) \neq (0,0)$,

where

$$\hat{t}(m,c,\hat{\sigma},\varepsilon) = \frac{1}{4(m+a)}\left\{E - \left(\frac{-G}{D}\right)^{\frac{m}{2m-1}}\right\}, \quad \hat{T}(m,\varepsilon) = \frac{E}{4(m+a)}, \tag{4.10}$$

$$S = \Big\{(t,x,y) \mid t \in \Big[\hat{t}(m,c,\hat{\sigma},\varepsilon),\ \hat{T}(m,\varepsilon)\Big) \ \text{and} \ x^2+y^2 \leq \{E-4(m+a)t\}^{\frac{1}{m}}\left[-G - D\{E-4(m+a)t\}^{\frac{2m-1}{m}}\right]\Big\}. \tag{4.11}$$

(See Fig. 4.1 in the next page).

By the simple calculations we can show that $v(t,x,y) = u(t,x,y)^{\frac{1}{m-1}}$ satisfies the first equation of (1.1) for $(t,x) \in ((0,\ \hat{T}(m,\varepsilon)) \times \mathbf{R}^2)\backslash\partial S$. Since $1 < m < 2$, it follows that $\dfrac{1}{m-1} > 1$ and $\dfrac{m}{m-1} > 2$, which implies that $v_t(t,x,y) = \Delta(v^m)(t,x,y) = 0$ hold on $(t,x,y) \in \partial S \backslash (\hat{T}(m,\varepsilon),0,0)$. Thus $v(t,x,y)$ becomes the solution of the first equation of (1.1) on $(0,\hat{T}(m,\varepsilon)) \times \mathbf{R}^2$.

Under the specific case where $m+p=2$ and $0<p<1$ we have

THEOREM 4.2. *Assume that $w(x,y)$ be the stationary solution of (2.1) with the non-empty region where $w(x,y)=0$. Let $v(t,x,y)$ be the solution of (1.1) with $\psi(t,x,y) = w(x,y)$ on $\partial\Omega$. Then the region where $v(t,x,y)=0$ appears in finite time.*

Proof. Without loss of generality we assume that $w(x,y) \equiv 0$ in the neighborhood of $(x,y)=(0,0)$. Let $GV(t,x,y;\ m,c,\hat{\sigma},\varepsilon)$ denote Galaktionov and Vazquez's solution written in the form of the term on the right side of (4.4). From the properties of this solution it is possible to take sufficiently large $\varepsilon(>0)$ so that there exists some constant $t_\varepsilon(>0)$ satisfying

$$GV(0,x,y;\ m,c,0,\varepsilon) \geq w^{m-1}(x,y) \quad \text{on } \bar{\Omega}, \tag{4.12}$$

$$GV(0,x,y;\ m,c,0,\varepsilon) > w^{m-1}(x,y) \quad \text{on } \partial\Omega, \tag{4.13}$$

$$GV(t,x,y;\ m,c,0,\varepsilon) > w^{m-1}(x,y) \quad \text{on } [0,\ t_\varepsilon) \times \partial\Omega. \tag{4.14}$$

Taking the positive number $\hat{\sigma} < DE$, we have $G<0$. Then the region where $GV(t,x,y\ ;\ m,c,\hat{\sigma},\varepsilon) = 0$ appears at $t = \hat{t}(m,c,\hat{\sigma},\varepsilon)(>0)$ by Lemma 4.1. Moreover, since $\hat{t}(m, c, \hat{\sigma}, \varepsilon) \searrow 0$ as $\hat{\sigma} \searrow 0$, we take $\hat{\sigma}$ sufficiently small so that $\hat{t}(m, c, \hat{\sigma}, \varepsilon) < t_\varepsilon$. We fix $\hat{\sigma}$ and ε. Then Theorem 3.1 (Stabilization) yields for sufficiently large t^*

$$GV(0, x, y;\ m, c, \hat{\sigma}, \varepsilon) > u(t^*, x, y) \equiv v^{m-1}(t^*, x, y) \quad \text{on } \bar{\Omega}. \tag{4.15}$$

We have from (4.14) and (4.4)

$$\begin{aligned} GV(t, x, y;\ m, c, \hat{\sigma}, \varepsilon) &> GV(t, x, y;\ m, c, 0, \varepsilon) \\ &> w^{m-1}(x, y) = u(t^* + t, x, y) \\ &\equiv v^{m-1}(t^* + t, x, y) \quad \text{in } [0,\ t_\varepsilon) \times \partial\Omega. \end{aligned} \tag{4.16}$$

Applying the comparison theorem [1], which is concerned with the initial and boundary data, to (4.15) and (4.16), we obtain

$$GV(t, x, y;\ m, c, \hat{\sigma}, \varepsilon) \geq u(t^* + t, x, y) \quad \text{on } [0,\ t_\varepsilon) \times \bar{\Omega}. \tag{4.17}$$

Thus the region where $v(t, \cdot) = u(t, \cdot)^{\frac{1}{m-1}} = 0$ appears at $t = t^* + \hat{t}(m, c, \hat{\sigma}, \varepsilon)$, and the proof is complete. ∎

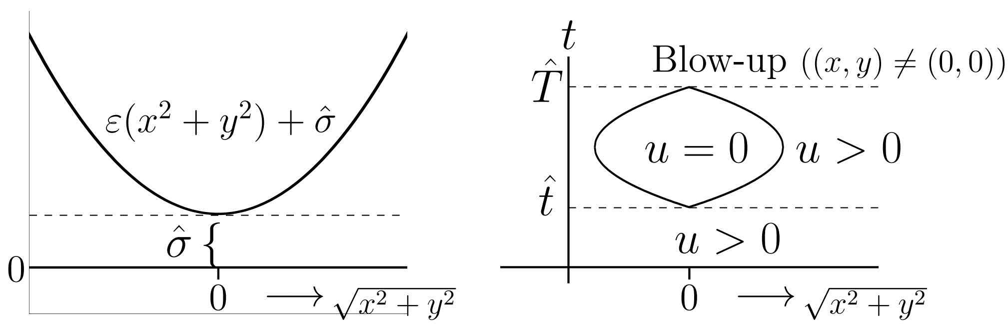

FIG. 4.1. *The initial function and the support of Galaktionov and Vazquez's solution (4.4).*

REFERENCES

[1] M. BERTSCH, *A class of degenerate diffusion equations with a singular nonlinear term*, Nonlinear Anal., 7 (1983), pp.117–127.

[2] E. DIBENEDETTO, *Continuity of weak solutions to a general porous medium equation*, Indiana Univ. Math. J., 32 (1983), pp.83–118.

[3] V. A. GALAKTIONOV AND J. L. VAZQUEZ, *Extinction for a quasilinear heat equation with absorption I. Technique of intersection comparison*, Commun. in Partial Differential Equation, 19 (1994), pp.1075–1106.

[4] V. A. GALAKTIONOV AND J. L. VAZQUEZ, *Extinction for a quasilinear heat equation with absorption II. A dynamical systems approach*, Commun. in Partial Differential Equation, 19 (1994), pp.1107–1137.

[5] R. KERSNER, *Degenerate parabolic equations with general nonlinearities*, Nonlinear Anal., 4 (1980), pp.1043–1062.

[6] T. NAKAKI AND K. TOMOEDA, *A finite difference scheme for some nonlinear diffusion equations in an absorbing medium: support splitting phenomena*, SIAM J. Numer. Anal., 40 (2002), pp.945–964.

[7] P.Y. Polubarinova-Kochina, *Theory of Ground Water Movement*, Princeton Univ. Press, 1962.
[8] P. Rosenau, S. Kamin, *Thermal waves in an absorbing and convecting medium*, Physica, 8D (1983), pp.273–283.
[9] A.E. Scheidegger, *The Physics of Flow through Porous Media*, Third edition, University of Toronto Press,1974.
[10] K. Tomoeda, *Numerically repeated support splitting and merging phenomena in a porous media equation with strong absorption*, Journal Math-for-Industry of Kyushu, 3 (2012), pp.61–68.
[11] K. Tomoeda, *Appearance of repeated support splitting and merging phenomena in a porous media equation with absorption*, Application of Mathematics in Technical and Natural Sciences (AMiTaNS'15), AIP Conference Proceedings, 1684 (2015), pp.080013-1–080013-9.
[12] D. Gilbarg and N. S. Trudinger *Elliptic Partial Differential Equations of Second Order*, Second Edition, Revised Third Printing 1998, Springer.