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A GENERALIZATION OF THE KELLER–SEGEL SYSTEM TO HIGHER DIMENSIONS FROM A STRUCTURAL VIEWPOINT*

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Abstract. We consider initial boundary problems of a two-chemical substances chemotaxis system. In the four-dimensional setting, it was shown that solutions exist globally in time and remain bounded if the total mass is less than $(8\pi)^2$, whereas the solution emanating from some initial data of large magnitude may blows up.

This result can be regarded as a generalization of the well-known 8π problem in the Keller–Segel system to higher dimensions. We will compare mathematical structures of the Keller–Segel system and our system and discuss the difference.

Key words. chemotaxis; global existence; Lyapunov functional; Adams' inequality

AMS subject classifications. 35B45, 35K45, 35Q92, 92C17

1. Problem. Consider the following fully parabolic system:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) & \text{in } \Omega\times(0,\infty),\\ \tau_1 v_t = \Delta v - v + w & \text{in } \Omega\times(0,\infty),\\ \tau_2 w_t = \Delta w - w + u & \text{in } \Omega\times(0,\infty), \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ $(n\in\mathbb{N})$ with smooth boundary $\partial\Omega$, where the parameters τ_1, τ_2, and χ are positive. Suppose that the boundary condition:

$$\frac{\partial u}{\partial\nu} - \chi u\frac{\partial v}{\partial\nu} = v = w = 0 \qquad \text{on } \partial\Omega\times(0,\infty). \tag{1.2}$$

Moreover assume that

$$u(\cdot,0) = u_0,\quad v(\cdot,0) = v_0,\quad w(\cdot,0) = w_0 \qquad \text{in } \Omega, \tag{1.3}$$

where the initial data (u_0, v_0, w_0) satisfies

$$\begin{cases} u_0\in C^0(\overline{\Omega}), & u_0\geq 0 \quad \text{in } \overline{\Omega},\\ v_0\in C^2(\overline{\Omega}), & v_0\geq 0 \quad \text{in } \overline{\Omega},\\ w_0\in C^2(\overline{\Omega}) & u_0\geq 0 \quad \text{in } \overline{\Omega} \end{cases} \tag{1.4}$$

and the boundary condition

$$v_0 = w_0 = 0 \qquad \text{on } \partial\Omega\times(0,\infty). \tag{1.5}$$

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2. Background and motivation. In 1970 Keller and Segel ([17]) proposed a mathematical model describing a movement of cells, which is the following reaction-diffusion system:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u. \end{cases} \tag{2.1}$$

Here functions u and v represent the population of cells and the density of a chemical substance, respectively. The term $-\chi \nabla \cdot (u \nabla v)$ represents the chemotaxis effect.

From a mathematical view point, the type of (2.1) has been studied well (see surveys [14, 12, 1]). Under suitable boundary conditions, smooth solutions of (2.1) conserve the total mass, i.e., $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ for all $t > 0$. Considering the simplified system of (2.1) such as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ v_t = \Delta v + u \end{cases}$$

in $\mathbb{R}^n$, we can confirm that the above system is invariant by the standard scaling $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t)$ and $v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$ with $\lambda > 0$ and

$$\|u_\lambda(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \lambda^{2-n} \|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \qquad t > 0.$$

Hence in the above sense, the two-dimensional setting is the critical case. Moreover, in [10, 19], it is shown that the system (2.1) has the particular mathematical structure, the Lyapunov functional:

$$\frac{d}{dt}\mathcal{F}(u(t), v(t)) + \mathcal{D}(u(t), v(t)) = 0 \qquad \text{for all } t \in (0, T),$$

where

$$\begin{aligned} \mathcal{F}(u,v) &= \int_\Omega (u \log u - \chi uv) + \frac{\chi}{2} \int_\Omega |\nabla v|^2 + \frac{\chi}{2} \int_\Omega v^2, \\ \mathcal{D}(u,v) &= \int_\Omega u |\nabla(\log u - \chi v)|^2. \end{aligned}$$

This Lyapunov functional is the key ingredient in the study of behaviors of solutions to the Keller–Segel system (2.1) ([19, 15, 25]). The Trudinger–Moser inequality ([5]): for all $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that for all $u \in H^1(\Omega)$,

$$\log \left(\int_\Omega e^{|u(x)|}\, dx \right) \le \left(\frac{1}{2 \cdot 8\pi} + \varepsilon \right) \|\nabla u\|^2_{L^2(\Omega)} + C_\varepsilon \|u\|_{L^1(\Omega)},$$

plays a role of judgement of the balance of terms in the Lyapunov functional in the critical case $n = 2$. This combination implies "8π-problem", which seems to be one of the main topic in the study of the Keller–Segel system ([16, 3, 18, 2]). Precisely, in the two-dimensional and radially symmetric setting, the behavior of radial solutions to the Neumann problem of (2.1) is classified as follows:

- if $\|u_0\|_{L^1\Omega} < 8\pi/\chi$ then the solution exists globally and remains bounded ([19]).

- there exists some initial data with $\|u_0\|_{L^1\Omega} > 8\pi/\chi$ such that the corresponding solution blows up in finite [11, 13].

As to nonradial solutions, the critical constant changed to $4\pi/\chi$ ([19, 15]). Here the critical constants $8\pi/\chi$ and $4\pi/\chi$ come from the critical constants in the Trudinger–Moser inequality. As to the subcritical case, in [20] it was established that for all regular initial data the system (2.1) has global bounded solution in the one-dimensional setting. As to the supercritical case, that is, the higher dimensional case $n \geq 3$, solutions of (2.1) exist globally in time and converge to the constant steady state provided that $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|\nabla v_0\|_{L^n(\Omega)}$ is sufficiently small ([4]). Moreover there are many finite time blowup radial solutions with $\|u_0\|_{L^1(\Omega)} = m$ for all $m > 0$ ([25]).

Motivation. The motivation of this study is to give a generalization of the Keller–Segel system (2.1) to higher dimensions in the sense of a mathematical structure. Indeed, the system (1.1) has a similar structural properties as the Keller–Segel system. Smooth solutions of (1.1) conserve the total mass, i.e., $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ for all $t > 0$. We confirm that the simplified system of (1.1) such as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \tau_1 v_t = \Delta v + w, \\ \tau_2 w_t = \Delta w + u \end{cases}$$

in $\mathbb{R}^n$ is invariant by the following standard scaling

$$\begin{cases} u_\lambda(x,t) & = \lambda^4 u(\lambda x, \lambda^2 t), \\ v_\lambda(x,t) & = v(\lambda x, \lambda^2 t), \\ w_\lambda(x,t) & = \lambda^2 w(\lambda x, \lambda^2 t) \qquad (\lambda > 0). \end{cases}$$

Moreover we have

$$\|u_\lambda(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \lambda^{4-n} \|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \qquad t > 0.$$

Hence the four-dimensional setting is the critical case in the above sense. Moreover the system (1.1) has a Lyapunov functional, which seems to be a natural generalization of one of the Keller–Segel system (2.1):

$$\frac{d}{dt}\mathcal{F}(u(t),v(t)) + \mathcal{D}(u(t),v(t)) = 0 \qquad \text{for all } t \in (0,T),$$

where

$$\mathcal{F}(u,v) = \int_\Omega (u \log u - \chi u v) + \frac{\tau_1 \tau_2 \chi}{2} \int_\Omega |v_t|^2 + \frac{\chi}{2} \int_\Omega |(-\Delta + 1)v|^2,$$

$$\mathcal{D}(u,v) = \chi(\tau_1 + \tau_2) \int_\Omega \left(|\nabla v_t|^2 + |v_t|^2 \right) + \int_\Omega u |\nabla (\log u - \chi v)|^2.$$

Now, in the critical case $n = 4$, an Adams type inequality, which is a generalization of the Trudinger–Moser inequality to higher derivatives, plays a key role to decide the balance of the Lyapunov functional in the same way that the Trudinger–Moser inequality does in the study of the Keller–Segel system. Hence the system (1.1) has a generalized mathematical structure of the Keller–Segel system.

3. Main results. Our main results read as follows.

THEOREM 3.1 ([8]). *Let $n \leq 3$. Suppose that (u_0, v_0, w_0) satisfies* (1.4) *and* (1.5). *Then the problem* (1.1)–(1.2)–(1.3) *has a unique classical positive solution, which exists globally in time. Moreover the solution is uniformly bounded in time in the sense that*

$$\sup_{t\in[0,\infty)} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{2,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \Big) < \infty.$$

REMARK 3.2. *This result corresponds to the study of the Keller–Segel system in the one-dimensional case. In [20] it is shown that for all regular initial data the Keller–Segel system* (2.1) *has global and bounded solution.*

THEOREM 3.3 ([8]). *Let $n = 4$. Suppose that the initial data (u_0, v_0, w_0) satisfies* (1.4), (1.5) *and*

$$\int_\Omega u_0 < \frac{(8\pi)^2}{\chi}.$$

Then the problem (1.1)–(1.2)–(1.3) *has a unique classical positive solution, which exists globally in time. Moreover the solution is uniformly bounded in time in the sense that*

$$\sup_{t\in[0,\infty)} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{2,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \Big) < \infty.$$

REMARK 3.4. *As to the initial-boundary problem of the Keller–Segel system with the mixed boundary condition, nonradial solutions exist globally in time and remain bounded if $\|u_0\|_{L^1(\Omega)} < 8\pi/\chi$. Hence the above theorem is regarded as a generalization of the study of the Keller–Segel system.*

REMARK 3.5. *By the standard compactness methods, we can show asymptotic behavior of the globally bounded solutions in Theorem 3.1 and Theorem 3.3. Precisely, there exists some increasing sequence $T_k \in (0, \infty)$ such that $(u(T_k), v(T_k), w(T_k))$ converges to a solution of the stationary problem.*

We consider blowup solutions to (1.1)–(1.2)–(1.3). The following is the definition of blowup of solutions.

DEFINITION 3.6. *We say that a solution (u, v, w) to* (1.1) *blows up, if the solution satisfies*

$$\limsup_{t\nearrow T_{max}} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}) = \infty,$$

where T_{max} is the maximal existence time of the classical solution (u, v, w).

THEOREM 3.7 ([9]). *Suppose $n = 4$, Ω be a convex bounded domain and $\Lambda \in ((8\pi)^2/\chi, \infty)\setminus\{(8\pi)^2/\chi\}\mathbb{N}$. Then there exist blowup solutions (u, v, w) to* (1.1)–(1.2)–(1.3) *satisfying $\|u(t)\|_{L^1(\Omega)} = \Lambda$.*

REMARK 3.8. *By Theorem 3.3 and Theorem 3.7, we established that the case where $n = 4$ and $\int_\Omega u_0 = (8\pi)^2/\chi$ is critical and that this case is corresponding to the case $n = 2$ and $\int_\Omega u_0 = 8\pi/\chi$ of the Keller–Segel system.*

4. Strategy and mathematical challenge. As compared with the Keller–Segel system, we should control the power balance between the terms $\int_\Omega u \log u + (\chi/2)\int_\Omega |(-\Delta+1)v|^2$ and $\chi\int_\Omega uv$. Instead of the Trudinger–Moser inequality, we will apply the Adams type inequality ([21, 24]): for all $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that for all $u \in H^2(\Omega)$,

$$\log\left(\int_\Omega e^{|u(x)|}\,dx\right) \le \left(\frac{1}{2(8\pi)^2}+\varepsilon\right)\|(-\Delta+1)v\|^2_{L^2(\Omega)} + C_\varepsilon.$$

We remark that the critical constant of the Adams type inequality implies the constant $(8\pi)^2/\chi$. Invoking the smallness of the mass, we can combine these estimates and deduce the lower estimate for the Lyapunov functional.

The mathematical challenge is also in regularity estimates. After deriving the energy estimate from the lower estimate for the Lyapunov functional, we will proceed to deduce L^p estimate for u. We cannot adopt the approach in the study of the Keller–Segel system to our system (1.1) because the four-dimensional setting disturbs the relationships of exponents in the Sobolev inequality. Moreover the particular structure of (1.1), i.e., the system (1.1) consists of three parabolic equations, causes a difficulty. From this reason, we use the localizing method, which is introduced in [22, 23, 6, 7].

As to the blowup result, our method has the same spirit in [13, 15]. We first consider a blowing up sequence of stationary solutions. Stationary solutions (u, v, w) to (1.1)–(1.2)–(1.3) satisfy that

$$\begin{cases} 0 = \Delta u - \chi\nabla\cdot(u\nabla v) & \text{in } \Omega, \\ 0 = \Delta v - v + w & \text{in } \Omega, \\ 0 = \Delta w - w + u & \text{in } \Omega, \\ u \ge 0,\ v \ge 0,\ w \ge 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} - \chi u\dfrac{\partial v}{\partial \nu} = v = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Put $\Lambda = \|u\|_{L^1(\Omega)} \in (0,\infty)$. The system (4.1) can be rewritten as the following:

$$\begin{cases} (-\Delta+1)^2 v = \dfrac{\Lambda}{\int_\Omega e^{\chi v}} e^{\chi v} & \text{in } \Omega, \\ u = \dfrac{\Lambda}{\int_\Omega e^{\chi v}} e^{\chi v}, \quad w = -\Delta v + v & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Here and henceforth, we say that (u, v, w, Λ) is a solution to (4.2), if the function (u, v, w) and the positive constant Λ satisfies (4.2). The following proposition plays a key role in our analysis. This claim is about a quantization property of solutions to (4.2).

Proposition 4.1 ([9]). *Let $\Lambda > 0$. Suppose that solutions $\{(u_k, v_k, w_k, \Lambda)\}_k$ to (4.2) satisfy that $\lim_{k\to\infty}\|v_k\|_{L^\infty(\Omega)} = \infty$. Then $J = \Lambda\chi/(8\pi)^2$ is a positive integer and there is a set of points $\{Q(j)\}_{j=1}^J \subset \Omega$ satisfying that*

$$u_k \to \sum_{j=1}^J \frac{(8\pi)^2}{\chi}\delta_{Q(j)} \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad \text{as } k\to\infty,$$

where $\delta_{Q(j)}$ is the delta function whose support is the point $Q(j)$ and $\mathcal{M}(\overline{\Omega})$ is a set of Radon measures on $\overline{\Omega}$.

For $\Lambda > 0$ put the set $\mathcal{S}(\Lambda)$ as

$$\big\{(u,v,w) \in C^2(\overline{\Omega}) : (u,v,w) \text{ is a stationary solution to (1.1)–(1.2)–(1.3)} \\ \text{with } \|u\|_{L^1(\Omega)} = \Lambda\big\}.$$

The following lemma is an immediate consequence of Proposition 4.1.

Lemma 4.2 ([9]). *For $\Lambda \in (0,\infty) \setminus \{(8\pi)^2/\chi\}\mathbb{N}$, there exists a constant $C > 0$ satisfying*

$$\sup\{\|(u,v,w)\|_{L^\infty(\Omega)} : (u,v,w) \in \mathcal{S}(\Lambda)\} \le C$$

and

$$F_*(\Lambda) := \inf\{\mathcal{F}(u,v,w) : (u,v,w) \in \mathcal{S}(\Lambda)\} \ge -C.$$

In order to find a blowup solution, we construct a triplet of nonnegative functions (u_0, v_0, w_0) satisfying

$$\mathcal{F}(u_0,v_0,w_0) < F_*(\Lambda) \qquad \text{for } \Lambda > (8\pi)^2/\chi \quad \text{with } \Lambda \notin \{(8\pi)^2/\chi\}\mathbb{N}.$$

5. Further comments and conjectures. Let us first give some comments on Neumann boundary case. Suppose that the following boundary conditions:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{5.1}$$

and the initial data satisfies the boundary condition

$$\frac{\partial v_0}{\partial \nu} = \frac{\partial w_0}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,\infty). \tag{5.2}$$

Moreover we assume the radial symmetry:

$$\begin{aligned} &\Omega = B(R) = \{x \in \mathbb{R}^4 \,|\, |x| \le R\} \quad \text{with} \quad R > 0 \qquad \text{and} \\ &(u_0, v_0, w_0) : \text{ radial symmetry.} \end{aligned}$$

Theorem 5.1 ([8]). *Let $n = 4$, $\Omega = B(R) = \{x \in \mathbb{R}^4 \,|\, |x| \le R\}$ $(R > 0)$. Suppose that (u_0, v_0, w_0) is radially symmetric and satisfies* (1.4), (5.2) *and*

$$\int_\Omega u_0 < \frac{(8\pi)^2}{\chi}.$$

Then the problem (1.1)–(5.1)–(1.3) *has a unique classical positive solution, which exists globally in time. Moreover the solution is uniformly bounded in time in the sense that*

$$\sup_{t\in[0,\infty)} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{2,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)}\Big) < \infty.$$

Remark 5.2. *Comparing with the study of the two-dimensional Keller–Segel system, the critical constant is changed from $8\pi/\chi$ to $(8\pi)^2/\chi$.*

Remark 5.3. *We used the assumption of radial symmetry to deduce an Adams type inequality in [8]. We conjecture that without this assumption the threshold constant seems to be $(8\pi)^2/2\chi$.*

As to blowup of solution, at least, the following questions have been left as an open (especially, the second one is related to the result [25]):

- does the blowup in Theorem 3.3 occur at finite time or infinite time?;
- does the solution blows up independently of the size of the initial data in the super critical case ($n \geq 5$)?

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