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STABILITY OF ALE SPACE-TIME DISCONTINUOUS GALERKIN METHOD

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Abstract. We assume the heat equation in a time dependent domain, where the evolution of the domain is described by a given mapping. The problem is discretized by the discontinuous Galerkin (DG) method in space as well as in time with the aid of Arbitrary Lagrangian-Eulerian (ALE) method. The sketch of the proof of the stability of the method is shown.

Key words. ALE formulation, discontinuous Galerkin method, discrete characteristic function, stability

AMS subject classifications. 65M60, 65M99

1. Introduction. Although many theoretical results are devoted to the numerical analysis of parabolic PDEs within a fixed domain, there are number of areas with many important applications of parabolic PDEs with time dependent domain. We can mention, for example, problems with moving boundaries, where the motion of the boundary is either prescribed or given by the PDE itself.

There are several approaches how to deal with problems in time dependent domains, e.g. fictitious domain method, see e.g. [21], or immersed boundary method, see e.g. [4]. A very popular technique is Arbitrary Lagrangian-Eulerian (ALE) method that is based on a one-to-one ALE mapping of the reference domain on the current one. ALE method is often applied in connection with conforming finite element method (FEM) in space and lower order time discretizations (backward Euler method, Crank-Nicolson method, BDF2) in time, see e.g. [18] or [19].

The class of discontinuous Galerkin methods seems to be one of the most promising candidates to construct high order accurate schemes for solving convection-diffusion problems, where narrow layers and steep gradients of the solution may appear. For a survey about DG space discretization, see [1], [10], [11]. The discontinuous Galerkin method could be applied for time discretization as well. For a survey about DG time discretization, see e.g. [23]. The discontinuous Galerkin method in space with BDF time discretization was applied with success to time dependent problems, see e.g. [7] or [22]. Moreover, in [8] space-time DG discretization was applied to the vibration of an airfoil problem and the results were compared with BDF time discretization. According to this comparison, DG time discretization seems to be more robust and accurate than BDF.

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Numerical analysis of stability and a priori error estimates of time dependent problems with divergence free domain velocity and discretized by the conforming FEM in space and by DG in time could be found in [5] and [6]. Finally, the stability analysis of space-time DG discretization of nonlinear convection-diffusion problems is studied in [2] for lower degree polynomial approximations in time and in [3] for general polynomial degree.

2. Continuous problem. Let $T > 0$. We consider the following initial–boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = f \quad &\text{in } \Omega_t \times (0,T), \\ u = 0 \quad &\text{in } \partial\Omega_t \times (0,T), \\ u = u^0 \quad &\text{in } \Omega_0, \end{aligned} \tag{2.1}$$

where $\Omega_t \subset R^d$ $(d = 1, 2, 3)$ is a bounded polyhedral time dependent domain with a Lipschitz continuous boundary $\partial\Omega_t$. We assume that the initial condition $u^0 \in L^2(\Omega_0)$ and the right-hand side $f \in L^2(0, T, L^2(\Omega_t))$. We denote by $(.,.)_t$ and $\|.\|_t$ the $L^2(\Omega_t)$ scalar product and norm, respectively.

The evolution of the domain Ω_t in time is described by a given regular one-to-one ALE mapping

$$\mathcal{A} : \overline{\Omega}_0 \times [0, T] \to \overline{\Omega}_t, \tag{2.2}$$

where $\overline{\Omega}_0$ or $\overline{\Omega}_t$ are closures of Ω_0 or Ω_t, respectively. For the purpose of the proof of the stability we introduce following regularity assumptions on the ALE mapping $\mathcal{A}$:

$$\mathcal{A} \in W^{1,\infty}(0, T, W^{1,\infty}(\Omega_0)), \quad \mathcal{A}^{-1} \in W^{1,\infty}(0, T, W^{1,\infty}(\Omega_t)). \tag{2.3}$$

Moreover, we denote the Jacobi matrix of $\mathcal{A}$ by $B = \frac{d\mathcal{A}}{dX}$, the corresponding determinant by $J = \det(B)$ and the *domain velocity* by $\omega = \frac{\partial \mathcal{A}}{\partial t} \circ \mathcal{A}^{-1}$. From the regularity assumptions (2.3) it is possible to show that B, B^{-1}, J, J^{-1}, ω and $\nabla \cdot \omega$ are bounded, i.e. there exists a constant $C_{\mathcal{A}} > 0$ such that

$$\begin{aligned} &\max(\|B\|_{L^\infty(0,T,L^\infty(\Omega_0))}, \|B^{-1}\|_{L^\infty(0,T,L^\infty(\Omega_0))}, \|J\|_{L^\infty(0,T,L^\infty(\Omega_0))} \\ &\|J^{-1}\|_{L^\infty(0,T,L^\infty(\Omega_0))}, \|\omega\|_{L^\infty(0,T,L^\infty(\Omega_t))}, \|\nabla \cdot \omega\|_{L^\infty(0,T,L^\infty(\Omega_t))}) \le C_{\mathcal{A}}. \end{aligned} \tag{2.4}$$

Problem (2.1) is usually transformed into the *ALE formulation*. To this end, we introduce ALE derivative

$$D_t u = \frac{\partial u}{\partial t} + \omega \cdot \nabla u. \tag{2.5}$$

Now we introduce the ALE formulation equivalent to problem (2.1)

$$\begin{aligned} D_t u - \Delta u - \omega \cdot \nabla u = f \quad &\text{in } \Omega_t \times (0,T), \\ u = 0 \quad &\text{in } \partial\Omega_t \times (0,T), \\ u = u^0 \quad &\text{in } \Omega_0. \end{aligned} \tag{2.6}$$

3. Discretization. In this section, we describe the interior penalty discontinuous Galerkin discretization in space variables together with the discontinuous Galerkin time discretization in the ALE framework.

We consider a space partition $\mathcal{T}_{h,0}$ consisting of a finite number of closed, d-dimensional simplices K with mutually disjoint interiors and covering $\overline{\Omega}_0$. We assume conforming properties, i.e. neighbouring elements share an entire edge or face. We set $h_K = \mathrm{diam}(K)$ and $h = \max_K h_K$. We assume that the mesh is quasi-uniform, i.e. there exists a constant $C_Q > 0$ such that $h_K \leq C_Q h_{\bar{K}}$ for all neighbouring elements K and $\bar{K}$. By ρ_K we denote the radius of the largest d-dimensional ball inscribed into K. We assume shape regularity of elements, i.e. $h_K/\rho_K \leq C$ for all $K \in \mathcal{T}_h$, where the constant does not depend on $\mathcal{T}_{h,0}$ for $h \in (0, h_0)$. By $\Gamma_{h,0}$ we denote the set of all edges of $\mathcal{T}_{h,0}$. We define a unit normal vector n to arbitrary edge from $\Gamma_{h,0}$. For inner edges the direction is arbitrary, for outer edges we assume that n is the unit outer normal vector.

Since the domain Ω_0 evolves into $\Omega_t = \mathcal{A}(\Omega_0, t)$, we define similarly the evolution of the mesh $\mathcal{T}_{h,t} = \mathcal{A}(\mathcal{T}_{h,0}, t)$, the evolution of the edges $\Gamma_{h,t} = \mathcal{A}(\Gamma_{h,0}, t)$.

We introduce the space for the semidiscrete solution on Ω_0

$$V_h = \{v \in L^2(\Omega_0) : v|_K \in P^p(K)\}, \tag{3.1}$$

where $P^p(K)$ denotes the space of polynomials up to the degree $p \geq 1$ on K. Functions from the space V_h are discontinuous across the edges of $\mathcal{T}_{h,0}$. For this reason we define one-sided limits

$$v_L(x) = \lim_{s\to 0+} v(x - ns), \quad v_R(x) = \lim_{s\to 0+} v(x + ns), \tag{3.2}$$

jumps and mean values

$$[v] = v_L - v_R, \quad \langle v\rangle = \frac{v_L + v_R}{2}. \tag{3.3}$$

For outer edges we define

$$[v] = \langle v\rangle = v_L = \lim_{s\to 0+} v(x - ns). \tag{3.4}$$

In order to discretize problem (2.6) in time, we consider a time partition $0 = t_0 < t_1 < \ldots < t_r = T$ with time intervals $I_m = (t_{m-1}, t_m)$, time steps $\tau_m = t_m - t_{m-1}$ and $\tau = \max_{m=1,\ldots,r} \tau_m$. We define the solution space

$$V_h^\tau = \{v \in L^2(0, T, L^2(\Omega_t)) : (v \circ \mathcal{A})|_{I_m} \in P^q(I_m, V_h)\}. \tag{3.5}$$

For a function $v \in V_h^\tau$ we define the one–sided limits

$$v_\pm^m = v(t_m\pm) = \lim_{t\to t_m\pm} v(t) \tag{3.6}$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m, \quad m \geq 1 \quad \text{and} \quad \{v\}_0 = v_+^0 - u^0. \tag{3.7}$$

We approximate the diffusion term by the *discontinuous Galerkin interior penalty* form

$$\begin{aligned} a_{h,t}(u, v) = &\sum_{K\in\mathcal{T}_{h,t}} \int_K \nabla u \cdot \nabla v dx - \sum_{e\in\Gamma_{h,t}} \int_e (\langle\nabla u\rangle \cdot n[v] + \theta\langle\nabla v\rangle \cdot n[u]) dS \\ &+ \sum_{e\in\Gamma_{h,t}} \int_e \sigma[u][v] dS. \end{aligned} \tag{3.8}$$

The choice of parameter $\theta = 1, 0, -1$ corresponds to SIPG, IIPG and NIPG formulation, respectively. Parameter σ is defined on the inner edges between elements K and $\bar{K}$ by

$$\sigma = \frac{C_W}{\frac{h_K + h_{\bar{K}}}{2}} \tag{3.9}$$

and on the boundary edges by

$$\sigma = \frac{C_W}{h_K}, \tag{3.10}$$

where the constant $C_W > 0$ needs to be chosen large enough to guarantee ellipticity of $a_{h,t}$. Lower bounds for C_W will be briefly discussed later. For more informations about different variants of discontinuous Galerkin method and their corresponding formulations approximating $(-\Delta u, v)_t$ see e.g. [1].

Now, we are able to formulate the fully discrete space-time discontinuous Galerkin scheme:

Definition 3.1. *We say that a function $U \in V_h^\tau$ is the discrete solution of problem (2.6) obtained by space-time discontinuous Galerkin method, if the following conditions are satisfied*

$$\begin{aligned} \int_{I_m} (D_t U, v)_t + a_{h,t}(U, v) - (\omega \cdot \nabla U, v)_t dt + (\{U\}_{m-1}, v_+^{m-1})_{t_{m-1}} \\ = \int_{I_m} (f, v)_t dt \quad \forall m = 1, \dots, r, \, \forall v \in V_h^\tau. \end{aligned} \tag{3.11}$$

The time discretization in (3.11) can be viewed as a generalization of some specific classical one–step methods for parabolic problems. It is possible to show that setting $q = 0$, i.e. piecewise constant approximation in time, is equivalent (up to suitable quadrature of the right–hand side) to backward Euler method in time and discontinuous Galerkin method in space. Similarly, the higher polynomial degree approximations in time lead to methods that are equivalent (up to suitable quadrature of the right–hand side) to Radau IIA Runge-Kutta methods. For details about the relations between Galerkin methods and Runge-Kutta methods see e.g. [15] and [20]. For the descriptions of Radau IIA Runge-Kutta methods see e.g. [12] or [16] and [17].

4. Stability. The aim of this section is to show that the numerical scheme (3.11) is stable, i.e. the approximate solution obtained from (3.11) can be bounded in terms of the data f and u^0 of the problem (2.1).

An important auxiliary tool for the analysis of problems in time-dependent domains is the *Reynolds transport formula*:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} v(x,t) dx &= \int_{\Omega_t} \frac{\partial v}{\partial t}(x,t) + \nabla \cdot (\omega v)(x,t) dx \\ &= \int_{\Omega_t} D_t v(x,t) + \nabla \cdot \omega(x,t) v(x,t) dx. \end{aligned} \tag{4.1}$$

For the purpose of the forthcoming estimates we define discontinuous Galerkin energy norm

$$\|u\|_{DG,t}^2 = \sum_{K \in \mathcal{T}_{h,t}} \|\nabla u\|_{L^2(K)}^2 + \sum_{e \in \Gamma_{h,t}} \|\sigma^{1/2}[u]\|_{L^2(e)}^2. \tag{4.2}$$

Using this norm we can summarize the properties of $a_{h,t}$ in following lemma.

LEMMA 4.1. *Let $U, v \in V_h^\tau$. Then there exists a constant $C_a > 0$ such that*

$$a_{h,t}(U,v) \le C_a \|U\|_{DG,t} \|v\|_{DG,t}. \tag{4.3}$$

Moreover, let the constant C_W satisfy

$$\begin{aligned} &C_W > 0, && \theta = -1, \quad NIPG,\\ &C_W \ge \frac{1}{2} C_M (C_I + 1)(C_Q + 1), && \theta = 0, \quad IIPG,\\ &C_W \ge C_M (C_I + 1)(C_Q + 1), && \theta = 1, \quad SIPG, \end{aligned} \tag{4.4}$$

where constant C_M and C_I come from the trace inequality and the inverse inequality, respectively, see [11]. Then

$$a_{h,t}(U,U) \ge \frac{1}{2}\|U\|^2_{DG,t}. \tag{4.5}$$

Proof. The ideas of the proof are well described in e.g. [11]. The generalization to the problems in the time dependent domains can be found in [2]. □

We need the estimate of the ALE derivative term.

LEMMA 4.2. *Let $U \in V_h^\tau$. Then*

$$\begin{aligned} &\int_{I_m} (D_t U, U)_t dt + (\{U\}_{m-1}, U_+^{m-1})_{t_{m-1}} \\ &\ge \frac{1}{2}\|U_-^m\|^2_{t_m} - \frac{1}{2}\|U_-^{m-1}\|^2_{t_{m-1}} - \frac{C_{\mathcal{A}}}{2}\int_{I_m} \|U\|_t^2 dt. \end{aligned} \tag{4.6}$$

Proof. At first, we will study relation (4.6) elementwise for each element $K \in \mathcal{T}_{h,0}$. Let us denote $K_t = \mathcal{A}(K,t)$. Applying Reynolds transport formula with $v = U^2$ we get

$$\begin{aligned} &\int_{I_m}\int_{K_t} U \cdot D_t U dx dt + \int_{K_{t_{m-1}}} \{U\}_{m-1} U_+^{m-1} dx \\ &= \frac{1}{2}\int_{I_m}\int_{K_t} D_t U^2 dx dt + \int_{K_{t_{m-1}}} \{U\}_{m-1} U_+^{m-1} dx \\ &= \frac{1}{2}\int_{I_m} \frac{d}{dt}\int_{K_t} U^2 dx dt - \frac{1}{2}\int_{I_m}\int_{K_t} (\nabla\cdot\omega) U^2 dx dt + \int_{K_{t_{m-1}}} \{U\}_{m-1} U_+^{m-1} dx \\ &= \frac{1}{2}\|U_-^m\|^2_{L^2(K_{t_m})} - \frac{1}{2}\|U_+^{m-1}\|^2_{L^2(K_{t_{m-1}})} + \|U_+^{m-1}\|^2_{L^2(K_{t_{m-1}})} \\ &\quad - \int_{K_{t_{m-1}}} U_-^{m-1} U_+^{m-1} dx - \frac{1}{2}\int_{I_m}\int_{K_t} (\nabla\cdot\omega) U^2 dx dt \\ &= \frac{1}{2}\|U_-^m\|^2_{L^2(K_{t_m})} - \frac{1}{2}\|U_-^{m-1}\|^2_{L^2(K_{t_{m-1}})} + \frac{1}{2}\|\{U\}_{m-1}\|^2_{L^2(K_{t_{m-1}})} \\ &\quad - \frac{1}{2}\int_{I_m}\int_{K_t} (\nabla\cdot\omega) U^2 dx dt \\ &\ge \frac{1}{2}\|U_-^m\|^2_{L^2(K_{t_m})} - \frac{1}{2}\|U_-^{m-1}\|^2_{L^2(K_{t_{m-1}})} - \frac{C_{\mathcal{A}}}{2}\int_{I_m}\int_{K_t} U^2 dx dt. \end{aligned} \tag{4.7}$$

The lemma is proved by summing this relation over all $K_t \in \mathcal{T}_{h,t}$. □

Setting $v = U$ in (3.11) we get the basic identity

$$\int_{I_m} (D_t U, U)_t + a_{h,t}(U,U) - (\omega \cdot \nabla U, U)_t dt + (\{U\}_{m-1}, U_+^{m-1})_{t_{m-1}} = \int_{I_m} (f, U)_t dt. \tag{4.8}$$

Since

$$\int_{I_m} (\omega \cdot \nabla U, U)_t dt \le C_{\mathcal{A}} \int_{I_m} \|U\|_{DG,t} \|U\|_t dt \le C_{\mathcal{A}}^2 \int_{I_m} \|U\|_t^2 dt + \frac{1}{4} \int_{I_m} \|U\|_{DG,t}^2 dt, \tag{4.9}$$

applying Lemma 4.1 and Lemma 4.2 we get

$$\begin{aligned} &\frac{1}{2}\|U_-^m\|_{t_m}^2 - \frac{1}{2}\|U_-^{m-1}\|_{t_{m-1}}^2 + \frac{1}{2}\int_{I_m} \|U\|_{DG,t}^2 dt \\ &\le \|f\|_{L^2(I_m, L^2(\Omega_t))} \|U\|_{L^2(I_m, L^2(\Omega_t))} + C_{\mathcal{A}}^2 \int_{I_m} \|U\|_t^2 dt + \frac{1}{4}\int_{I_m} \|U\|_{DG,t}^2 dt \\ &\quad + \frac{C_{\mathcal{A}}}{2} \int_{I_m} \|U\|_t^2 dt \\ &\le \|f\|_{L^2(I_m, L^2(\Omega_t))}^2 + \frac{1}{4}\int_{I_m} \|U\|_{DG,t}^2 dt + \tau_m C_1 \sup_{t \in I_m} \|U\|_t^2, \end{aligned} \tag{4.10}$$

where the constant $C_1 = 1/4 + C_{\mathcal{A}}/2 + C_{\mathcal{A}}^2$.

To be able to get rid of the last supremum term, we need to derive a technique for estimating the values of the discrete solution inside of intervals I_m.

4.1. Discrete characteristic function. The concept of the discrete characteristic function comes from [9]. As we have seen in (4.10), application of the test function $v = U$ naturally leads to the nodal estimate. Setting $v = \chi_{(t_{m-1},s)} U$, where $\chi_{(t_{m-1},s)}$ is characteristic function of the interval (t_{m-1}, s) for $s \in [t_{m-1}, t_m]$, will lead to a similar estimate for $\|U(s)\|_s$ instead of $\|U_-^m\|_{t_m}$. Unfortunately, it is not possible to do it, since $\chi_{(t_{m-1},s)} U \notin V_h^\tau$. The idea of the discrete characteristic function is based on the construction of $U_s \in V_h^\tau$ for given $U \in V_h^\tau$ and $s \in [t_{m-1}, t_m]$ such that U_s will preserve similar properties to the classical characteristic function. For applications of the discrete characteristic function see, e.g. [11] or [24].

We will use a notation $\tilde{v} = v \circ \mathcal{A}$ for transformation of functions from the evolving space-time cylinder to the reference space-time cylinder. From the assumptions on the ALE mapping $\mathcal{A}$ and according to the definition of space V_h^τ it is possible to see that this transformation is bijection between V_h^τ and $\tilde{V}_h^\tau$, where

$$\tilde{V}_h^\tau = \{v \in L^2(0, T, L^2(\Omega_0)) : v|_{K \times I_m} \in P^q(I_m, P^p(K))\}, \tag{4.11}$$

i.e. $\tilde{V}_h^\tau$ represents the space of classical piecewise polynomial functions.

We define the discrete characteristic function for time dependent domains in three steps. At first, the given function $U \in V_h^\tau$ is transformed onto the reference domain, i.e. $\tilde{U} = U \circ \mathcal{A} \in \tilde{V}_h^\tau$. Second step is the construction of discrete characteristic function in fixed domains, i.e. $\tilde{U}_s \in \tilde{V}_h^\tau$ such that

$$\begin{gathered}\tilde{U}_{s+}^{m-1} = \tilde{U}_{+}^{m-1},\\ \int_{I_m} \left(\tilde{U}_s, \frac{\partial v}{\partial t}\right)_0 dt = \int_{t_{m-1}}^{s} \left(\tilde{U}, \frac{\partial v}{\partial t}\right)_0 dt \quad \forall v \in \tilde{V}_h^\tau.\end{gathered} \tag{4.12}$$

The last step is the transformation back to the current domain, i.e. $U_s = \tilde{U}_s \circ \mathcal{A}^{-1} \in V_h^\tau$.

Now, we want to show a similar relation to the relation from Lemma 4.2 that will also describe the *contraction* property of the discrete characteristic function.

LEMMA 4.3. *Let $U \in V_h^\tau$ and $U_s \in V_h^\tau$ be its discrete characteristic function associated with $s \in I_m$. Then there exists a constant $C_D > 0$ depending only on the polynomial degree q and on the regularity of the ALE mapping (2.3) such that*

$$\begin{aligned}&\int_{I_m} (D_t U, U_s)_t dt + (\{U\}_{m-1}, U_{s+}^{m-1})_{t_{m-1}}\\ &\geq \frac{1}{2} \sup_{I_m} \|U(t)\|_t^2 - \frac{1}{2}\|U_-^{m-1}\|_{t_{m-1}}^2 - C_D \tau_m \sup_{t\in I_m} \|U\|_t^2\end{aligned} \tag{4.13}$$

Proof. Since the proof is long and technical, it is skipped in this paper. The proof will be contained in [3]. □

Using Lemma 4.3, it is possible to deal with the ALE derivative term. For all the other terms we need to show that the process of creating the discrete characteristic function is stable with a constant independent of the parameter $s \in I_m$.

LEMMA 4.4. *Let $U \in V_h^\tau$ and $U_s \in V_h^\tau$ be its discrete characteristic function associated with $s \in I_m$. Then there exists a constant $C_{ST} > 0$ depending only on the polynomial degree q and on the regularity of ALE mapping (2.3) such that*

$$\int_{I_m} \|U_s(t)\|_t^2 dt \leq C_{ST} \int_{I_m} \|U(t)\|_t^2 dt, \tag{4.14}$$

$$\int_{I_m} \|U_s(t)\|_{DG,t}^2 dt \leq C_{ST} \int_{I_m} \|U(t)\|_{DG,t}^2 dt. \tag{4.15}$$

Proof. Since the proof is long and technical, it is skipped in this paper. The proof will be contained in [3]. □

4.2. Main result. Now, we are ready to formulate the main result.

THEOREM 4.5. *Let the parameter C_W satisfy (4.4) and let $U \in V_h^\tau$ be an approximate solution obtained by scheme (3.11). Then there exist constants $C > 0$ and $C^* > 0$ such that $\tau \leq C^*$ implies*

$$\sup_{I_m} \|U\|_t^2 \leq C(\|f\|_{L^2(0,T,L^2(\Omega_t))}^2 + \|u^0\|_0^2). \tag{4.16}$$

Proof. Setting $v = U_s$ in the left-hand side of (3.11), where $s \in [t_{m-1}, t_m]$ such that $\|U(s)\|_s = \sup_{t\in I_m} \|U\|_t$, and using Lemma 4.1, Lemma 4.3 and Lemma 4.4 we get

$$
\begin{aligned}
(4.17)\quad & \int_{I_m} (D_t U, U_s)_t + a_{h,t}(U, U_s)_t - (\omega \cdot \nabla U, U_s)_t dt + (\{U\}_{m-1}, U_+^{m-1})_{t_{m-1}} \\
& \geq \frac{1}{2}\|U(s)\|_s^2 - \frac{1}{2}\|U_-^{m-1}\|_{t_{m-1}}^2 - C_D \tau_m \sup_{t \in I_m} \|U\|_t^2 \\
& \quad - \int_{I_m} C_a \|U\|_{DG,t} \|U_s\|_{DG,t} dt - C_{\mathcal{A}} \int_{I_m} \|U\|_{DG,t} \|U_s\|_t dt \\
& \geq \frac{1}{2} \sup_{I_m} \|U\|_t^2 - \frac{1}{2} \sup_{I_{m-1}} \|U\|_t^2 - C_D \tau_m \sup_{t \in I_m} \|U\|_t^2 - \frac{C_a}{2} \int_{I_m} \|U\|_{DG,t}^2 dt \\
& \quad - \frac{C_a C_{ST}}{2} \int_{I_m} \|U\|_{DG,t}^2 dt - \frac{1}{2} \int_{I_m} \|U\|_{DG,t}^2 dt - \frac{C_{\mathcal{A}}^2 C_{ST}}{2} \int_{I_m} \|U\|_t^2 dt,
\end{aligned}
$$

where we use the notation $\sup_{I_0} \|U\|_t^2 = \|u^0\|_0^2$. Similarly, setting $v = U_s$ in the right-hand side of (3.11) we get

$$
(4.18)\qquad \int_{I_m} (f, U_s)_t dt \leq \frac{1}{2}\|f\|_{L^2(I_m, L^2(\Omega_t))}^2 + \frac{C_{ST}}{2} \int_{I_m} \|U\|_t^2.
$$

Using these relations we get

$$
\begin{aligned}
(4.19)\qquad & \frac{1}{2} \sup_{I_m} \|U\|_t^2 - \frac{1}{2} \sup_{I_{m-1}} \|U\|_t^2 \leq \frac{1}{2}\|f\|_{L^2(I_m, L^2(\Omega_t))}^2 \\
& + C_2 \tau_m \sup_{t \in I_m} \|U\|_t^2 + C_3 \int_{I_m} \|U\|_{DG,t}^2 dt,
\end{aligned}
$$

where $C_2 = C_D + (C_{\mathcal{A}}^2 + 1)C_{ST}/2$ and $C_3 = (1 + C_a + C_a C_{ST})/2$.

Multiplying (4.10) by $4C_3$ and summing with (4.19) we get

$$
\begin{aligned}
(4.20)\qquad & \frac{1}{2}\left(4C_3\|U_-^m\|_{t_m}^2 + \sup_{I_m} \|U\|_t^2\right) - \frac{1}{2}\left(4C_3\|U_-^{m-1}\|_{t_m}^2 + \sup_{I_{m-1}} \|U\|_t^2\right) \\
& \leq \frac{8C_3 + 1}{2}\|f\|_{L^2(I_m, L^2(\Omega_t))}^2 + (4C_1C_3 + C_2)\tau_m \sup_{t \in I_m} \|U\|_t^2.
\end{aligned}
$$

Setting $C^* = 8C_1C_3 + 2C_2$ we get we get $(4C_1C_3 + C_2)\tau_m < 1/2$ and the statement of the theorem follows from the application of the discrete Gronwall lemma. □

5. Conclusion. We presented a higher order method for the heat equation in a time dependent domain based on the space-time discontinuous Galerkin method. For this problem, the idea of the proof of the unconditional stability for any polynomial degree is shown. There are several items for the future work.

- The extension of the discontinuous Galerkin discretization and the stability analysis to nonlinear problems.
- Deriving a priori error estimates.
- Investigating other suitable higher order time discretizations for problems with a time dependent domain, e.g. continuous Galerkin method, DIRK, etc.
- The numerical analysis of coupled problems, where the ALE mapping depends on the solution of the problem.

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REFERENCES

[1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. *Unified analysis of discontinuous Galerkin methods for elliptic problems.* SIAM J. Numer. Anal., 39(5):1749–1779, 2002.

[2] M. Balázsová, and M. Feistauer. *On the stability of the space-time discontinuous Galerkin method for nonlinear convection-diffusion problems in time-dependent domains.* Appl. Math., 60:501–526, 2015.

[3] M. Balázsová, M. Feistauer, and M. Vlasák. *Stability of the ALE space-time discontinuous Galerkin method for nonlinear convection-diffusion problems in time-dependent domains.* (in preparation).

[4] D. Boffi, L. Gastaldi, and, L. Heltai. *Numerical stability of the finite element immersed boundary method.* Math. Models Methods Appl. Sci., 17:1479–1505, 2007.

[5] A. Bonito, I. Kyza, and, R. H. Nochetto. *Time-discrete higher-order ALE formulations: Stability.* SIAM J. Numer. Anal., 51(1):577–604,2013

[6] A. Bonito, I. Kyza, and, R. H. Nochetto. *Time-discrete higher order ALE formulations: a priori error analysis.* Numer. Math., 125:225–257,2013

[7] J. Česenek, M. Feistauer, J. Horáček, V. Kučera, and J. Prokopová. *Simulation of compressible viscous flow in time-dependent domains.* Appl. Math. Comput., 219:7139–7150,2013

[8] J. Česenek, M. Feistauer, and A. Kosík. *DGFEM for the analysis of airfoil vibrations induced by compressible flow.* ZAMM Z. Angew. Math. Mech., 93 No. 6-7:387–402,2013

[9] K. Chrysafinos, and N. J. Walkington. *Error estimates for the discontinuous Galerkin methods for parabolic equations.* SIAM J. Numer. Anal., 44:349–366,2006

[10] B. Cockburn, G. E. Karniadakis. and C.-W. Shu. *Discontinuous Galerkin methods.* In Lecture Notes in Computational Science and Engineering 11. Springer, Berlin, 2000.

[11] V. Dolejší and M. Feistauer. *Discontinuous Galerkin method, Analysis and applications to compressible flow.* Cham: Springer, 2015.

[12] B. L. Ehle. *On Padé approximations to the exponential function and A-stable methods for the numerical solution of initial value problems.* Research report CSRR 2010, Dept. AACS, Univ. of Waterloo, Ontario, Canada, 1969.

[13] L. Formaggia, and F. Nobile. *A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements.* East-West J. Numer. Math., 7(2):105–131,1999

[14] L. Gastaldi. *A priori error estimates for the Arbitrary Lagrangian Eulerian formulation with finite elements.* East-West J. Numer. Math., 9(2):123–156,2001

[15] A. Guillo, and J. L. Soulé. *La résolution numérique des problemes différentiels aux conditions initiales par des méthodes de collocation.* R.A.I.R.O., R-3:17–44,1969

[16] E. Hairer, S. P. Norsett, and G. Wanner. *Solving ordinary differential equations I, Nonstiff problems.* Number 8 in Springer Series in Computational Mathematics. Springer Verlag, 2000.

[17] E. Hairer and G. Wanner. *Solving ordinary differential equations II, Stiff and differential-algebraic problems.* Springer Verlag, 2002.

[18] C. W.Hirt, A. A. Amsdem, and J. L, Cook. *An arbitrary Lagrangian-Eulerian computing method for all flow speeds.* J. Comput. Phys., 135(2):198–216, 1997.

[19] T. J. R. Hughes, W. K. Liu, and T. K, Zimmermann. *Lagrangian-Eulerian finite element formulation for incompressible viscous flows.* Comput. Methods Appl. Mech. Eng., 29(3):329–349, 1981.

[20] B. L. Hulme. *One step piecewise polynomial Galerkin methods for initial value problems.* Math. Comp., 26:415–424,1972

[21] K. Khadra, P. Angot, S. Parneix, and J. -P. Caltagirone. *Fictitious domain approach for numerical modelling of Navier-Stokes equations.* Int. J. Numer. Methods Fluids., 34(8):651–684, 2000.

[22] A. Kosík, M. Feistauer, M. Hadrava, and J. Horáček. *Numerical simulation of the interaction between a nonlinear elastic structure and compressible flow by the discontinuous Galerkin method.* Appl. Math. Comput., 267:382–396,2015

[23] V. Thomeé. *Galerkin finite element methods for parabolic problems. 2nd revised and expanded ed.* Springer, Berlin, 2006.

[24] M. Vlasák, V. Dolejší, and J. Hájek. *A Priori Error Estimates of an Extrapolated Space-Time Discontinuous Galerkin Method for Nonlinear Convection-Diffusion Problems.* Numer. Methods Partial Differ. Equations, 27(6):1453–1482,2011