COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

PERPETUAL AMERICAN OPTIONS AND REAL OPTIONS

BACHELOR THESIS

2014 **Marek ŠEBO**

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FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

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THESIS ASSIGNMENT

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Aim: Real Options refers to the modelling of capital budgeting decisions with the use of models usually developed for the valuation of financial options such as the perpetual american options. In this thesis we shall review some of the models built to price these options and study theirs application in the field of real options. A particular case study will be developed in order to understand the major advantages and challenges of this approach in real world applications.

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Abstrakt v štátnom jazyku

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Práca predstavuje úvod do metódy reálnej opcie - prístupu k investičnému rozhodovaniu, ktorý je založený na analógii medzi investičnou príležitosťou a finančnou Americkou opciou. Uvedieme čitateľa do problematiky tohto konzervatívneho prístupu k investovaniu, ktorý berie na zreteľ hodnotu odloženia investície a čakania na novú, nikdy však nie úplnu informáciu, ktorá by mohla ovplyvniť akceptovanie/neakceptovanie danej nezvratnej investície v prostredí neistoty. S použitím základných vedomostí z oblasti stochastického kalkulu uvedieme Samuelsonov-McKeanov model oceňovania warrantov, upravený na účely oceňovania reálnych opcií, a ukážeme jeho použitie na reálnej prípadovej štúdii z prostredia realitného biznisu. Nakoniec zdôrazníme, že model a samotný prístup metódou reálnej opcie nám môže pomôcť pochopiť niektoré empiricky odsledované schémy správania sa firiem, ktoré nie sú v súlade s masovo používaným prístupom založeným na čistej súčasnej hodnote (NPV).

Kľúčové slová: reálne opcie, Samuelson-McKean model, geometrický Brownov pohyb

Abstract

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The thesis presents an introduction to the real option approach to capital budgeting - the real option valuation, that is based on an analogy between an investment opportunity and a financial American option. We introduce the reader to this conservative approach, that takes into consideration the value of postponing the investment and waiting for new, but yet never complete information, that could affect the desirability of the irreversible investment under uncertainty. Using the fundamental knowledge of stochastic calculus, we present the Samuelson-McKean model of warrant pricing, modified for pricing real options and show its application to the real world of business on a case study from the real estate development industry. Finally, we highlight, that the model and real option approach itself help us understand some of the patterns in behavior of firms, that are inconsistent with the widely used approach based on the net present value (NPV).

Keywords: Real Options, Samuelson-McKean Model, Geometric Brownian Motion

Contents

Introduction

The conventional methods of modeling capital budgeting decisions, such as the net present value (NPV) criterion, are widely taught and used because of the simplicity and intuitiveness of the underlying calculations. In most real world situations, however, they fall short, as it operates with several oversimplifying assumptions, such as deterministic path of the input parameters or the right to invest only now or never.

The NPV fails to explain some of the empirically observed patterns in investment behavior of companies in an uncertain environment, such as why sometimes firms don't invest, even if the NPV of the project is far above zero, and don't abandon their projects, even though the NPV is significantly negative. It also doesn't justify why the interest rate cuts have a weak, if not ambiguous, effect on aggregate investment, as Dixit and Pindyck observe in [1].

The real options approach to capital budgeting, based on the analogy between an investment opportunity and an American call option, eliminates many of the drawbacks of the conventional methods. The aim of this thesis is to provide an introduction to the real options approach to investment decision-making and to introduce the reader the application of the Samuelson-McKean model of warrant pricing to capital budgeting, that draws analogy between the investment opportunity and a perpetual American call option. A particular case study in real estate business will be developed to demonstrate the real world application of the model.

The thesis is divided in five chapters. The first chapter introduces the reader to the approach, points out the key differences to the NPV based methods and identifies, when it is suitable to treat investment as a real option. The second chapter contains a brief overview of the stochastic calculus, building on the reader's fundamental knowledge of probability theory. It offers the necessary mathematical background needed to construct the model. The construction of Samuelson-McKean model is the content of the third chapter. We will identify the key assumptions we will work with and derive the formulas of the model. The fourth chapter is dedicated to examining how a shift in one of the parameters affects the output. We will inspect the comparative statics of the model, confront it with the basic intuition and hopefully also explain some of the phenomenons mentioned above. The final part applies the theory to resolve a dilemma of a real estate company. A fictional investment opportunity in a real economic environment demonstrates the possible application of the model to the world of business.

1 Investment criteria

1.1 Investment

Investment is omnipresent. Building a factory, attending an accountancy course, going to a gym, buying new hardware for your company, or even brushing your teeth in the evening. All these fall under the definition of investment - sacrificing today's benefits in exchange for future rewards. Somehow less evident, shutting down an unprofitable factory is an investment, too, as the management decides to suffer immediate costs of severance payments to the labor, in prospect of reducing the future losses. According to Dixit and Pindyck in [1], most investment opportunities have three important aspects in common:

- *Irreversibility* - Once the investment is done, at least part of the initial costs is sunk, i.e. cannot be retrieved if the investor decides to abandon the project
- *Uncertainty* - Profits, sometimes also costs, interest rates, inflation rate and other parameters, that affect value of the project, are stochastic variables, the value of which at a certain point in the future may be only estimated on a probability basis.
- *Timing* - Decision whether to invest is usually not a *now-or-never* one. In most cases there is a possibility of delaying the investment in order to wait for further information about the future.

In this thesis, we will focus on mathematical modeling of the investment decisions of companies. The role of the manager there is to appraise (value) all possible projects to invest the firm's capital in, and to choose the one(s), that the company will undertake. Good appraisal is a crucial prerequisite for making a good investment decision. We will bear the three aspects of an investment decision in mind while searching for the optimal method of appraisal.

1.2 NPV

The most widely used technique, the simple net present value (NPV) rule, is based on a fairly simple principle. Summing the expected discounted costs of the project (C_i) and subtracting it from the sum of the expected discounted profits (PV_i), we obtain the net present value (NPV): [1]

$$NPV = \sum_i C_i - PV_i \tag{1}$$

The NPV investment criterion then is:

Theorem 1.1 (NPV rule)**.** *Invest in the project if its* $NPV \geq 0$

The NPV rule is taught at every business school. It is popular for the simplicity of the underlying calculations and theory. However, it operates with several oversimplifying assumptions, which collide with all the three above mentioned aspects, causing inaccuracy.

First of all, the simple NPV approach either assumes, that the expenditures on the project can be recovered, in case the situation turns out to be worse than expected, or, if the investment is irreversible, it assumes that it can be undertaken only now or never. But the ability to delay and irreversibility seems to be two of the crucial aspects of an investment opportunity and the fact that the simple NPV calculation doesn´t take them into consideration turns out to be its big drawback. We will return to those two aspects later and examine them more rigorously.

Furthermore, all sources of uncertainty are treated as deterministic parameters. Stochastic variables are replaced by their expected values, what also negatively affects the flexibility and accuracy of the simple NPV calculations. Aware of the shortcomings of the simple NPV and the derived rules, a new theoretical approach has been developed. The theory of real options applies the tools and framework developed for pricing financial derivatives to investment decision-making.

[1] When we compare an investment opportunity with an another one, we further subtract the opportunity cost of not accepting the alternative (the potential NPV of the alternative). This is equivalent to simply comparing the NPV of the two investment opportunities.

1.3 The real options

In the financial markets, ownership of an *American call option* gives its owner a right (but not an obligation) to pay a pre-set exercise price in order to buy an underlying asset at any time until the expiration date of the option.

Analogically, a firm with an opportunity (real option) to invest has a right to make an investment expediture (exercise price) and receive a project (asset), which is supposed to yield cashflow of some value, at any time until the expiration of the investment opportunity. We will call the right to invest a *real option.*

The firm exercises the option by investing in the project. By doing so, it gives up the opportunity of waiting for more information, that could have affected the desirability and timing of the investment. Choice of the right moment to invest is an option, not an obligation and therefore it is of a nonzero value. Thus any rule that ignores this option value, like the NPV rule, involves an inaccuracy of some extent. Taking this knowledge into consideration, the new investment criterion would be:

Theorem 1.2 (Real options investment rule)**.** *Invest in the project if its $NPV \geq F$, where F is the value of the option to invest in the project.*

Introducing an option value, one might succeed to understand why real firms don´t undertake an investment project immediately after the expected return turns equal to the cost of capital. Instead, they set up their own minimum required rate of return, that tends to be significantly higher than the actual cost of capital. Speaking in the language of the NPV, the firm doesn't undertake the project, until its NPV exceeds zero by certain constant.

Authors like Summers [11, p. 300] observe the so called *hurdle rates* typically three or four times higher than the cost of capital, as the hurdle rates in their observations ranged from 8 to 30 percent with a median of 15 percent and a mean of 17 percent, while the nominal interest rate was 4 percent and the real rate close to zero. Even in projects with very high systematic risk, the difference between the two rates is too big to be sufficiently explained by the standard theory of investment, which omits the real option value.

On the other hand, the option approach also gives us an explanation, why companies don´t abandon their projects immediately as they begin generating operation losses.

Making an irreversible decision to abandon the project is connected with losing the prospects of possible future profit, should the market conditions improve. Furthermore, it entails the costs of abandonment (e.g. severance payments, legal costs, costs of demolition), that are also usually far from insignificant.

1.4 Example: NPV versus real option valuation

To illustrate the difference between the NPV and Real option valuation, we offer a simple example by Michnová [5, p. 8].

Imagine that when digging a well in the garden we came across a vein of gold of 1,000 troy ounces (1 troy ounce = 31.10348 grams) . We assume that the price of gold is an outcome of a random process : today its value is \$ 500 for an ounce. Next year will rise to \$ 600 per ounce with probability 0.6, or decrease to \$ 350 per ounce with a probability of 0.4 . Exploitation costs are \$ 450 per ounce. We stand in front of a decision whether exploit the gold today (Strategy I) or postpone the mining to the next year (Strategy II), while assuming that all the gold will be sold immediately for the market price. Furthermore, we consider that the price risk is perfectly diversified, and thus future cash flows are discounted using the risk-free rate of 5 percent per annum.

1.4.1 Decision based on the NPV valuation

We will examine this case using the conventional NPV valuation method first. As was mentioned before, NPV is obtained summing the discounted costs for a project and subtracting it from the sum of discounted cash flows. According to the NPV investment criterion, we choose the one out of the two strategies, which offers higher NPV.

Strategy I

$$NPV^{I,NPV} = 1000 \times (500 - 450)\$ = 50000\$ \tag{2}$$

Strategy II

If the price of gold rises to 600:

$$NPV_{up}^{II,NPV} = 1000 \times \frac{600 - 450}{1.05}\$ \doteq 142860\$ \tag{3}$$

If the price of gold falls to \$ 350:

$$NPV_{down}^{II,NPV} = 1000 \times \frac{350 - 450}{1.05}\$ \doteq -95240\$ \tag{4}$$

Then the total NPV of the Strategy II is the expected value, considering the two possible scenarios:

$$NPV^{II,NPV} = 0.6 \times NPV_{up}^{II,NPV} + 0.4 \times NPV_{down}^{II,NPV} = 47620\$ \tag{5}$$

Comparing the NPV of the two strategies, Strategy I seems to be more profitable, we should exploit the gold today.

1.4.2 Decision based on the Real Option Valuation

Strategy I

The net present value of the Strategy I is the same as the one obtained by the NPV valuation

$$NPV^{I,ROV} = NPV^{I,NPV} = 50000\$ \tag{6}$$

StrategyII

Assume for a while that we knew that the price of gold would fall to \$ 350 next year. The NPV of the Strategy II would then be \$ $-95240 < 0$, and we would not choose this strategy. But if we appraise the project using the NPV valuation, we assume that the decision, whether or not exploit the gold the next year, is made today with no option of changing it in case of the negative scenario.

In most real world situations, including this example, nothing obliges us to make the decision now. The real option approach takes into consideration, that in if we decide to postpone the the mining to the next year, we obtain a new information about the price of gold, and we can make a whole new decision, whether to start mining or not. In this particular case, if the price fell down to \$ 350 and the value of the project turned negative, we would decide not to start mining (and rather accept the NPV=0).

The NPVs in the two possible scenarios would be:

$$NPV_{up}^{II,ROV} = NPV_{up}^{II,NPV} \doteq 142860\$ \tag{7}$$

$$NPV_{down}^{II,ROV} = 0\$ \tag{8}$$

Thus the final appraisal of the project by the real option valuation would be:

$$NPV^{II,ROV} = 0.6 \times NPV_{up}^{II,ROV} + 0.4 \times NPV_{down}^{II,ROV} = 85716\$ \tag{9}$$

Hence if we based our decision on the ROV, we would postpone the mining.

1.4.3 Conclusion

Michnová in [5] concludes: "Through this example we can see, that the NPV and ROV approaches not only appraise the investment opportunity differently (according to the NPV valuation the investment has the value of \$ 50 000, ROV returns the value of \$ 85 716), but even consider optimal the two diametrically different strategies". That is partially due to the fact, that the two numbers actually *do* stand for the value of two different investment opportunities, where \$ 50 000 is the value of the opportunity to invest with a decision made *now*, while \$ 85 716 is the value of the opportunity to invest in the same project, but with the ability to make the decision *now or next year*.

The difference $85716\$ - 50000\$ = 35716\$$ represents the *value of waiting* and postponing the very decision to invest to the future, in this simple case represented by a single additional decision point a year from now. This value is sometimes far from insignificant, as future brings new information that could strongly affect the profitability of the investment.

In this particular example, the added flexibility in case of postponing the investment decision caused the investor not to invest in the project today. The acceptance of the option value, entailed in the ROV, makes our investment decisions more *conservative*, advising to immediately undertake less projects than we would if we followed the conventional NPV criterion. On the other hand, the valuation of the investment opportunity is usually higher, increased by the value of waiting.

1.5 Using the real option approach

1.5.1 Conditions

What types of investment decisions can be covered by real option approach? As was mentioned before, the real option valuation (ROV) is based on techniques and framework, developed for valuation of financial options. Therefore it is obvious, that our investment opportunity must share a few important characteristics with those financial derivatives. Using the real options, we can approach almost every investment decision, that is at least partially irreversible, the future profits and (or) costs are largely uncertain and which involves some possibility to be postponed (timing).

The more an investment opportunity fits these characteristics, the more suitable it is for us to use the methodology for pricing the derivatives, the time value of which disappears as we exercise them, the payoff in uncertain and can be exercised at any point in time until the expiration.

It is especially desirable to use the real option approach instead of the NPV where the final yield from an investment is far from deterministic and strongly depends on the firm's future managerial decisions. The flexible frame of an option value is far more suitable to include the variety of possibilities and parameters that affect the decision than the rigid sum of cashflows of the NPV calculation.

The extensions of the Samuelson-McKean appraising model, presented in this thesis, enable us to take into account various options of the firm, such as: option to stage the investment in several phases, each of them containing an additional information value, temporary shutdown of production and option to restart it later or an option to expand or contract the production capacity.

1.5.2 Creating real options

How do firms acquire real options? Broyles in [4] highlights, that some can be literally bought, like patents or land to build buildings on, market research before introduction of a new product or clinical testing of a drug. In all of these we can clearly recognise the option premium as the purchase price of the service, patent etc. In Chapter 4 we will demonstrate, how can the real option approach be used to estimate the value of a parcel to build a real estate complex on.

But every company also creates real options. They, according to Dixit and Pindyck [1, p. 9] "arise from a firm's managerial resources, technological knowledge, reputation, market position, and possible scale, all of which may have been built up over time, and which enable the firm to productively undertake investments that individuals or other firms cannot undertake".

Senel [13, p. 13] highlights, that especially high proportion of the real option value on the market capitalization of a company can be seen in the segment of internet companies. This value is often based on a strong brand, unique service or highly skilled team of employees, and therefore is hard to evaluate using the conventional valuation methods. That implies, that it is especially desirable to use the real option valuation for estimating the market value of those firms instead.

2 Stochastic calculus

If we want to construct a model for pricing financial options, and hence also real options, we have to take into consideration, that the future price of underlying asset and the future values of other variables, that determine the option value (interest rates, inflation), are an outcome of a stochastic process. In this chapter we will review the basics of stochastic calculus, which is the necessary building block to construct a relevant pricing model. We will build on the publication by Ševčovič, Stehlíková and Mikula [2], but staging the chapter into more gradual steps, inspired by Melicherčík, Olšarová and Úradníček [3] and Michnová [5].

2.1 The Ito processes

Definition 2.1 (Stochastic process). *Stochastic process is a t-parametric system of random variables* $\{X(t), t \in I\}$, *where* I *is an interval or a discrete set of indices.*

Definition 2.2 (Wiener process). *The Wiener process* $\{W(t), t \in [0, \infty)\}$ *is a stochastic process with following properties:*

- *W(0)=0*
- *W(t+s)-W(s)* $\sim$ *N(0,t)*
- *W(t) has independent increments,i.e.* $W(t_1), W(t_2) - W(t_1), \dots W(t_k) - W(t_{k-1})$ *are independent for all* $0 < t_1 < t_2 < \cdots < t_k$

where $W(u)$ *stands for the value of the random variable in time* u *and* $N(\mu, \sigma^2)$ *denotes normal distribution with a mean* μ *and variance* σ^2.

We will further denote dw *the increments of the Wiener process in an infinitesimal time* $dt \to 0$, *i.e.* $dw = W(t + dt) - W(t)$.

The formal notation of the Wiener process would be:

$$dw = \epsilon_t \sqrt{dt} \tag{10}$$

where $\epsilon_t \sim N(0, 1)$, $\sigma(\epsilon_t, \epsilon_s) = 0$, *for* $t \neq s$

The Wiener process is one of the most basic continuous time stochastic processes, and will serve us as a building block for construction of more complex processes, namely the Geometric Brownian motion, which we will make use of in our model.

Definition 2.3 (Brownian motion). *The Brownian motion is a stochastic process, derived from the Wiener process. It can be expressed by a differential equation:*

$$dx = \mu dt + \sigma dw \tag{11}$$

where $\mu \in \mathbb{R}$ is called drift, $\sigma \in [0, \infty)$ is the standard deviation and dw is an increment of the Wiener process. [2]

Definition 2.4 (Ito process). *A generalization of the Brownian motion, where a(x,t), b(x,t) are known (non-random) functions, is called* Ito process. *The differential formulation is given by:*

$$dx = a(x,t)dt + b(x,t)dw \tag{12}$$

Definition 2.5 (Geometric Brownian motion). *The Geometric Brownian motion is an Ito process $\{X(t), t > 0, X(0) > 0\}$, following the differential equation:*

$$dx = \mu x dt + \sigma x dw \tag{13}$$

where $\mu \in \mathbb{R}$, $\sigma \in [0, \infty)$ and dw is an increment of the Wiener process.

2.2 Properties of Ito processes

The stochastic processes are used to model the path of the selected variable. In our model, in order to make the calculations doable, we assume the *weak form of market efficiency.* This means that the markets are rational and all the information that might affect the future price of the given asset is already involved in its present price. This is translated mathematically as the *Markov property* of a stochastic process. The property states, that the future values of a variable depend only on its present value, and do not depend on the past values of the variable.

[2]Some authors, like Melicherčík [3] or Dixit and Pindyck [1] use the terms *Wiener process* and *Brownian motion* as synonyms to denote what we call the Wiener process. They introduce the notion *Brownian motion with drift* to denote the latter. We will, however, use the terminology of Ševčovič [2] and keep the two terms separately to avoid confusion.

Definition 2.6 (The Markov property). *A stochastic process $\{X(s) \in \Omega, s \in I\}$ has the Markov property, if:*
$\forall k \in \mathbb{N}, \forall(x_1, x_2, ...x_k), \forall(t_1 \leq t_2 \leq ... \leq t_k), x_i \in \Omega, t_i \in I, i \in \mathbb{N}:$
$P[X(t_k) = x_k | X(t_1) = x_1, X(t_2) = x_2, ...X(t_{k-1}) = x_{k-1}] = P[X_k = x_k | X_{k-1} = x_{k-1}]$

That is, the distribution of future values of the stochastic process is dependent only on the current value of the process and not on it's past values.

The model, that we will introduce in the next chapter, uses the Geometric Brownian motion (GBM) to model the price of an asset. Let us justify its usage and give a hint of explanation of its enormous popularity in modeling in general. Empirical studies of financial markets have shown, that in the long run the changes in prices of assets approximately follow the normal distribution. Thus their absolute values are roughly lognormally distributed, and therefore suitable to be modeled by GBM. The second benefit of the Geometric Brownian motion lies in it generating positive values, what is useful in modeling variables, that shouldn't take negative values (e.g. price, amount). And last, but not least, it is generated by a differential equation, that can be solved analytically, what is not true for most of the more advanced processes.

2.3 Ito Lemma

In the previous section we introduced the reader the basic theory of stochastic processes and pointed out several of them, that we might find useful as approximations of the future asset value in our model. We defined the processes in form of stochastic differential equations.

The next step in the option pricing is the analysis of the functions, where one of the variables solves the stochastic differential equation. The crucial question there is, whether it is always possible to construct a stochastic differential equation f(x,t), where x is a solution of a given stochastic differential equation.

Lemma 2.7 (Ito lemma)**.** *Let* $\{X(t), t \geq 0\}$ *be an Ito process given by a stochastic differential equation* $dx = \mu(x,t)dt + \sigma(x,t)dw$*, let* $f(x,t) \in C^2(\mathbb{R} \times [0,\infty))$*. Then the first differential of a function* $f(X(t))$ *is given by formula:*

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x,t)dt \tag{14}$$

which implies, that the function f solves the stochastic differential equation: [5]

$$df = (\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2})dt + \sigma(x,t)\frac{\partial f}{\partial x}dw \tag{15}$$

This lemma carries a very powerful result, as it allows us to expand the ordinary calculus into a stochastic setting, what we will find useful in the next chapter.

3 Samuelson-McKean model

As we have mentioned in the Chapter 1, models for appraising investment opportunities with real options are based on the tools and framework developed for pricing financial derivatives. In this chapter, we will base our valuation of the real options on the model developed for pricing financial perpetual American warrants (PAW)[3] - the famous Samuelson-McKean model (1965), [6].

In the decades following, several authors, namely Tourinho [7], Brennan and Schwartz [8] and McDonald and Siegel [9] subsequently altered the original model to cover real option valuation of an irreversible investment project, mostly focusing on an application in evaluation of natural resource deposits. In this chapter, we will introduce the reader the altered model, following the derivation by Björk [10] while aiming to apply it on valuation of real options such as the option to convert land into a real estate or an option to start exploiting a natural resource.

3.1 Inputs and assumptions

The model provides an analytical solution for the value of the PAW or the real option. There are five inputs to it: the construction cost (K), initial value (V_0), the yield (δ) and the expected volatility (σ) of the value of the asset (e.g. built property, commodity) and the risk-free rate (r).

The model simulates a decision of a company with an opportunity to invest (investor). Paying the immediate costs $K \in \mathbb{R}$ at any time t, an investor obtains a project - an asset of a value V [4]. K is known or deterministic. $V = V(t)$ is a random variable, that follows a geometric Brownian motion:

$$dV = \mu V dt + \sigma V dw \tag{16}$$

$$V(0) = V_0 > 0 \tag{17}$$

where $\mu \in \mathbb{R}$ is the drift rate, $\sigma \in \mathbb{R}^+$ is the standard deviation, dw is an increment of a Wiener process and V_0 is the initial value (the value of the project today).

[3] i.e. special types of financial perpetual American call options, that are issued and guaranteed by the company [14].

[4] meaning the spot price of the asset on the market at the certain point in time

We assume that there is a convenience yield δ (continuous rate) associated with the asset. This yield can be either positive (as in the case of a rent of a real estate project) or negative (as in the case of storage cost of given commodity (e.g. gold)). The investment is to take place in an environment with a constant continuous risk-free rate of r.

3.2 Outputs and the final proposition

The model returns two important outputs. $F = (V(V_0), K, \sigma, \delta, r) = F(V)$ is the value of an option to invest. Notice, that the the immediate payoff from undertaking the investment is equal to $V - K$. For the sake of identifying the boundary point between not investing (if *option value* $>$ *immediate payoff*) and investing (once *option value* $=$ *immediate payoff*), the model offers a welcome by-product: the *trigger value* V*.

Definition 3.1 (The (ROV) trigger value). *The ROV* trigger value V^**is the lowest possible value of the asset, at which the value of the option to invest F equals the immediate payoff from exercising it $V - K$. Only once the value of the asset V reaches the trigger V^*, the approach based on the real option valuation justifies the investment - exercising the option.*

And one final assumption: the value of an option to invest in an asset never increases with a decrease in value of the asset. In mathematical terms: F is a non decreasing function of V.

Having the assumptions on both inputs and outputs set up, we can finally state the theorem of the model.

Theorem 3.2 (Samuelson-McKean model). *Consider a firm with an investment opportunity. Let K be the investment cost, V the value of the completed project, σ the volatility of V, δ the rate of convenience yield on the project and r the risk-free interest rate. Then the real option valuation of the investment opportunity $F(V)$ and the trigger value V^* are:*

$$F(V) = \begin{cases} (V^* - K)(\frac{V}{V^*})^{\beta_1} & \text{if } V \leq V^* \\ V - K & \text{if } V > V^* \end{cases}$$

where

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{(\frac{r-\delta}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} \tag{18}$$

and

$$V^* = AK = \frac{\beta_1}{\beta_1 - 1}K \tag{19}$$

We will now define the trigger value for the NPV valuation for a brief comparison of the two methods.

Definition 3.3 (The Marshallian trigger value). *The Marshallian (NPV) trigger value V_M^* is the (lowest possible) value of an asset, at which the NPV of the investment equals 0. Only once the value of the asset V reaches the trigger V_M^*, the approach based on the NPV valuation justifies the investment.*

Notice, that the trigger value in NPV valuation with these conditions in place is equal to K. Hence $A = \frac{\beta_1}{\beta_1 - 1}$ tell us the relation between the outcomes of the two methods. As $\beta_1 > 1$, it is clear that the ROV trigger is always higher than the Marshallian trigger, the real option valuation therefore present a more conservative approach to investment.

3.3 Derivation of the formulas

We will now prove Theorem (3.2).

3.3.1 Obtaining an ordinary differential equation (ODE)

The value of our asset V evolves in a stochastic manner, following the geometric Brownian motion

$$dV = \mu V dt + \sigma V dw \tag{20}$$

Additionally, we denote the convenience yield in in a short amount of time as

$$dC = \delta V dt \tag{21}$$

It is important to note that this yield is a deterministic rate on a stochastic quantity V. Bearing that in mind, we can define a *gain* process that represents the change in the total wealth generated by owning the project of a value V:

$$dG = (\mu + \delta)V dt + \sigma V dw \tag{22}$$

Now we have an option on the asset of a value V. Its price will be a function of the asset value, $F(V)$. We build up a portfolio, the value of which we denote π, composed of a long position on the option and n short positions on the asset

$$\pi = F(V) - nV \tag{23}$$

We now take the full differential of the portfolio value π (but we take into account that here we are working with the process G and not V to find:[5]

$$d\pi = dF(V) - ndG \tag{24}$$

Applying Ito lemma, we can easily find $dF(V)$.

$$dF(V) = \left(\mu V \frac{\partial F}{\partial V} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial V^2} V^2\right) dt + \left(\sigma \frac{\partial F}{\partial V}\right) dw \tag{25}$$

[5] We consider the *full gain*, that is a sum of the convenience yield (at rate δ) and capital gains (at rate μ) on an asset. We included the convenience yield in order to be able to compare the total wealth generated by the portfolio to the total wealth on another portfolio.

Putting it all together with dG we find,

$$d\pi = \left(\mu V \frac{\partial F}{\partial V} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial V^2} V^2 - n(\mu + \delta)V\right) dt \tag{26}$$

$$+ \left(\sigma V \frac{\partial F}{\partial V} - n\sigma V\right) dw \tag{27}$$

$$\tag{28}$$

Now, up to the moment we have a completely unspecified n, we can of course study a particular portfolio, one with exactly $\frac{\partial F}{\partial V}$ options (short). This will have a very interesting effect, with $n = \frac{\partial F}{\partial V}$ one finds that the stochastic term (that is, the term multiplying with dw) vanishes, and we are let with a completely deterministic portfolio with dynamics,

$$d\pi = \left(\frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial V^2} V^2 - \delta \frac{\partial F}{\partial V} V\right) dt \tag{29}$$

$$\tag{30}$$

Finally, since we assume no arbitrage in this market, we have that this portfolio must evolve in the same way as a risk-free asset, that is

$$d\pi = r\pi dt = r\left(F(V) - \frac{\partial F}{\partial V} V\right) dt \tag{31}$$

Therefore

$$r(F(V) - \frac{\partial F}{\partial V} V) = \left(\frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial V^2} V^2 - \delta \frac{\partial F}{\partial V} V\right) \tag{32}$$

Recollecting the terms we obtain the second order ordinary differential equation (ODE) for the value of the option F:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial V^2} V^2 + (r - \delta)\frac{\partial F}{\partial V} V - rF(V) = 0 \tag{33}$$

3.3.2 Solving the ODE

The second order ordinary differential equation ODE (33) has the characteristic equation:

$$\frac{1}{2}\sigma^2\beta(\beta-1)+(r-\delta)\beta-r=0 \tag{34}$$

with the roots β_1, β_2:

$$\beta_1=\frac{1}{2}-\frac{r-\delta}{\sigma^2}+\sqrt{[\frac{r-\delta}{\sigma^2}-\frac{1}{2}]^2+\frac{2r}{\sigma^2}} \tag{35}$$

and

$$\beta_2=\frac{1}{2}-\frac{r-\delta}{\sigma^2}-\sqrt{[\frac{(r-\delta)}{\sigma^2}-\frac{1}{2}]^2+\frac{2r}{\sigma^2}} \tag{36}$$

Therefore the ODE has a solution:

$$F(V)=c_1V^{\beta_1}+c_2V^{\beta_2} \tag{37}$$

where c_1, c_2 are arbitrary constants. One can easily verify, that $\beta_1>1$ and $\beta_2<0$.

3.3.3 Finding the c_1, c_2

We start with c_2. The reader can easily notice that as $\beta_2<0$ and $V>0$ (since it follows a GBM) the term V^{β_2} increases with a downward move of V. The value of the option, however, is an non decreasing function of V [6]. This implies c_2 in term $c_2V^{\beta_2}$ has to be less or equal to zero. But the value of an option is never negative. Therefore $c_2=0$. Knowing that fact, the value of an option would be:

$$F(V)=c_1V^{\beta_1} \tag{38}$$

where $\beta_1>1$ Let us consider c_1 next. Setting two trivial boundary conditions on F(V), we are able to find c_1 and as a bonus, we obtain also the formula for the ẗriggerv̈alue V* of the project.

[6]see section 3.2

First boundary condition: Value matching

As we mentioned in section 3.2, V^* is the value of the project at which its NPV equals the option value:

$$V^* - K = c_1(V^*)^{\beta_1} \tag{39}$$

Second boundary condition: Smooth pasting

The second condition is the equality of derivatives of the two functions from (39) at the trigger value V^*.

$$\beta_1 c_1 V^{*^{\beta_1 - 1}} = 1 \tag{40}$$

Solving the system of (39) and (40) we get the trigger value and the arbitrary constant c_1:

$$c_1 = \frac{(\beta_1 - 1)^{\beta_1 - 1}}{\beta_1^{\beta_1} K^{\beta_1 - 1}} = \frac{K}{(\beta_1 - 1)V^{*\beta_1}} = \frac{V^* - K}{V^{*\beta_1}} \tag{41}$$

$$V^* = \frac{\beta_1}{\beta_1 - 1} K \tag{42}$$

And eventually, substituting (41) for c_1 in (38), we obtain the formula for the value of the option to invest:

$$F(V) = \frac{K}{\beta_1 - 1}(\frac{V}{V^*})^{\beta_1} = (V^* - K)(\frac{V}{V^*})^{\beta_1} \tag{43}$$

4 Application

In this Chapter, using the Samuelson-McKean model with real market data, we will apply the ROV to resolve an investment dilemma of a real estate developer.

4.1 The case study

BTI, ltd. is a real estate development company, that considers building a residence complex in Northern Iowa, US. The estimated construction costs of the project are estimated to be $K = \$30.700$ million, and the built property has the market value of V, that fluctuates stochastically and currently has the value of $V_0 = \$31.000$ million.

If firm kept the complex in its portfolio, it would be yearly receiving approximately $\delta = 0.03$ of the value of the building on rent (after discounting depreciation of value of the property). The firm, however, plans to sell the building immediately after finalization of the building process. The full construction cost K is paid to a building company at the beginning of the construction phase. We assume, that the construction process is fast and we can neglect its duration.

The nominal risk free discount rate r is only 0.0065 % per annum (continuous rate)[7]. An investment fund P&P&P Inc. is interested in buying the parcel for the price $P = \$5.4$ million.

Should BTI undertake the project now? Should it sell the land? What is the minimum market value of the complex, at which construction should commence?

4.2 The ROV-based analysis

As the firm's financial analysts we estimate, that the value of the built property follows the Geometric Brownian motion with parameter $\sigma = 0.1826$. This estimation is based on variance of the historical values of the iShares US Real Estate (IYR) ETF from Aug 17 2009 to May 25 2014 [8]. Our assumption of normality of changes in V can be

[7]on 29th May 2014 the highest yield on an 6 months CD on the US market by MetLife Bank: http://us.deposits.org/accounts/metlife-bank-6-month-cd-rates.html

[8]http://finance.yahoo.com/echarts?s=IYR+Interactive#symbol=IYR;range=1d We decided to start at this date, because the months preceding August 2009 were a turbulent period with extreme volatility because of the broken bubble on the real estate market, connected with a huge downturn

considered feasible, see Appendix.

First we compute the coefficient β_1 and the trigger value V^*:

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{(\frac{r-\delta}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} \doteq 2.562 \tag{44}$$

and

$$V^* = AK = \frac{\beta_1}{\beta_1 - 1} K \doteq 1.640K \doteq 50.357 \tag{45}$$

As $V^* \doteq \$50.357$ million$>V_0 = \$30.700$ million, the ROV orders the company to delay the investment. What is more, it claims, that the property would have to rise almost 62.5% in market value, if the investment was to be profitable. On the contrary, according to the NPV valuation we are above the trigger point $V_M^* = \$31.000$ million.

Notice, that $A \doteq 1.640$, meaning the trigger value in ROV is *1.640 times higher* than the Marshallian trigger. The huge difference between the outcomes of the two methods implies, that at least one of them greatly inaccurate. In the next section we will explain, why the one is probably not ROV.

The option value (value of the opportunity to build the complex = the value of the underlying parcel and the rights to build the complex) is:

$$F(V) = \begin{cases} (V^* - K)(\frac{V}{V^*})^{\beta_1} \doteq 0.00086V^{2.562} & \text{if } V \leq 50.357 \\ V - 30.700 & \text{if } V > 50.357 \end{cases}$$

With the particular $V_0 = \$31.700$ million, F is worth approximately \$ 5.672 million. Therefore the ROV prefers keeping the parcel over selling it to P&P&P for \$ 5.400 million. However, keeping the option open entails a significant level of liquidity risk, as it is uncertain, how long will it take the value of the building to cross the trigger value, so that the investment could commence or when will BTS once more have the opportunity to sell the land.

in real estate prices. The data from those months would exaggerate the sigma, as such an enormous downturn is not likely to repeat again.

Moreover, the real estate market is one of the least liquid ones in general, therefore the selling process would not be easy and could take long time, negatively affecting also the accuracy of the ROV. Neglecting the duration of the construction is also a potential source of inaccuracy. All three sources of liquidity risk and uncertainty can be lowered by signing a forward contract. However, it is highly probable, especially in the first case, that the price of such forward would exceed the difference $P - F(V)$ and therefore make the investment unprofitable. Thus it depends on the BTI's acceptance of liquidity risk, whether it keeps the parcel or sells it for a lower, but fixed price.

4.3 Further observations

The Figure 1 illustrates the relationship between V and $F(V)$ for different levels of σ. Observe the smooth pasting of the two pieces of the $F(V)$ in the trigger value V^*.

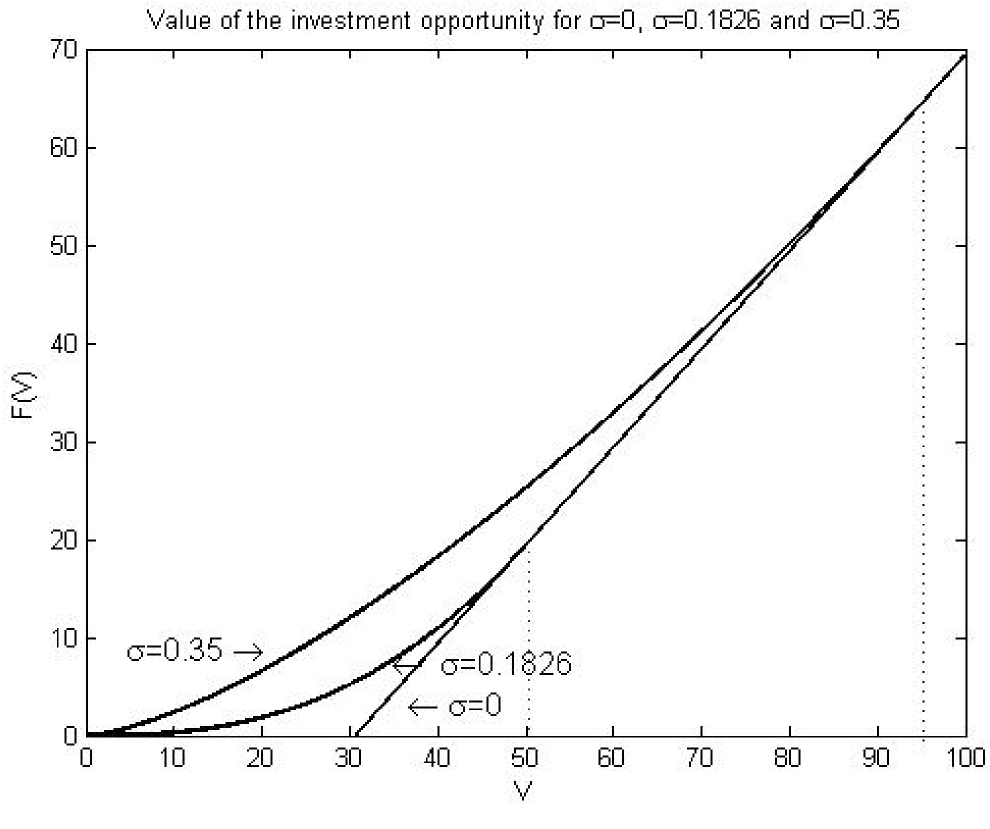

Figure 1: Value of the investment opportunity for $\sigma = 0$, $\sigma = 0.1826$ and $\sigma = 0.35$

Note that when $\sigma = 0$, $V^* = K$ and therefore $F(V) = 0$ for $V < K$. The ROV and NPV valuation are equivalent in case of no uncertainty. This idea can be easily generalized by a brief look at the formula. The higher the volatility, the bigger the option value (opportunity cost of investment), and hence also the difference between the two approaches.

Notice that both F and V^* are increasing functions of σ. The higher the volatility, the higher the probability, that V exceeds any given value in a given amount of time, therefore it becomes more desirable to wait rather than invest immediately. Hence the firm's investment opportunities have higher value, but it becomes more conservative in their exploitation. Dixit and Pidyck in [1] spot an interesting implication: "As a result, when a firm's market or economic environment becomes more uncertain, the market value of the firm can go up, even though the firm does less investment and perhaps produces less."

The following figure shows more directly, how V^* depends on σ. The function is steep, and leaves a clear message. Uncertainty represses investment. Regardless of the correlation of the particular project with the market, regardless of the firm's aversion to risk, regardless of the hedging in place.

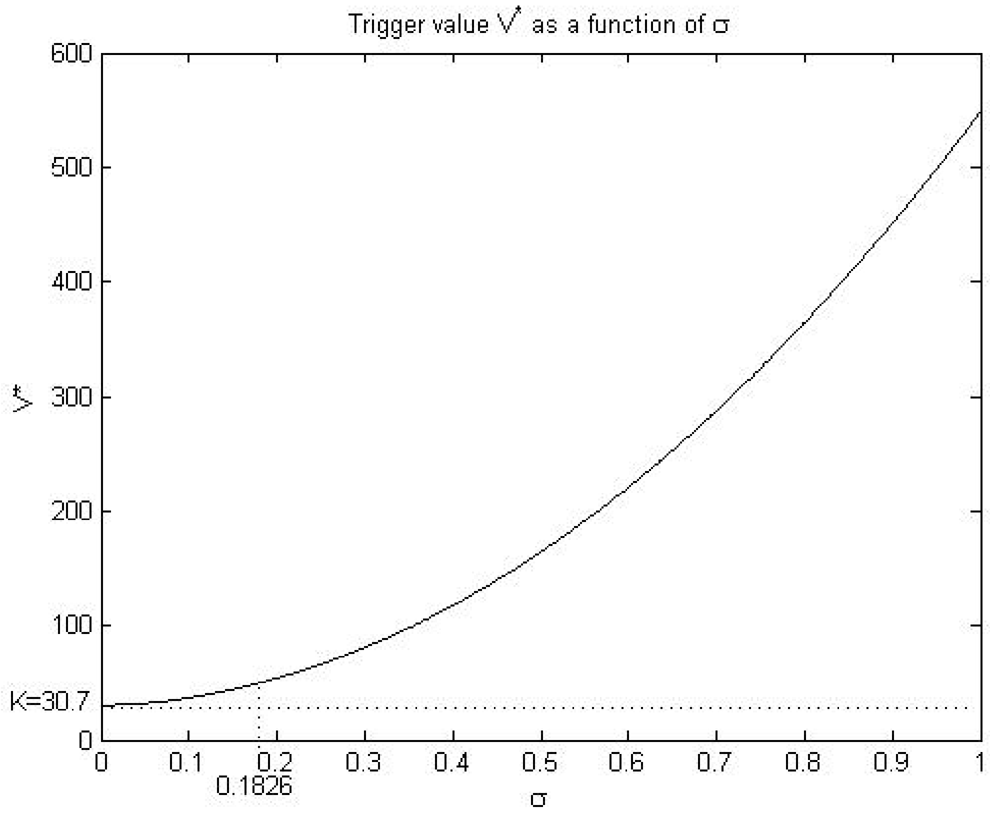

Figure 2: Trigger value V^* as a function of σ

Conclusion

Drawing an analogy between an irreversible investment opportunity in an uncertain environment with an ability to be delayed and a financial perpetual American option, the real option approach presents a whole new view of investment decision-making.

Taking into account the stochastic character and volatility of the future values of a given source of uncertainty (in our case the value of the project) and the ability to decide optimally in the future, the ROV, especially in a highly uncertain environment, recognizes the value of waiting for more, but yet never complete information, that could affect the desirability of an investment. Therefore it presents a more conservative view of investment than the conventional methods like the NPV. The real option approach, being either purposely or intuitively used by real world companies, could be the explanation of phenomenons such as, the sharp decrease in amount of investment made with an increase of market volatility or the hurdle rates high above the IRR.

The Samuelson-McKean model is a basic model for appraising investment opportunities, which carry the characteristics of a real option. Its application is very straightforward, as it requires just an analytical solution, and the process of its derivation is understandable for a person with solid mathematical background. The model operates with several simplifications, such as a single source of uncertainty, its value performing the geometric Brownian motion, or the perfectly liquid markets, that ensure we pay the cost and receive the value in an instant. Even though, its output is usually far more feasible than that of any of the conventional valuation methods, and the difference is often significant.

The model can be further improved by incorporating a broad class of real life options such as an option to stage an investment into several phases or the temporary shutdown of production of a factory. There is also a variety of other stochastic processes, if the GBM was not the right one to use, and a possibility to work with more than one source of uncertainty. The real option valuation is a highly flexible framework with a wide field of a application, ranging from pricing the parcels and patents to everyday decision-making of an individual and is definitely a potentially fertile field for further research.

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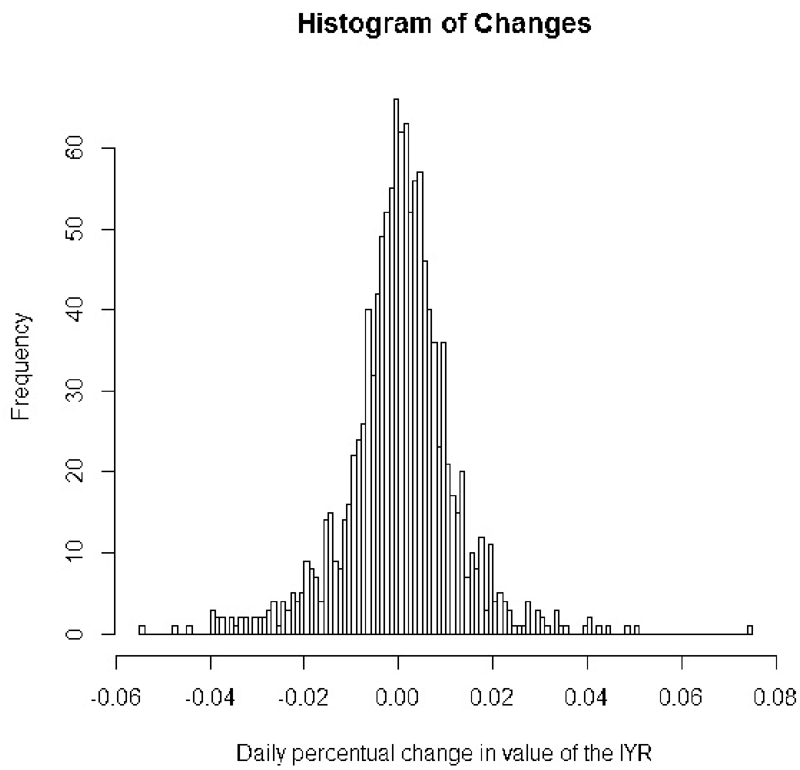
Appendix

Normality of daily changes in value of the iShares US Real Estate (IYR) ETF

The data range from Aug 17 2009 to May 25 2014. We decided to start on Aug 17 2009, because the months preceding August 2009 were a turbulent period with extreme volatility because of the broken bubble on the real estate market, connected with a huge downturn in real estate prices. The data from those months would exaggerate the sigma, as such an enormous downturn is very unlikely to repeat again.

Source: http://finance.yahoo.com/echarts?s=IYR+Interactive#symbol=IYR;range=1d

The histogram hints the normal distribution:



We will perform the normality testing with Kolmogorov-Smirnoff test on the level of significance of $\alpha = 5\%$. We set the mean equal to $\mu_d = 0.0003157429$ and standard deviation to 0.01155032 (this is a daily standard deviation σ_d, as the yearly sigma is proportional to the square root of time (250 days), $\sigma = \sqrt{250}\sigma_d$). The computed p-value is 0.2705, that does not contradict our hypothesis of normality.