A REMARK ON THE LARGE TIME BEHAVIOR OF SOLUTIONS OF VISCOUS HAMILTON-JACOBI EQUATIONS

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1. Introduction and main result

Consider the viscous Hamilton-Jacobi equation

\[
\begin{aligned}
&u_t - \Delta u = |\nabla u|^q, & t > 0, & x \in \mathbb{R}^N \\
&u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{aligned}
\]

(1.1)

where \( q > 0 \) and \( u_0 \in C_b(\mathbb{R}^N) \). It is known [6] that (1.1) admits a unique classical solution, global for \( t > 0 \).

The large time behavior of solutions of problem (1.1) has been studied recently by several authors, see [1]–[5], [7, 8] and the references therein. In particular it was shown by Gilding [5] that the large time limits

\[
\omega := \liminf_{t \to \infty} v(x, t) \leq \overline{\omega} := \limsup_{t \to \infty} v(x, t)
\]

are independent of \( x \in \mathbb{R}^N \). One of the main results of [5] is the following.

**Theorem A.** Assume \( 0 < q < 2 \) and \( u_0 \in C_b(\mathbb{R}^N) \). Then \( \omega = \overline{\omega} \).

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It was known that Theorem A fails for the linear heat equation and, moreover, Gilding observed that it fails for $q = 2$. The aim of this short note is to show that the assumption $q < 2$ in Theorem A is actually necessary.

**Theorem 1.** Assume $q \geq 2$. Then there exists $u_0 \in C_b(\mathbb{R}^N)$ such that $\omega < \overline{\omega}$.

**Proof.** It is known (see e.g. [5, Proposition H1]) that there exists $v_0 \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that the solution $v$ of the heat equation

\[
\begin{cases}
  v_t - \Delta v = 0, & t > 0, \ x \in \mathbb{R}^N \\
  v(0, x) = v_0(x), & x \in \mathbb{R}^N
\end{cases}
\]  

(1.2)

satisfies

\[
\omega^* := \liminf_{t \to \infty} v(x, t) < \overline{\omega}^* := \limsup_{t \to \infty} v(x, t), \ x \in \mathbb{R}^N.
\]  

(1.3)

Moreover, upon replacing $v_0$ by $\lambda v_0 + \mu$ for suitable constants $\lambda, \mu$, one can assume that

\[
\omega^* = 0
\]  

(1.4)

and

\[
\|v_0\|_\infty \leq 1/2, \quad \|\nabla v_0\|_\infty \leq 1/2.
\]

(1.5)

Now, set

\[
u_0(x) := e^{v_0(x)} - 1.
\]

(1.5)

The function $w := e^v - 1$ satisfies

\[
\begin{cases}
  w_t - \Delta w = |\nabla w|^2, & t > 0, \ x \in \mathbb{R}^N \\
  w(0, x) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\]  

(1.6)
Let $u$ be the solution of (1.1) with initial data $u_0$ defined by (1.5). We note that
\[ \|\nabla u_0\|_{\infty} \leq \|\nabla v_0\|_{\infty} \|e^{v_0}\|_{\infty} \leq (1/2) e^{1/2} < 1. \]

Since it is known (see e.g. [5, Lemma 2]) that $|\nabla u|$ satisfies a maximum principle, it follows that
\[ |\nabla u| \leq \|\nabla u_0\|_{\infty} < 1 \quad \text{in } Q := (0, \infty) \times \mathbb{R}^N. \]

Due to $q \geq 2$, we deduce that
\[ u_t - \Delta u = |\nabla u|^q \leq |\nabla u|^2 \quad \text{in } Q. \]

In view of (1.6), it follows from the comparison principle that
\[ u \leq w = e^v - 1 \quad \text{in } Q. \]

In particular, there holds
\[ (1.7) \quad \omega \leq e^{\omega^*} - 1 = 0. \]

But on the other hand, we have $u_0 \geq v_0$ due to (1.5). In view of (1.2), the maximum principle implies that $u \geq v$, hence
\[ (1.8) \quad \overline{\omega} \geq \overline{\omega}^*. \]

Combining (1.3), (1.4), (1.7) and (1.8), we conclude that
\[ \overline{\omega} \geq \overline{\omega}^* > \overline{\omega}^* = 0 \geq \omega \]

and the proof of Theorem 1 is complete. \qed


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