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NONLINEAR OSCILLATIONS OF COMPLETELY RESONANT WAVE EQUATIONS*

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Abstract. We present recent existence and multiplicity results of small amplitude periodic solutions of completely resonant nonlinear wave equations with frequencies ω belonging to a Cantor-like set of asymptotically full measure. The proofs rely on a suitable Lyapunov-Schmidt decomposition, a variant of the Nash-Moser Implicit Function Theorem and Variational Methods.

Key words. Nonlinear Wave Equation, Infinite Dimensional Hamiltonian Systems, Periodic Solutions, Variational Methods, Lyapunov-Schmidt reduction, Small Divisors, Nash-Moser Theorem.

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1. Completely resonant PDEs. We consider completely resonant nonlinear wave equations like

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.1}$$

where the nonlinearity

$$f(x,u) = a_p(x)u^p + O(u^{p+1}), \qquad p \geq 2, \tag{1.2}$$

vanishes at least quadratically at $u = 0$.

Equation (1.1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$. Any solution

$$v = \sum_{j\geq 1} a_j \cos(jt + \theta_j)\sin(jx)$$

of the linearized equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.3}$$

is 2π-periodic in time. For this reason, equation (1.1)–(1.2) is called a *completely resonant* PDE.

- *Question*: there exist periodic solutions of the nonlinear equation (1.1)–(1.2) close to the equilibrium solution $u = 0$?

For finite dimensional Hamiltonian systems, existence of periodic solutions close to a completely resonant elliptic equilibrium has been proved by Weinstein [14], Moser [12] and Fadell-Rabinowitz [10]. The proofs are based on the classical Lyapunov-Schmidt decomposition which splits the problem into

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(i) the *range equation*, solved through the standard Implicit Function Theorem,
(ii) the *bifurcation equation*, solved via variational arguments.

To extend these results for completely resonant Hamiltonian PDEs like (1.1)–(1.2), the main difficulties to be overcome are

(i) a "*small divisors problem*" which prevents, in general, to use the implicit function theorem to solve the range equation,
(ii) the presence of an *infinite dimensional* bifurcation equation: which solutions v of the linearized equation (1.3) can be continued to solutions of the nonlinear equation (1.1)?

The "small divisors problem" (i) arises as follows. Since Equation (1.1) is autonomous, the frequency ω of the periodic solution is not a-priori fixed. We introduce ω as a free parameter looking for 2π-periodic solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \,. \end{cases} \tag{1.4}$$

The eigenvalues of the linear operator

$$L_\omega := \omega^2 \partial_{tt} - \partial_{xx}$$

in a space of functions $u(t,x)$, 2π-periodic in time and valued in $H_0^1(0,\pi)$ (because of Dirichlet boundary conditions) are

$$\sigma(L_\omega) \equiv \Big\{ -\omega^2 l^2 + j^2 \,,\, l \in \mathbb{Z} \,,\, j \geq 1 \Big\} \,. \tag{1.5}$$

Therefore, for almost every $\omega \in \mathbb{R}$, the eigenvalues of L_ω accumulate to 0, implying that the inverse operator of L_ω is unbounded. For this reason the standard Implicit Function Theorem is, in general, not applicable.

The first existence result of small amplitude periodic solutions of (1.1)–(1.2) has been obtained in [3] for $f = u^3 + O(u^5)$, imposing on the frequency ω the "strongly non-resonance" condition $|\omega l - j| \geq \gamma/l, \forall l \neq j,\, l \geq 0$, which is satisfied in a *zero measure* set accumulating at $\omega = 1$. For such ω the spectrum of L_ω does not accumulate to 0 because $|-\omega^2 l^2 + j^2| = |\omega l - j||\omega l + j| \geq \gamma\omega$, and so the small divisor problem (i) does not appear. Next, the bifurcation equation (problem (ii)) is solved proving that the 0th-order bifurcation equation possesses *non-degenerate* periodic solutions.

In [4]–[5], for the same set of strongly non-resonant frequencies, existence and multiplicity of periodic solutions has been proved for *any* nonlinearity $f(u)$. The novelty of [4]–[5] was to solve the infinite dimensional bifurcation equation via a variational principle at fixed frequency (in the spirit of Fadell-Rabinowitz [10]) which, jointly with min-max arguments, enables to find solutions of (1.1) as critical points of the Lagrangian action functional (mountain pass critical points of a "reduced" action functional).

We now want concentrate on the small divisors problem (i) in order to find periodic solutions of (1.1) for positive measure (actually asymptotically full measure) sets of frequencies close to $\omega = 1$, presenting the recent results of [6]–[7].

Previous results in this direction have been obtained in [8] (for periodic spatial boundary conditions) and in [11] with the Lindsted series method, for $f = u^3 +$ h.o.t.. Again the dominant term u^3 garantees a non-degeneracy property for the solutions of the 0th order bifurcation equation.

2. Existence of periodic solutions for asympotically full measure sets of frequencies. Instead of looking for solutions of (1.4) in a shrinking neighborhood of $u = 0$, let perform the rescaling

$$u \to \delta u\,, \qquad \delta > 0\,,$$

obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \varepsilon g(\delta, x, u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.1}$$

where

$$\varepsilon := \delta^{p-1} \qquad \text{and} \qquad g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^p} = a_p(x) u^p + \delta a_{p+1}(x) u^{p+1} + \dots\,.$$

We look for solutions of (2.1) in the Hilbert algebra ($\sigma > 0$, $s > 1/2$)

$$X_{\sigma,s} := \Big\{ u = \sum_{l \geq 0} \cos(lt)\ u_l(x) \ \Big|\ u_l \in H_0^1(0,\pi),$$
$$\|u\|^2_{\sigma,s} := \sum_{l\geq 0} \mathrm{e}^{2\sigma|l|}(l^{2s}+1)\|u_l\|^2_{H^1} < +\infty \Big\}$$

of 2π-periodic, even, σ-analytic in time functions valued in $H_0^1(0,\pi)$ (we can look for even solutions because equation (1.1) is reversible).

The solutions of the linear equation (1.3) that belong to $H_0^1(\mathbf{T} \times (0,\pi), \mathbf{R})$ and are even in time form the infinite dimensional linear space

$$V := \Big\{ v(t,x) = \sum_{l \geq 1} \cos(lt) u_l \sin(lx) \ \mid\ \sum_{l\geq 1} l^2 |u_l|^2 < +\infty \Big\}\,.$$

We endow V with the H^1-topology in view of the variational arguments used for the bifurcation equation.

Let implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s})$$

where

$$W := \Big\{ w = \sum_{l\geq 0} \cos(lt)\ w_l(x) \in X_{0,s} \quad \text{such that} \quad \int_0^\pi w_l(x)\sin(lx) = 0 \Big\}\,.$$

Projecting Equation (1.1), setting $u = v + w$, $v \in V$, $w \in W$, and imposing the "frequency-amplitude" relation

$$\frac{\omega^2 - 1}{2} = s^* \varepsilon$$

with $s^* = \pm 1$ to be chosen later (see (2.5)), yields

$$\begin{cases} -\Delta v = s^* \Pi_V g(\delta, x, v + w) & \textit{bifurcation equation} \\ L_\omega w + \varepsilon \Pi_W g(\delta, x, v+w) = 0 & \textit{range equation} \end{cases} \tag{2.2}$$

where $\Delta v := v_{xx} + v_{tt}$ and Π_V, Π_W denote the projectors respectively on V and W.

2.1. The finite dimensional reduction. Since V is infinite dimensional a serious difficulty arises in the application of the method of [9]: if $v \in V \cap X_{\sigma,s}$ then the solution $w(\delta, v)$ of the range equation, obtained with any Nash-Moser iteration scheme will have a lower regularity, e.g. $w(\delta, v) \in X_{\sigma/2,s}$. Therefore, in solving next the bifurcation equation substituting $w = w(\delta, v)$, the best estimate we can obtain is $v \in V \cap X_{\sigma/2,s+2}$ which makes the scheme incoherent. In [9] this problem does not arise since, dealing with nonresonant or partially resonant Hamiltonian PDEs like $u_{tt} - u_{xx} + a_1(x)u = f(x, u)$, the bifurcation equation is finite dimensional.

Moreover we have to ensure that the 0th-order bifurcation equation[1] (obtained setting $\delta = 0$ in the bifurcation equation)

$$- \Delta v = s^* \Pi_V \Big(a_p(x) v^p \Big) \tag{2.3}$$

has solutions $v \in V$ which are analytic, a necessary property to initiate an analytic Nash-Moser scheme.

We overcome both these difficulties thanks to a reduction to a *finite dimensional* bifurcation equation on a subspace of V of dimension N independent of ω. This reduction can be implemented, in spite of the complete resonance of equation (1.1)–(1.2), thanks to the compactness of the operator $(-\Delta)^{-1}$. Let decompose

$$V = V_1 \oplus V_2$$

where

$$\begin{cases} V_1 := \Big\{ v \in V \mid v(t,x) = \sum_{l=1}^{N} \cos(lt) u_l \sin(lx) \Big\}, & \text{“}\textit{low Fourier modes}\text{”}, \\ V_2 := \Big\{ v \in V \mid v(t,x) = \sum_{l>N} \cos(lt) u_l \sin(lx) \Big\}, & \text{“}\textit{high Fourier modes}\text{”}. \end{cases}$$

Setting $v := v_1 + v_2$, $v_1 \in V_1, v_2 \in V_2$, system (2.2) is equivalent to

$$\begin{cases} -\Delta v_1 = s^* \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) & (Q1) \\ -\Delta v_2 = s^* \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) & (Q2) \\ L_\omega w + \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w) = 0 & \textit{range equation} \end{cases} \tag{2.4}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$ $(i = 1, 2)$ denote the projectors on V_i.

Our strategy to find solutions of system (2.4) is the following.

Step 1: Solution of the (Q2)-equation. The solution $v_2(\delta, v_1, w)$ of the $(Q2)$-equation is found as a fixed point of $v_2 = s^*(-\Delta)^{-1} \Pi_{V_2} g(\delta, x, v_1 + v_2 + w)$ by a Contraction mapping argument. We obtain, if $w \in W \cap X_{\sigma,s}$, a solution $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s+2}$, provided N is large enough and $0 < \sigma \le \overline{\sigma}$ is small enough, depending only on the nonlinearity f. To clarify this point note that equation (2.3) is the Euler Lagrange equation of the functional $\Phi_0 : V \to \mathbb{R}$

$$\Phi_0(v) := \frac{\|v\|^2_{H^1}}{2} - s^* \int_\Omega a_p(x) \frac{v^{p+1}}{p+1}\,, \qquad \Omega := \mathbf{T} \times (0, \pi)$$

[1] We assume for simplicity that $\Pi_V(a_p(x)v^p) \not\equiv 0$ or, equivalently, $\int_\Omega a_p(x)v^{p+1} \not\equiv 0$, which is verified for p odd, iff $a(x)$ is not antisymmetric w.r.t. to $x = \pi/2$, and for p even, iff $a(x)$ is not symmetric w.r.t. to $x = \pi/2$. If not verified – like for $f = u^2$, u^4 – the $0th$-order non-trivial bifurcation equation will involve the higher order terms of the nonlinearity.

where $\|v\|_{H^1}^2 = \int_\Omega v_t^2 + v_x^2$. Choose

$$s^* = \begin{cases} 1 & \text{if } \exists \tilde v \in V \quad \text{such that} \displaystyle\int_\Omega a_p(x) \tilde v^{p+1} > 0 \\ -1 & \text{if } \exists \tilde v \in V \quad \text{such that} \displaystyle\int_\Omega a_p(x) \tilde v^{p+1} < 0 \end{cases} \tag{2.5}$$

and let $\widetilde t > 0$ be large enough such that $\Phi_0(\widetilde t \tilde v) < 0$. The mountain pass value

$$c := \inf \Big\{ \max_{t \in [0,1]} \Phi_0(\gamma(t)) \mid \gamma \in C([0,1], V)\,, \gamma(0) = 0\,, \gamma(1) = \widetilde t \tilde v \Big\}$$

($c > 0$) is a critical level[2] with a non-trivial critical set

$$K_c := \Big\{ v \in V \mid \Phi_0(v) = c\,, \Phi_0'(v) = 0 \Big\}$$

which is compact for the H^1-topology [4], in particularly K_c is bounded $\|v\|_{H^1} \le R$, $\forall v \in K_c$. N must be chosen large enough depending only on K_c: euristically, to find solutions of the complete bifurcation equation close to the solutions K_c of the 0th order bifurcation equation (2.3), N must be taken large enough so that the majority of the H^1-"mass" of the solutions of K_c is "concentrated" on the first N Fourier modes.

Step 2: Solution of the range equation. We solve next the range equation

$$L_\omega w + \varepsilon \Pi_W \Gamma(\delta, v_1, w) = 0 \tag{2.6}$$

where

$$\Gamma(\delta, v_1, w) := g(\delta, x, v_1 + w + v_2(\delta, v_1, w))$$

by means of a *Nash-Moser* Implicit Function Theorem for (δ, v_1) belonging to some Cantor-like set of parameters.

THEOREM 2.1 ([6]). *For $\|v_1\|_{H^1} \le 2R$ and δ small enough, there is a C^∞ function $\widetilde w(\delta, v_1) \in W \cap X_{\overline\sigma/2,s}$ and the "large" Cantor-like set*

$$B_\infty := \Big\{ (\delta, v_1) \mid \Big| \omega l - j - s^* \varepsilon \frac{M(\delta, v_1, \widetilde w(\delta, v_1))}{2j} \Big| \ge \frac{\gamma}{l^\tau}\,, \ \Big| \omega l - j \Big| \ge \frac{\gamma}{l^\tau}\,, \ \forall l \ne j\,,\ l, j \ge \frac{1}{3\varepsilon} \Big\}$$

where $M(\delta, v_1, w) := (1/|\Omega|) \int_\Omega (\partial_u g)(\delta, x, u)$, such that, $\forall (\delta, v_1) \in B_\infty$, $\widetilde w(\delta, v_1)$ solves the range equation (2.6).

To understand how such Cantor set B_∞ arises, we recall that the core of any Nash-Moser convergence method (based on a Newton's iteration scheme) is the proof of the invertibility of the linearized operators

$$\mathcal{L}(\delta, v_1, w)[h] := L_\omega h + \varepsilon \Pi_W D_w \Gamma(\delta, v_1, w)[h]$$

where w is the approximate solution obtained at a given stage of the Nash-Moser iteration. The eigenvalues $\{\lambda_{lj}(\delta, v_1),\, l \ge 0, j \ge 1\}$ of $\mathcal{L}(\delta, v_1, w)$ are, in general, dense on $\mathbb{R}$ (as the spectrum of the unperturbed operator L_ω in (1.5)) and depend in a very sensitive way on the parameters (δ, v_1). We estimate perturbatively $\lambda_{lj}(\delta, v_1)$ and, imposing restrictions

[2] Actually Φ_0 has an unbounded sequence of critical levels tending to plus infinity [1].

on (δ, v_1) like $|\lambda_{lj}(\delta, v_1)| \geq |l|^{-(\tau-1)}$, we obtain the invertibility of $\mathcal{L}(\delta, v_1, w)$ with a controlled loss of analiticity to obtain the convergence of the iterative scheme.

Our approach is different than in [9] and works also for *not* odd nonlinearities f with low spatial regularity, unlike [9] works only for nonlinearities which are odd and analytic in (x, u).

Note that the difficulty mentioned at the beginning of the subsection is overcome because, since $v_2(\delta, v_1, w)$ has always the same regularity of $w \in X_{\sigma,s}$ (actually 2 derivatives more because of the regularizing effect of $(-\Delta)^{-1}$), during the Nash-Moser iteration $v_2(\delta, v_1, w_n)$ will "adjust" its regularity as the iterates of w_n (which decrease its analyticity).

Step 3: solution of the $(Q1)$-equation. Finally remains the finite dimensional bifurcation equation

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(\delta, v_1) \tag{2.7}$$

where $\mathcal{G}(\delta, v_1) := g(\delta, x, v_1 + \widetilde{w}(\delta, v_1) + v_2(\delta, v_1, \widetilde{w}(\delta, v_1)))$.

We have to ensure that for a positive measure set of δ there are solutions $v_1(\delta)$ of (2.7) with $(\delta, v_1(\delta))$ belonging to the Cantor set B_∞ where also the range equation was solved. Although B_∞ is – in a measure theoretic sense – a "large" set this property is *not* obvious because there are "gaps" in B_∞.

The first way to proceed is to assume a classical non-degeneracy condition (analogue to the KAM-Arnold condition) stating the existence of a non-degenerate solution $\overline{v}$ of (2.3), i.e. $\mathrm{Ker} D^2 \Phi_0(\overline{v}) = \{0\}$. In this case, by the implicit function theorem, there is a C^∞-curve $(\delta \to v_1(\delta))$ of solutions of (2.7) (with $v_1(0) = \Pi_{V_1} \overline{v}$) which intersects transversally, and so in a asympotically full measure set, the Cantor set B_∞. This non-degeneracy condition can be verified for several nonlinearities [6]–[2].

THEOREM 2.2. **([6])** *Assume that*[3]

$$f(x,u) = \begin{cases} a_2 u^2 + \sum_{k \geq 4} a_k(x) u^k\,, & a_2 \neq 0 \\ a_3(x) u^3 + \sum_{k \geq 4} a_k(x) u^k\,, & \langle a_3 \rangle := \pi^{-1} \int_0^\pi a_3(x) \neq 0 \\ a_4 u^4 + \sum_{k \geq 8} a_k(x) u^k\,, & a_4 \neq 0 \end{cases}$$

where $a_k(x) \in H^1(0,\pi)$ satisfy $\sum_k \|a_k\|_{H^1} \rho^k$ for some $\rho > 0$.

Then, $s > 1/2$ being given, there exist $\delta_0 > 0$, $\overline{\sigma} > 0$ and a C^∞-curve $[0, \delta_0) \ni \delta \to u(\delta) \in X_{\overline{\sigma}/2,s}$ with the following properties:

- *(i) $\left\| u(\delta) - \delta \overline{v} \right\|_{\overline{\sigma}/2,s} = O(\delta^2)$ for some $\overline{v} \in V \cap X_{\overline{\sigma},s}$, $\overline{v} \neq 0$;*
- *(ii) There exists a Cantor set $\mathcal{C} \subset [0, \delta_0)$ of asymptotically full measure, i.e. satisfying*

$$\lim_{\eta \to 0^+} \frac{\mathrm{meas}(\mathcal{C} \cap (0,\eta))}{\eta} = 1 \tag{2.8}$$

such that, for all $\delta \in \mathcal{C}$, $u(\delta)(\omega(\delta)t, x)$ is a $2\pi/\omega(\delta)$-periodic solution of (1.1) with

$$\omega(\delta) = \begin{cases} \sqrt{1 - 2\delta^2} \\ \sqrt{1 + 2\delta^2 \mathrm{sign}\langle a_3 \rangle} \\ \sqrt{1 - 2\delta^6}. \end{cases}$$

[3] For some more general nonlinearities see [2].

Corollary 2.3 (Multiplicity [6]). *There exists a Cantor-like set* $\mathcal{W}$ *of asymptotically full measure at* $\omega = 1$ *such that,* $\forall \omega \in \mathcal{W}$ *equation* (1.1) *possesses geometrically distinct periodic solutions*

$$u_1, \ldots, u_n, \ldots u_{N_\omega}, \qquad N_\omega \in \mathbf{N}$$

with the same period $2\pi/\omega$*. Their number increases arbitrarily as* ω *tends to* 1*:* $\lim_{\omega \to 1} N_\omega = +\infty$*.*

2.2. A variational principle on a Cantor set. The previous Theorem 2.2 was proved under the non-degeneracy of the solutions of the 0th order bifurcation equation (2.3). We want to relax this condition finding solutions of the finite dimensional bifurcation equation (2.7) through *variational* methods. Aside for aims of generality, it is also a conceptually important problem for understanding how to use variational methods in problems with small divisors.

By classical variational bifurcation theory, it is easy to define a smooth functional $\Phi(\delta, v_1)$ such that any critical point $v_1(\delta)$ of $\Phi(\delta, \cdot)$ is a solution of the finite dimensional bifurcation equation (2.7). Morevoer it is easy to prove the existence, for any δ small enough, of a mountain pass critical point $v_1(\delta)$ of $\Phi(\delta, \cdot)$ close to $\Pi_{V_1} K_c$. However, the big difficulty is that – if the critical set K_c does not reduce to a non-degenerate solution – $v_1(\delta)$ could vary in a highly irregular way as $\delta \to 0$. The only information available in general is that $v_1(\delta) \to K_c$ as $\delta \to 0$. Therefore for each δ the mountain pass critical point $v_1(\delta)$ could belong to the complementary of the Cantor set B_∞ where the range equation (2.6) was solved (actually B_∞^c is even arcwise connected!). This is the common difficulty in applying variational methods in a problem with small divisors.

The main point is to have a sufficiently good control on how $v_1(\delta)$ varies with δ. In [7] we establish that, if, for some $M > 0$, $(\delta \to \delta^M v_1(\delta))$ is a BV function then $(\delta, v_1(\delta))$ belongs to B_∞ for δ in a set of asymptotically full measure (note that the function $\delta \to v_1(\delta)$ can be discontinuous and with large oscillations of order δ^{-M}). Next we verity that this property holds for several nonlinearities.

The information of how the critical points of a family of parameter dependent functionals varies with the parameters is in general very hard to obtain. On the contrary, the critical values behave in general rather smoothly on the parameters. We want to find BV-path of critical points by informations on critical levels (this is somehow related to the Struwe monotonicity method [13]). We are not able to ensure the BV-property for any $f = a_p(x)u^p + O(u^p)$, but rather for parameter depending nonlinearities

$$f(x, u, \lambda) = a_p(x)u^p + \sum_{i=1}^{M} \lambda_i b_i(x) u^{q_i} + r(x, u), \qquad q_i > p \tag{2.9}$$

where $r(x, u) := \sum_{k>p} a_k(x)u^k$ satisfy $\sum_{k>p} \|a_k\|_{H^1} \rho^k$ for some $\rho > 0$.

Theorem 2.4 ([7]). *For any* $\overline{q} > p$ *there exist* $\overline{q} \le q_1 \le \ldots \le q_M$ *and* $b_1(x), \ldots, b_M(x) \in H^1(0, \pi)$ *such that, for any* $r(x, u)$*, for almost every* $\lambda = (\lambda_1, \ldots, \lambda_M)$*,* $|\lambda| \le 1$*, equation* (1.1) *with the nonlinearity* $f(x, u, \lambda)$ *like in* (2.9) *possesses small amplitude periodic solutions for an asymptotically full measure set of frequencies close to* $\omega = 1$*.*

We remark that, since $q_i > p$, the nonlinearities $\lambda_i b_i(x) u^{q_i}$ do not change the 0th-order bifurcation equation (2.3), which keeps being in particular degenerate. Actually, since we can choose the exponents $q_i > \overline{q}$ arbitrarily large, we are adding arbitrarily small corrections $b_i(x)u^{q_i} = o(u^p)$ for $u \to 0$. Moreover we underline that, given $a_p(x)u^p$, $b_i(x)u^{q_i}$, the existence result of Theorem 2.4 holds for *any* nonlinear term $r(x, u) =$

$\sum_{k>p} a_k(x)u^k$, changing only the full measure set of parameters λ; in this sense Theorem 2.4 is a genericity result.

REFERENCES

[1] A. Ambrosetti, P. Rabinowitz, *Dual Variational Methods in Critical Point Theory and Applications*, Journ. Func. Anal, **14** (1973), 349-381.

[2] P. Baldi, M. Berti, *Periodic solutiond of wave equations for asymptotically fulll measure sets of frequencies*, Rend. Mat. Acc. Naz. Lincei , **17**(3) (2006), 257–277.

[3] D. Bambusi, S. Paleari, *Families of periodic solutions of resonant PDEs*, J. Nonlinear Sci. **11** (2001), 69–87.

[4] M. Berti, P. Bolle, *Periodic solutions of nonlinear wave equations with general nonlinearities*, Comm. Math. Phys. **243** (2003), 315–328.

[5] M. Berti, P. Bolle, *Multiplicity of periodic solutions of nonlinear wave equations*, Nonlinear Analysis **56** (2004), 1011–1046.

[6] M. Berti, P. Bolle, *Cantor families of periodic solutions for completely resonant nonlinear wave equations*, Duke Mathematical Journal , **134**(2) (2006), 359–419.

[7] M. Berti, P. Bolle, *Cantor families of periodic solutions for wave equations via a variational principle*, preprint 2006.

[8] J. Bourgain, *Periodic solutions of nonlinear wave equations*, Harmonic analysis and partial differential equations, 69–97, Chicago Lectures in Math., Univ. Chicago Press, 1999.

[9] W. Craig, C. E. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appl. Math **46** (1993), 1409–1498.

[10] E. R. Fadell, P. H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. **45** (1978), 139–174.

[11] G. Gentile, V. Mastropietro, M. Procesi, *Periodic solutions for completely resonant nonlinear wave equations*, Comm. in Math. Physics, 2005.

[12] J. Moser, *Periodic orbits near an equilibrium and a theorem by Alan Weinstein*, Comm. Pure Appl. Math. **29** (1976), 724–747.

[13] M. Struwe, *The existence of surfaces of constant mean curvature wit free boundaries*, Acta Math., 1988, 19–64.

[14] A. Weinstein, *Normal modes for nonlinear Hamiltonian systems*, Invent. Math **20** (1973), 47–57.