Proceedings of Equadiff-11
2005, pp. 135–140

ON A BLOW-UP ESTIMATE FOR A HIGHER ORDER SEMILINEAR PARABOLIC EQUATION

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Abstract. We study the blow-up rate of the solutions of a higher order semilinear parabolic equation. We first deal with several applications in the second order case. Then we show that although new crucial ingredients are needed, some key ideas involved in the order-preserving case remain valid when analyzing the higher-order case.

Key words. Blow-up solution, parabolic higher order equations, rate of blow-up

AMS subject classifications. 35K57, 35B40

1. Introduction. We consider the Cauchy problem for the $2m$-th order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + f(u) \quad \text{in } Q = \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$

where $f(u) = u \ln^\gamma(1+|u|)$ with $\gamma > 1$ and initial data $u_0 \in X = L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We note that for $m = 1$ this is a classical semilinear reaction-diffusion equation from Combustion Theory. For higher-order generalizations $m > 1$, semilinear and quasilinear diffusion operators of this type arise in several applications including thin film theory, flame and wave propagation, phase transition at critical Lifschitz points and bi-stable systems (e.g., the Kuramoto-Sivashinskii equation and the extended Fisher-Kolmogorov equation).

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Let $u(x,t)$ be a solution of (1.1) which *blows up* in finite time $T = T(u_0) < \infty$. By blow-up we mean that u is a bounded classical solution in $Q_\tau = \mathbb{R}^N \times (0, \tau]$ for any $\tau \in (0, T)$ and cannot be extended as a bounded one beyond $t = T$. By the classical parabolic regularity theory [6], [9], this means that

$$\sup_x |u(x,t)| \to \infty \quad \text{as} \quad t \to T^-.$$

The limit is understood in the usual sense (one can see that if $\liminf_{t \to T^-}$ were finite, then the solution could be extended for all $t - T > 0$ small as a classical bounded solution). General results on global nonexistence for higher-order parabolic equations are well known from 70's, see a survey paper [18] and references to [21, Chapt. 4].

We point out that the study of singularities in higher-order heat equations, quite well understood for second order reaction-diffusion equations, remains an open problem of the general theory of higher-order parabolic equations.

It is known, see [2], that solutions of (1.1) blow up at finite time. In [2], the blow-up behaviour of the solutions of (1.1) was also analyzed for $\gamma = 2m$ which corresponds to regional blow-up phenomena and a classification of other types of blow-up in terms of the exponent γ was given. The asymptotic results, in spite of the significant differences between the higher-order and second-order cases, are similar: asymptotic simplification to a Hamilton-Jacobi equation occurs in both cases, see [15] and [2].

In this work we deal with an a priori estimate from below of the blow-up rate of the solutions of equation (1.1), see [2] for a general approach. We show that such estimate is given, except a multiplicative constant D, by the blow-up rate of the homogeneous in space blow-up solutions of the ordinary differential equation:

$$V_t = f(V). \tag{1.2}$$

The derivation of this estimate is well known in the second order case $m = 1$, where the multiplicative constant becomes $D = 1$. Such result is obtained via a particular way of comparison, by using the Maximum Principle. We remark that an important clue in the proof, as well as in in the general study of blow-up phenomena, relies on the standard comparison and the intersection comparison between solutions having the same blow-up time. However, these ideas strongly rely on the order-preserving property of the equation when $m = 1$ which does not remain valid for the higher order equations (1.1). We show that in this case, the construction of the majorizing equation associated to (1.1) introduced in [14] plays an important role in obtaining this blow-up estimate and makes it possible to follow a similar approach to that used in the second order one.

The plan of the paper is as follows. In Section 2 we briefly comment on how to obtain the lower blow-up estimate under consideration in the second order case. In particular, we deal with solutions of a class of quasilinear second-order parabolic equations. Then, we focus on the higher-order case by introducing in Section 3 the main ideas involved in the construction of the Majorizing equation. Finally, Section 4 is devoted to analyzing the rate of blow-up of the solutions of (1.1).

2. A blow-up rate estimate for a quasilinear reaction-absorption diffusion equation. Many results concerning the asymptotic theory of blow-up have been obtained for semilinear and quasilinear second-order parabolic equations, $m = 1$, where the Maximum Principle applies and became an essential tool in the asymptotic analysis. We refer to the book [21] (quasilinear equations) and to papers [7], [16], [19], [23] (semilinear equations with $f(u) = u^p$ and $f(u) = e^u$), see also references in [21].

As we mention above, the general blow-up analysis in the second-order case, is essentially based on different types of comparison. In particular, the lower estimate under consideration follows in many cases by a straightforward comparison with a flat solution $U(t)$ independent of x and satisfying the ODE

$$U' = f(U) \quad \text{for} \quad t \in (0, T), \quad U(T) = \infty,$$

so that we compare solution $u(x, t)$ and $U(t)$ having the same blow-up time (then they must intersect each other whence an estimate from below: $\sup_x u(x, t) > U(t)$ for any $t \in (0, T)$.

We next show that similar ideas apply even when asymptotic simplification phenomena occur which makes necessary a comparison between solutions of different equations.

We consider a class of second order quasilinear parabolic equations in which blow-up phenomena occur. It corresponds to the well-known m-laplacian equation where reaction and absorption terms appear:

$$u_t = (|u_x|^m u_x)_x + u^p - \lambda u^q \quad m > 1, \quad p > 1, \quad q > 1, \tag{2.1}$$

in the range of parameters $1 < q < p < m$. We assume that the initial data $u_0(x)$ is continuous and bounded with compact support. It is known, see [3], that there exists a class of initial data such that the corresponding solution blows-up at finite time for every $x \in \mathbb{R}^N$. It was proved that in this case, asymptotic simplification occurs and the asymptotic behaviour of the solutions close to blow-up is described by the pure reaction-diffusion equation, the absorption term being negligible. This makes necessary a kind of like-intersection comparison between solutions of different equations. Let $u(x,t)$ be a solution of (2.1) which blows-up at finite time T. We have the following.

Proposition 2.1. *If T is the blow-up time of the solution $u(x,t)$ of the Cauchy problem corresponding to equation* (2.1) *then*

$$\|u(t)\| \geq U_T(t) = [(p-1)(T-t)]^{-1/(p-1)}, \qquad t \in [0,T).$$

Proof. Assume for contradiction that the inequality fails at $t = t_0$. Then, by the geometry of $U(t)$ this implies that $u(x,t_0) < U(t_0)$ for every $x \in \mathbb{R}$ and by continuity there exists ϵ small enough such that $u(x,t_0) < U_\epsilon(t_0)$ where $U_\epsilon(t) = [(p-1)(T+\epsilon-t)]^{-1/(p-1)}$. Since $U_\epsilon(t)$ is a supersolution of (2.1), we have from the Maximum Principle that the same inequality holds for every $t \geq t_0$. Hence the solution $u(x,t)$ is bounded and does not blow-up at time T, whence a contradiction follows. □

We remark that the same estimate can be obtained without changes in the N-dimensional case and for other ranges of the parameters where blow-up occurs at finite time.

3. The Majorizing order-preserving equation. In this section we are going to use the integral representation of the solution:

$$u(t) = \mathbf{M}_{2m}(u) \equiv b(t) * u_0 + \int_0^t b(t-s) * f(u(s))\,\mathrm{d}s, \quad u_0 \in X, \tag{3.1}$$

where $b(t) * u_0 \equiv \mathrm{e}^{\mathbf{A}t}\, u_0$ is the convolution representation of the continuous semigroup $\mathrm{e}^{\mathbf{A}t}$ with the infinitesimal generator $\mathbf{A}$, i.e., $u(t)$ is a fixed point of operator $\mathbf{M}_{2m}$ for any $t > 0$. Then

$$b(x,t) = t^{-N/2m} g(\xi), \qquad \xi = x/t^{1/2m}, \tag{3.2}$$

is the fundamental solution (the self-similar kernel) of the linear operator $\partial/\partial t - \mathbf{A}$. The function g is the unique radial solution of the elliptic equation

$$\mathbf{C}g \equiv \mathbf{A}g + \frac{1}{2m}\xi \cdot \nabla g + \frac{N}{2m} g = 0 \quad \text{in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} g(\xi)\,\mathrm{d}\xi = 1. \tag{3.3}$$

On the interval $(0,T)$ of the classical solvability, both integral and differential equations give the same solution $u(x,t)$.

Now, as in [14], given the integral evolution equation (3.1), we construct the corresponding *order-preserving majorizing equation.* The function g satisfies the following estimate, see [6].

PROPOSITION 3.1. *Let $m > 1$. There exist constants $D > 1$ and $d > 0$ depending on m and N such that*

$$|g(\eta)| < D\,F(\eta) \equiv D\omega_1\, \mathrm{e}^{-d|\eta|^\alpha} \qquad \text{in } \ \mathbb{R}^N, \tag{3.4}$$

where

$$\alpha = \frac{2m}{2m-1} \in (1,2) \quad \text{and} \quad \omega_1 = \left(\int_{\mathbb{R}^N} \mathrm{e}^{-d|\eta|^\alpha}\, d\eta \right)^{-1}.$$

We next introduce the *majorizing kernel*

$$\bar{b}(x,t) = t^{-N/2m} F(\eta), \qquad \eta = x/t^{1/2m}, \tag{3.5}$$

which is *strictly positive* for $t > 0$. Therefore, the corresponding majorizing integral equation

$$v(t) = \bar{b}(t) * v_0 + \int_0^t \bar{b}(t-s) * (Df(v(s)))\,\mathrm{d}s, \qquad t > 0, \tag{3.6}$$

describes an *order-preserving* evolution with the usual partial order: given two solutions $v(t)$ and $\tilde{v}(t)$ of (3.6),

$$v_0 \le \tilde{v}_0 \;\Rightarrow\; v(t) \le \tilde{v}(t) \;\text{ for }\; t > 0.$$

The nonlinear term is a locally Lipschitz map and local existence and uniqueness are straightforward, see [22, Chapt. 15]. In particular, we have the *positivity property*:

$$v_0 \ge 0,\, v_0 \not\equiv 0 \implies v(t) > 0 \;\text{ for }\; t > 0. \tag{3.7}$$

By comparing (3.1) and (3.6), we deduce the following result on the Comparison Principle duality between the semilinear parabolic equation and the integral majorizing one, cf. [14].

Theorem 3.2. *The integral equation (3.6) is majorizing for the $2m$-th order parabolic equation (1.2) in the following sense:*

$$D|u_0(x)| \le v_0(x) \;\textit{ in }\; \mathbb{R}^N \implies |u(x,t)| \le v(x,t) \;\textit{ for }\; x \in \mathbb{R}^N, \;\; t > 0. \tag{3.8}$$

Proof. We have that $v(t)$ solves the integral inequality

$$v(t) \ge \bar{b}(t) * (D|u_0|) + \int_0^t \bar{b}(t-s) * (Df(v(s)))\,\mathrm{d}s, \quad t > 0, \tag{3.9}$$

On the other hand, it follows from (3.1) and Proposition 3.1 that

$$\begin{aligned} |u(t)| &\le |b(t)| * |u_0| + \int_0^t |b(t-s)| * |f(u(s))| \, \mathrm{d}s \\ &\le |D\bar{b}(t)| * |u_0| + \int_0^t |D\bar{b}(t-s)| * f(|u(s)|) \, \mathrm{d}s. \end{aligned} \tag{3.10}$$

Using (3.9), (3.10), we conclude that the difference $w = v - |u|$ satisfies the linear integral inequality

$$w(t) \ge \int_0^t D\bar{b}(t-s) * (f(v(s)) - f(|u(s)|) \, \mathrm{d}s \equiv \int_0^t K(t)w(s) \, \mathrm{d}s, \ \ t > 0, \tag{3.11}$$

with the positive integral operator

$$K(t)w(s) = D\bar{b}(t-s) * [f'(\xi(s))w(s)],$$

with strictly positive kernel. Here $\xi(s) \in (|u(s)|, v(s))$ denotes intermediate points obtained via Lagrange's formula of finite increments. Since $D > 1$, we have $w(0) = v_0 - |u_0| \ge v_0 - D|u_0| \ge 0$. If $w(0) > \epsilon > 0$ in $\mathbb{R}^N$, then the integral inequality with positive operator implies that $w(t) \ge 0$ for all $t > 0$, see below. See [2] for more details. □

4. A lower blow-up rate estimate via the majorizing order equation. For $m > 1$, the majorizing integral equation (3.6) is not generated by a semigroup unlike the second-order case $m = 1$, where $b(t) * u_0 = \mathrm{e}^{\Delta t} u_0 > 0$ and the semigroup is order-preserving since $b(t) > 0$ for $t > 0$. Therefore, solutions $v(x,t)$ are not time-translational invariant. Nevertheless, the spatially homogeneous solutions $V = V(t)$ satisfying the integral equation (we recall that $\int \bar{b}(t) \equiv 1$)

$$V(t) = V(0) + \int_0^t Df(V(s)) \, \mathrm{d}s, \ \ t > 0,$$

and hence the ODE

$$V' = Df(V), \quad t > 0; \quad V(0) > 0, \tag{4.1}$$

admit translation in time, so that $V(t+\tau)$ is a solution of the majorizing equation for any constant time-shift τ.

In view of the order-preserving majorizing evolution, we now compare $u(x,t)$ with such spatially homogeneous solutions $V = V(t)$ and obtain the following simple estimate from above of the solutions and hence a lower estimate of the blow-up time.

Corollary 4.1. *Denote* $m_0 = \sup |u_0(x)| > 0$. *Then*

$$|u(x,t)| \le V(t) < \infty, \quad 0 < t < t_0 = \frac{1}{D} \int_{m_0 D}^{\infty} \frac{\mathrm{d}z}{f(z)}, \tag{4.2}$$

where $V(t) > 0$ *is the solutions of the ODE* (4.1) *with initial data* $V(0) = Dm_0$.

Finally, we establish the lower estimate by means of an *intersection-like* property of the solutions $u(x,t)$ and $V_T(t)$, $V_T(T) = \infty$, having the same blow-up time T. As we mention above, the same estimates based on such intersection properties play a fundamental role for the blow-up analysis in the second-order case $m = 1$, see [21, Chapt. 4] and references therein.

Theorem 4.2. *Let* $u(x,t)$ *and* $V = V_T(t)$ *solving* (4.1) *have the same finite blow-up time* $T > 0$. *Then, for any* $t \in [0,T)$, *the function* $D|u(\cdot,t)|$ *intersects* $V_T(t)$, *so that*

$$\|u(t)\|_\infty > D^{-1} V_T(t), \quad t \in [0,T). \tag{4.3}$$

Proof. Assume for contradiction that $D|u(x,t_0)| < V_T(t_0)$ in $\mathbb{R}^N$ for some $t_0 \in [0,T)$. By continuity, there exists a small $\epsilon > 0$ such that

$$D|u(x,t_0)| < V_T(t_0 - \epsilon) \equiv V_{T+\epsilon}(t_0), \quad x \in \mathbb{R}^N.$$

By the comparison THEOREM 3.2 we have that $|u(x,t)| < V_{T+\epsilon}(t)$ for all $t \in [t_0, T)$. This means that at $t = T$, the solution satisfies $|u(x,T)| < V_{T+\epsilon}(T) < \infty$ and hence is uniformly bounded at its blow-up time, whence a contradiction follows. ∎

As a consequence of the result we obtain the following lower L^∞-estimate on blow-up solutions:

$$\|u(\cdot,t)\|_\infty > D^{-1}V_T(t) = D^{-1}\exp\{[(\gamma-1)D(T-t)]^{-1/(\gamma-1)}(1+o(1))\},$$

as $t \to T^-$.

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