Proceedings of Equadiff-11
2005, pp. 67–76
ISBN 978-80-227-2624-5

UNIQUENESS/NONUNIQUENESS FOR NONNEGATIVE SOLUTIONS OF A CLASS OF SECOND-ORDER PARABOLIC EQUATIONS*

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Abstract. We investigate uniqueness and nonuniqueness for nonnegative solutions of the equation

$$\begin{aligned} &u_t = Lu + Vu - \gamma u^p && \text{in } \mathbb{R}^n \times (0,\infty);\\ &u(x,0) = f(x), && x \in \mathbb{R}^n;\\ &u \ge 0, \end{aligned}$$

where $\gamma > 0,\ p > 1,\ \gamma, V \in C^\alpha(\mathbb{R}^n),\ 0 \le f \in C(\mathbb{R}^n)$ and $L = \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$ with $a_{i,j}, b_i \in C^\alpha(\mathbb{R}^n)$.

Key words. semilinear parabolic equations, uniqueness/nonuniqueness, Cauchy problem, reaction-diffusion equations

AMS subject classifications. 35K15, 35K55

1. Introduction.

In this article we study uniqueness for nonnegative solutions $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ to the semilinear equation

$$\begin{aligned} &u_t = Lu + Vu - \gamma u^p && \text{in } \mathbb{R}^n \times (0,\infty);\\ &u(x,0) = f(x), && x \in \mathbb{R}^n;\\ &u \ge 0, \end{aligned} \tag{1.1}$$

where γ, $V \in C^\alpha(\mathbb{R}^n)$, $\gamma > 0$, $p > 1$, $0 \le f \in C(\mathbb{R}^n)$ and

$$L = \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i},$$

with $a_{i,j}, b_i \in C^\alpha(\mathbb{R}^n)$ and $\sum_{i,j=1}^n a_{i,j}(x)\nu_i\nu_j > 0$, for all $x \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n - \{0\}$.

In the case that V is bounded from above, it will be useful to compare uniqueness in the class of nonnegative solutions for the semilinear equation with uniqueness in the class of bounded solutions $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ for the corresponding linear equation:

$$\begin{aligned} &u_t = Lu + Vu && \text{in } \mathbb{R}^n \times (0,\infty);\\ &u(x,0) = f(x);\\ &\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^n} |u(x,t)| < \infty, && \text{for all } T > 0, \end{aligned} \tag{1.2}$$

where $f \in C(\mathbb{R}^n)$.

*The research of the second author was supported by the Fund for the Promotion of Research at the Technion.

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Notation. In the sequel, when referring to the *nonnegative semilinear* problem (1.1), we will sometimes use the notation NS_f, $NS(L, V, \gamma)$ or $NS_f(L, V, \gamma)$ to specify the dependence respectively on the initial condition, on the particular operator or on both the initial condition and the particular operator. Similarly, when referring to the *bounded linear* problem. (1.2), we will sometimes use the notation $BL(L, V)$. (In the linear case, the initial condition is of course irrelevant with regard to the question of uniqueness.)

REMARKS.

(i) Because of the page limit we will *not be able to provide proofs* except for some very short ones. A longer version of this paper with complete proofs is [4].

(ii) Further results have been achieved in the subject recently in the follow-up paper [5]. In that paper V is assumed to be bounded from above and the probabilistic point of view is emphasized. For example it has been shown in [5] that uniqueness for the semilinear parabolic equation is not affected by a bounded change in β. Moreover, assuming $\inf_{\mathbb{R}^n} \gamma > 0$, it was shown that whenever uniqueness fails for $BL(L, V)$, uniqueness also fails for NS_f for all f (cf. THEOREM 3.4).

In Section 2 we present a basic result asserting the existence of a minimal and a maximal solution to the nonnegative semilinear equation NS_f. For some related results in the case $L = \Delta$, see [12] and [1]. This result, of interest in its own right, is also useful for the study of uniqueness – indeed, uniqueness occurs if and only if the minimal and maximal solutions coincide. (The nonlinear reaction term in this paper is $Vu - \gamma u^2$. In [15], a quite precise growth condition is given on the nonlinear reaction term $f(x, u)$ in order to determine whether or not a maximal nonnegative solution exits for $u_t = Lu + f(x, u)$.)

In section three, we begin the study of uniqueness for the semilinear equation. One of the two main results in that section is a sufficiency condition for uniqueness which is given in terms of pointwise bounds on the coefficients of the semilinear operator. The other main result in that section is a sufficiency condition for nonuniqueness which states that if $\inf_{x \in \mathbb{R}^n} \frac{V(x)}{\gamma(x)} > 0$ and if nonuniqueness holds for the linear problem $BL(L, 0)$, then nonuniqueness also holds for $NS_0(L, V, \gamma)$. In order to implement this result, we also present a result on uniqueness for the linear problem.

In Section 4, we develop a connection between uniqueness for the semilinear parabolic problem and uniqueness for the corresponding steady state elliptic equation, which turns out to be very useful in applications. For additional results in this direction, see [15]. In Sections 5, we apply the results of Sections 3 and 4 to two specific classes of problems. We also show how our results can be used to give an alternative proof to a classical result of Ni [13], Kenig and Ni [9] and Lin [11] on uniqueness/nonuniqueness of positive solutions to the semilinear elliptic equation $\Delta w - \gamma w^p = 0$ in $\mathbb{R}^n$, for $n \geq 3$, and how they lead to a new result for this equation when $n = 1, 2$.

2. Existence of a Maximal and a Minimal Solution. In this section we present the following theorem on the existence of minimal and maximal solutions.

THEOREM 2.1. *Let $f \in C(\mathbb{R}^n)$. There exist solutions $u_{f;\min}$ and $u_{f;\max}$ of NS_f with the property that any solution u to NS_f satisfies*

$$u_{f;\min} \leq u \leq u_{f;\max}.$$

We now present a standard semilinear parabolic maximum principle which is applied in [4] to obtain an a priori estimate on the size of any solution to NS.

PROPOSITION 2.2. *Let $D \subset \mathbb{R}^n$ be a bounded domain and let $0 \le u_1, u_2 \in C^{2,1}(D \times (0,\infty)) \cap C(\bar{D} \times [0,\infty))$ satisfy*

$$\begin{aligned} &Lu_1 + Vu_1 - \gamma u_1^p - \frac{\partial u_1}{\partial t} \le Lu_2 + Vu_2 - \gamma u_2^p - \frac{\partial u_2}{\partial t}, && \text{for } (x,t) \in D \times (0,\infty), \\ &u_1(x,t) \ge u_2(x,t) && \text{for } (x,t) \in \partial D \times (0,\infty) \end{aligned}$$

and

$$u_1(x,0) \ge u_2(x,0), \qquad \text{for } x \in D.$$

Then $u_1 \ge u_2$ in $D \times (0,\infty)$.

Proof. Let $W = u_1 - u_2$ and define $H(x) = \frac{u_1^p(x) - u_2^p(x)}{W(x)}$, if $W(x) \neq 0$, and $H(x) = 0$ otherwise. We have $LW + (V - H)W - \frac{\partial W}{\partial t} \le 0$ in $D \times (0,\infty)$, $W(x,0) \ge 0$ in D, and $W(x,t) \ge 0$ on $\partial D \times (0,\infty)$. Thus, by the standard linear maximum principle, $u_1 \ge u_2$. □

In the sequel we will frequently use the notation

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}.$$

PROPOSITION 2.3. *Let $u \in C^{2,1}(B_R \times (0,\infty)) \cap C(\bar{B}_R \times [0,\infty))$ satisfy*

$$\begin{aligned} &u_t = Lu + Vu - \gamma u^p && \text{in } B_R \times (0,\infty); \\ &u(x,0) = f(x), && x \in \bar{B}_R; \\ &u \ge 0, \end{aligned}$$

where $f \in C(\bar{B}_R)$. Let $V_R = \sup_{x \in B_R} V(x)$, if $\sup_{x \in B_R} V(x) > 0$, and let $V_R > 0$ be arbitrary otherwise. Let $\gamma_R = \inf_{x \in B_R} \gamma(x)$. Then there exists a constant K_R such that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} u(x,t) \le{} & \left(\frac{V_R}{\gamma_R}\right)^{\frac{1}{p-1}} (1 - \exp(-(p-1)V_R(t+\varepsilon)))^{-\frac{1}{p-1}} \\ & + ((R+\varepsilon)^2 - |x|^2)^{-\frac{2}{p-1}} \exp(K_R(t+1)), \qquad \text{for } (x,t) \in \bar{B}_R \times [0,\infty). \end{aligned}$$

3. Uniqueness/Nonuniqueness for the Semilinear Parabolic Equation. Note that by THEOREM 2.1, uniqueness follows for NS_f if $u_{f;\min} \equiv u_{f;\max}$.

We begin with a couple of useful comparison results.

PROPOSITION 3.1. *Let $0 \le f_1 \le f_2$. If uniqueness holds for NS_{f_1}, then it also holds for NS_{f_2}.*

REMARK. In particular, it follows from the proposition that if uniqueness holds for $f \equiv 0$, then it holds for all $0 \le f \in C(\mathbb{R}^n)$. In fact, we suspect that uniqueness either holds for all f or no f.

PROPOSITION 3.2. *Assume that*

$$V_1 \le V_2$$

and

$$0 < \gamma_2 \le \gamma_1.$$

If uniqueness holds for $NS_0(L, V_2, \gamma_2)$, then uniqueness also holds for $NS_f(L, V_1, \gamma_1)$, for all f.

We now come to our first main result, which guarantees uniqueness for NS if the coefficients satisfy appropriate pointwise estimates.

Theorem 3.3. *Assume that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\nu_i\nu_j \le C|\nu|^2(1+|x|)^2; \tag{3.1}$$

$$|b(x)| \le C(1+|x|); \tag{3.2}$$

$$V(x) \le C, \tag{3.3}$$

for some $C > 0$. Assume in addition that

$$\inf_{x\in\mathbb{R}^n} \gamma(x) > 0.$$

Then uniqueness holds for NS_f, for all f.

The second main result in this section relates nonuniqueness of the semilinear equation to nonuniqueness of the corresponding linear problem obtained by setting both γ and V equal to 0.

Theorem 3.4. *Assume that uniqueness does not hold for $BL(L, 0)$ and that*

$$\inf_{x\in\mathbb{R}^n} \frac{V(x)}{\gamma(x)} > 0.$$

Then uniqueness does not hold for $NS_0(L, V, \gamma)$.

Remark. For an example where the condition $\inf_{x\in\mathbb{R}^n} \frac{V(x)}{\gamma(x)} > 0$ holds and there is uniqueness for BL but not for NS, one can turn to the applications in section five and take the class of equations in (5.2) with $V = C > 0$ and γ as in Theorem 5.3-(ii).

In order for Theorem 3.4 to be useful, we need to know when uniqueness holds for the bounded linear problem $BL(L, 0)$. Hence, before proceeding further, we make a small digression to study the linear problem. We have the following result which actually considers more generally $BL(L, V)$.

Proposition 3.5.

- (i-a) *If V is bounded from above and uniqueness holds for $BL(L, 0)$, then uniqueness holds for $BL(L, V)$.*
- (i-b) *If V is bounded from below and uniqueness holds for $BL(L, V)$, then uniqueness holds for $BL(L, 0)$.*
- (ii-a) *If there exist $m_0, \lambda > 0$ and a positive function ϕ satisfying $L\phi \le \lambda\phi$ in $\mathbb{R}^n - B_{m_0}$ and $\lim_{|x|\to\infty} \phi(x) = \infty$, then uniqueness holds for $BL(L, 0)$.*
- (ii-b) *If there exist $m_0, \lambda > 0$, an $x_0 \in \mathbb{R}^n$ satisfying $|x_0| > m_0$, and a bounded, positive function ϕ satisfying $L\phi \ge \lambda\phi$ in $\mathbb{R}^n - B_{m_0}$ and $\phi(x_0) \ge \sup_{|x|=m_0} \phi(x)$, then uniqueness does not hold for $BL(L, 0)$.*

REMARK 1. Recall from THEOREM 3.3 that if the pointwise bound (3.1)–(3.3) on the coefficients of the linear part of the semilinear equation is in effect along with the condition $\inf_{x\in\mathbb{R}^n}\gamma(x)>0$ on the nonlinear part, then uniqueness holds for the semilinear equation. It is interesting to note how (3.1)–(3.3) relates to uniqueness for the linear equation. Using the function $\phi(x)=|x|^2$ in part (ii-a) of PROPOSITION 3.5 and then using part (i-a) shows that if (3.1)–(3.3) is in force, then uniqueness holds for $BL(L,V)$. As far as pointwise polynomial-type bounds are concerned, condition (3.1)–(3.2) is sharp for the uniqueness of $BL(L,0)$. Indeed, applying part (ii-b) with the function $\phi(x)=1-|x|^{-l}$, where $l>0$ is sufficiently small, shows that uniqueness does not hold for $BL(L,0)$ in the following two cases: (1) $L=(1+|x|)^{2+\delta}\Delta$ with $\delta>0$ and $n\geq 3$; (2) $L=\Delta+b\nabla$ and $n\geq 1$, where $b(x)\cdot\frac{x}{|x|}\geq c|x|^{1+\delta}$ for large $|x|$ and some $\delta,c>0$.

In passing, we note that the question of uniqueness of *positive* solutions to the linear equation has a long history in the partial differential equations literature, going back to Widder. It is known that uniqueness of positive solutions holds if (3.2)–(3.3) is in force and if (3.1) is replaced by a two-sided bound of the form $C_1|\nu|^2(1+|x|)^q\leq\sum_{i,j=1}^n a_{ij}(x)\nu_i\nu_j\leq C_2|\nu|^2(1+|x|)^q$, for some $q\in[0,2]$. See, for example, [8] and references therein.

REMARK 2. It's well-known in the probability literature that uniqueness holds for $BL(L,0)$ if and only if the Markov diffusion process corresponding to the operator L is *nonexplosive*; that is, the process does not run out to infinity in finite time. In the case that $p\in(1,2]$, the equation NS is also connected with a Markov process; namely, with a measure-valued diffusion process. The so-called *compact support property* for measure valued diffusions can be thought of as the parallel to nonexplosiveness for ordinary diffusions. We have shown elsewhere that uniqueness for NS_0 is equivalent to the compact support property holding [3]. (Actually, the case $p=2$ is treated in [3] but it extends immediately to $p\in(1,2]$.) Certain results in this paper appeared in the case $p=2$ with probabilistic proofs in [3] or [2].

4. The Interplay Between Uniqueness/Nonuniqueness of the Parabolic Equation and of the Corresponding Steady-State Elliptic Equation. Consider the elliptic semilinear equation corresponding to steady state solutions of NS:

$$Lw+Vw-\gamma w^p=0 \text{ and } w\geq 0 \text{ in } \mathbb{R}^n. \tag{4.1}$$

The next theorem gives conditions for uniqueness/nonuniqueness in terms of solutions to the elliptic equation. As we shall see in the next section, this result can be very useful.

THEOREM 4.1.

(i) *Let* $\{f_m\}_{m=1}^\infty\subset C(\mathbb{R}^n)$ *be an increasing sequence of nonnegative compactly supported functions satisfying* $\lim_{m\to\infty}f_m=\infty$. *Let* $u_{f_m;\min}$ *denote the minimal solution to* NS_{f_m}. *Then*

$$w^*(x)\equiv\lim_{t\to\infty}\lim_{m\to\infty}u_{f_m;\min}(x,t) \tag{4.2}$$

exists and is a nonnegative solution to (4.1). *There exists a maximal solution* $w_{\max}$ *to* (4.1), *and if* $w_{\max}\gneq w^*$, *then uniqueness does not hold for* NS_f, *for any* f. *Furthermore, if* $\inf_{x\in\mathbb{R}^n}\gamma(x)>0$, *then* w^* *satisfies the bound*

$$\sup_{x\in\mathbb{R}^n}w^*(x)\leq\left(\frac{\sup_{x\in\mathbb{R}^n}V^+(x)}{\inf_{x\in\mathbb{R}^n}\gamma(x)}\right)^{\frac{1}{p-1}}, \tag{4.3}$$

where $V^+=\max(V,0)$.

(ii) *If $w = 0$ is the only solution to* (4.1)*, then uniqueness holds for NS_f, for all f.*

For the proof of THEOREM 4.1 we needed in [4] the following result which is of independent interest.

PROPOSITION 4.2. *Let $\{f_m\}_{m=1}^{\infty}$ be an increasing sequence of nonnegative compactly supported functions satisfying $\lim_{m\to\infty} f_m = \infty$. Then*

$$u_{\infty;\min} \equiv \lim_{m\to\infty} u_{f_m;\min}$$

and

$$u_{\infty;\max} \equiv \lim_{m\to\infty} u_{f_m;\max}$$

exist and are independent of the particular sequence $\{f_m\}$. They solve NS with initial condition $f = \infty$ and they are monotone nonincreasing in t. Furthermore

$$w^*(x) \equiv \lim_{t\to\infty} u_{\infty;\min}(x,t) \tag{4.4}$$

is a solution to (4.1) *and*

$$w_{\max}(x) \equiv \lim_{t\to\infty} u_{\infty;\max}(x,t) \tag{4.5}$$

is the maximal, nonnegative solution to (4.1).

5. Applications. In this section we use the array of results in sections three and four to prove theorems on uniqueness/nonuniqueness for two classes of semilinear parabolic equations (again, the missing parts can be found in [4]). We will also show how some of the results in this paper can be used to give an alternative proof and an extension of a classical result in semilinear elliptic theory.

We will determine how uniqueness depends on α for the following class of equations:

$$\begin{aligned} &u_t = \alpha\Delta u - u^p && \text{in } \mathbb{R}^n \times (0,\infty);\\ &u(x,0) = f(x), && x \in \mathbb{R}^n;\\ &u \ge 0. \end{aligned} \tag{5.1}$$

And with a relatively generic V we will determine how uniqueness depends on γ for the following class of equations:

$$\begin{aligned} &u_t = \Delta u + Vu - \gamma u^p && \text{in } \mathbb{R}^n \times (0,\infty);\\ &u(x,0) = f(x), && x \in \mathbb{R}^n;\\ &u \ge 0. \end{aligned} \tag{5.2}$$

Concerning the class of equations appearing in (5.1), we have the following result.

THEOREM 5.1.

(i-a) *Let $n \ge 2$. If*

$$\alpha(x) \le C(1+|x|)^2,$$

for some $C > 0$, then uniqueness holds in (5.1) *for all f.*

(i-b) *Let* $n \geq 2$. *If*

$$\alpha(x) \geq C(1+|x|)^{2+\varepsilon},$$

for some $\varepsilon, C > 0$, *then uniqueness does not hold in* (5.1) *for any* f.

(ii-a) *Let* $n = 1$. *If*

$$\alpha(x) \leq C(1+|x|)^{1+p},$$

for some $C > 0$, *then uniqueness holds in* (5.1), *for all* f.

(ii-b) *Let* $n = 1$. *If*

$$\alpha(x) \geq C(1+|x|)^{1+p+\varepsilon},$$

for $x > 0$ *or for* $x < 0$ *and some* $\varepsilon, C > 0$, *then uniqueness does not hold in* (5.1) *for any* f.

Before turning to (5.2), we will show how Theorems 3.3 and 4.1 can be used to obtain an alternate proof of a classical result concerning nonexistence of nontrivial solutions of a certain semilinear elliptic equation in dimension $n \geq 3$, and how these theorems along with Theorem 5.1 can be used to extend that result to appropriate corresponding results in the cases $d = 1, 2$.

It was shown by Ni [13] and Kenig and Ni [9] that the equation $\Delta w - \gamma w^p = 0$ in $\mathbb{R}^n, n \geq 3$, has no nontrivial, nonnegative solution if $\gamma(x) \geq C(1+|x|)^{-2+\varepsilon}$, for some $C, \varepsilon > 0$, and that nontrivial, nonnegative solutions do exist if $\gamma(x) \leq C(1+|x|)^{-2-\varepsilon}$.

Lin [11] extended the nonexistence result to the borderline case: there is no nontrivial solution if $\gamma(x) \geq C(1+|x|)^{-2}$. Here is a quick proof of this last result: Let $C > 0$. By Theorem 3.3, uniqueness holds for $NS((1+|x|)^2\Delta, 0, C)$. From (4.3) in Theorem 4.1, it follows that $w^* \equiv 0$. But then since uniqueness holds and $w^* = 0$, it follows again from Theorem 4.1 that there is no nontrivial nonnegative solution to $(1+|x|)^2\Delta w - Cw^p = 0$.

Note that the above proof is independent of dimension and works just as well for $n = 1, 2$. Using Theorem 5.1(i), we can also give an alternative proof of the existence part of the above result, and more importantly, we can extend the existence/nonexistence dichotomy to dimensions $n = 1, 2$.

Theorem 5.2. *Let* $p > 1$.

(i) *Consider the equation*

$$u'' - \gamma u^p = 0 \qquad \text{in } \mathbb{R}. \tag{5.3}$$

There exists a positive solution to (5.3) *if* $\gamma(x) \leq C(1+|x|)^{-1-p-\varepsilon}$, *for some* $C, \varepsilon > 0$, *and there is no positive solution to* (5.3) *if* $\gamma(x) \geq C(1+|x|)^{-1-p}$, *for some* $C > 0$.

(ii) *Consider the equation*

$$\Delta u - \gamma u^p = 0 \qquad \text{in } \mathbb{R}^n,\ n \geq 2. \tag{5.4}$$

There exists a positive solution to (5.4) *if* $\gamma(x) \leq C(1+|x|)^{-2-\varepsilon}$, *for some* $C, \varepsilon > 0$, *and there is no positive solution to* (5.4) *if* $\gamma(x) \geq C(1+|x|)^{-2}$, *for some* $C > 0$.

Proof. Part (i). Consider the semilinear equation

$$u_t = \alpha u'' - u^p \qquad \text{in } \mathbb{R} \times (0, \infty). \tag{5.5}$$

If $\alpha(x) \le C(1+|x|)^{1+p}$, then it follows from THEOREM 5.1(ii-a) that uniqueness holds for (5.5). Also, by (4.3) we have $w^* = 0$ for equation (5.5). Thus, we conclude from THEOREM 4.1(i) that there is no positive solution to $\alpha u'' - u^p = 0$ in $\mathbb{R}$. This is equivalent to the nonexistence statement in (i). On the other hand, if $\alpha(x) \ge C(1+|x|)^{1+p+\varepsilon}$, then by THEOREM 5.1(ii-b) uniqueness does not hold for (5.5). Thus, it follows from THEOREM 4.1(ii) that a positive solution exists for $\alpha u'' - u^p = 0$ in $\mathbb{R}$, which is equivalent to the existence statement in (i).

Part (ii) is proven in exactly the same manner. □

We now turn to the class of equations in (5.2).

THEOREM 5.3.

(i) *Let V be bounded from above. If*

$$\gamma(x) \ge C_1 \exp(-C_2|x|^2),$$

for some $C_1, C_2 > 0$, then uniqueness holds in (5.2) for all f.

(ii) *Let $V \ge 0$. If*

$$\gamma(x) \le C \exp(-|x|^{2+\varepsilon}),$$

for some $C, \varepsilon > 0$, then uniqueness does not hold in (5.2) for $f \equiv 0$.

REMARK. Equation (5.2) with $0 \le V \le C$ and $\gamma(x) \le C\exp(-|x|^{2+\varepsilon})$, with $C, \varepsilon > 0$ is an example where uniqueness holds for BL but not for NS. For another example, consider $L = (1+|x|)^l \Delta$ with $n = 2$ and $l > 2$ or with $n = 1$ and $l > 1+p$. Let $V = 0$ and $\gamma = 1$. Applying PROPOSITION 3.5-(ii-a) with $\phi(x) = \log|x|$ if $n = 2$ and with $\phi(x) = |x|$ if $n = 1$ shows that uniqueness holds for BL. On the other hand, by (4.3), we have $w^* = 0$ while by THEOREM 5.1, $w_{\max} \ne 0$. Thus, by THEOREM 4.1(i), uniqueness does not hold for NS.

For an example where uniqueness holds for NS but not for BL, consider the operator $L = (1+|x|)^l \Delta$ in $\mathbb{R}^n$, $n \ge 3$, for $l > 2$, and let $V = 0$. Then uniqueness does not hold for BL – see REMARK 1 after PROPOSITION 3.5. On the other hand, if $\gamma \ge (1+|x|)^{l-2}$, then uniqueness does hold for NS. Indeed, by PROPOSITION 3.2 and THEOREM 4.1(ii), it suffices to show that there is no nontrivial, nonnegative solution w to $Lw - \gamma w^p = 0$ in $\mathbb{R}^n$, or equivalently, to

$$\Delta w - \frac{\gamma(x)}{(1+|x|)^l}\, w^p = 0 \qquad \text{in } \mathbb{R}^n.$$

But this follows from THEOREM 5.2. Note that in this example, $\inf_{x\in\mathbb{R}^n} \frac{V}{\gamma}(x) = 0$, as must be the case in light of THEOREM 3.4.

For the proofs in [5] one also needs the following semilinear elliptic maximum principle.

PROPOSITION 5.4. *Let $D \subset \mathbb{R}^n$ be a bounded domain and let $0 \le u_1, u_2 \in C^2(D) \cap C(\bar{D})$ satisfy*

$$Lu_1 + Vu_1 - \gamma u_1^p \le Lu_2 + Vu_2 - \gamma u_2^p \qquad \text{in } D,$$

and

$$u_1 \ge u_2 \qquad \text{on } \partial D.$$

Assume that $V \le 0$. Then $u_1 \ge u_2$ in D.

Proof. Let $W = u_1 - u_2$ and define $H(x) = \frac{u_1^p(x)-u_2^p(x)}{W(x)}$, if $W(x) \neq 0$, and $H(x) = 0$ otherwise. Then $H \geq 0$ and we have $LW + (V-H)W \leq 0$ in D and $W \geq 0$ on ∂D. Since $V - H \leq 0$, it follows from the standard linear elliptic maximum principle that $W \geq 0$ in D. □

Proof of THEOREM 5.3. (i) Let $U(x,t) = u_{0;\max}(x,t)\exp(-C|x|^2(t+\delta))$, for some $C, \delta > 0$. Then U satisfies

$$\begin{aligned}&\Delta U + 4C(t+\delta)x\cdot\nabla U + (4|x|^2(t+\delta)^2C^2 + 2nC(t+\delta) + V - C|x|^2)U\\&\quad - C_1\exp(-C_2|x|^2)\exp(C(p-1)|x|^2(t+\delta))U^p - U_t \geq 0 \qquad \text{in } \mathbb{R}^n\times(0,\infty).\end{aligned}$$

Fixing $\delta = \frac{C_2}{C(p-1)}$ and $C \geq \frac{16C_2^2}{p-1}$, we obtain

$$\begin{aligned}&\Delta U + 4C(t+\delta)x\cdot\nabla U\\&\quad + (2nC(t+\delta)+V)U - U^p - U_t \geq 0 \qquad \text{in } \mathbb{R}^n\times(0,\delta).\end{aligned} \tag{5.6}$$

Note that the coefficients of the operator on the left hand side of (5.6) satisfy the requirements in THEOREM 3.3. (They depend on t unlike in THEOREM 3.3, but this is not important.) Thus, it follows from the maximum principle that for any $R > 1$, the super solution in $B_R \times (0,\infty)$ constructed in the proof of [6, Theorem 3.3] is larger or equal to U in $B_R \times (0,\delta)$. That is,

$$U(x,t) \leq (1+|x|)^{\frac{2}{p-1}}(R-|x|)^{-\frac{2}{p-1}}\exp(K(t+1)) \qquad \text{in } B_R\times(0,\delta).$$

Letting $R \to \infty$ shows that $U \equiv 0$ in $\mathbb{R}^n \times (0,\delta)$, and thus the same is true for $u_{0;\max}$. As the original equation was time homogeneous, it is clear that in fact $u_{0;\max} \equiv 0$ in $\mathbb{R}^n \times (0,\infty)$.

(ii) Writing $u(x) = \exp((1+|x|^2)^{1+\frac{\varepsilon}{4}})\hat{u}$ and dividing through by $\exp((1+|x|^2)^{1+\frac{\varepsilon}{4}})$ one sees that nonuniqueness for the initial condition $f = 0$ in (5.2) is equivalent to nonuniqueness for the initial condition $f = 0$ in an equation of the form

$$u_t = \Delta u + B\nabla u + \hat{V}u - \hat{\gamma}u^p, \tag{5.7}$$

where $B(x)\cdot\frac{x}{|x|} \geq C_1|x|^{1+\frac{\varepsilon}{2}}$, $\hat{V} \geq C_1$ and $\hat{\gamma} \leq C$, for constants $C_1, C > 0$. Uniqueness does not hold for $BL(\Delta + B\nabla, 0)$ as was shown in the remark following PROPOSITION 3.5. Thus, by THEOREM 3.4, uniqueness does not hold for the initial condition $f = 0$ in (5.7). □

Acknowledgement. The first author owes thanks to the organizers of the Equadiff 11 Conference for their invitation and for the welcoming and stimulating atmosphere.

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