THE HELMHOLTZ DECOMPOSITION IN ARBITRARY UNBOUNDED DOMAINS – A THEORY BEYOND $L^2$

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Abstract. It is well known that the usual $L^q$-theory of the Stokes operator valid for bounded or exterior domains cannot be extended to arbitrary unbounded domains $\Omega \subset \mathbb{R}^n$ when $q \neq 2$. One reason is given by the Helmholtz projection which fails to exist for certain unbounded smooth planar domains unless $q = 2$. However, as recently shown [6], the Helmholtz projection does exist for general unbounded domains in $\mathbb{R}^3$ if we replace the space $L^q_{\sigma}(\Omega)$ by $L^2 \cap L^q$ for $q > 2$ and by $L^q + L^2$ for $1 < q < 2$. In this paper, we generalize this new approach from the three-dimensional case to the $n$-dimensional case, $n \geq 2$.

Key words. Helmholtz decomposition, Helmholtz projection, general unbounded domains, domains of uniform $C^1$-type, intersection spaces, sum spaces

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and let $1 < q < \infty$. Then the classical Helmholtz projection $P_q$ on $L^q(\Omega)^n$ defines a topological and algebraic decomposition of $L^q(\Omega)^n$ into the direct sum of the solenoidal subspace $L^q_{\sigma}(\Omega) = C^\infty_{0,\sigma}(\Omega)$ and the space of gradients $G^q(\Omega)$

\[
L^q(\Omega)^n = C^\infty_{0,\sigma}(\Omega) \oplus G^q(\Omega),
\]

where $C^\infty_{0,\sigma}(\Omega) = \{ u \in C^\infty(\Omega)^n : \text{div } u = 0 \}$, and the space of gradients $G^q(\Omega) = \{ \nabla p \in L^q(\Omega)^n : p \in L^q_{\text{loc}}(\Omega) \} = \text{Ker}(P_q)$.

Hence every vector field $u \in L^q$ (here $L^q$ stands for $L^q(\Omega)^n$) has a unique decomposition $u = u_0 + \nabla p$ where $u_0 = P_q u \in L^q_{\sigma}$ and

\[
\|u_0\|_q + \|\nabla p\|_q \leq c\|u\|_q
\]

(1.1)

with a constant $c = c(q, \Omega) > 0$. The existence of $P_q$ is well known for several classes of domains with boundary of class $C^1$, namely for bounded domains, for exterior domains, aperture domains, layers, tubes, half spaces and perturbations of them, see e.g. [3], [4], [5], [7], [8], [10]. However, the decomposition

\[
L^q(\Omega)^n = L^q_{\sigma}(\Omega) \oplus G^q(\Omega), \quad 1 < q < \infty,
\]

(1.2)

no longer holds for infinite cones in $\mathbb{R}^2$ with “smoothed vertex” at the origin and of opening angle larger than $\pi$ when $q \neq 2$, see [2], [9].

On the other hand, an $L^2$-theory works for every bounded and unbounded domain without any assumptions on the boundary. Actually, the decomposition $u = u_0 + \nabla p$ can be found by solving the weak Neumann problem

\[
\Delta p = \text{div } u \quad \text{in } \Omega, \quad \frac{\partial p}{\partial N} = u \cdot N \quad \text{on } \partial \Omega,
\]

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where \( N \) denotes the exterior normal unit vector on \( \partial \Omega \); i.e., \( \nabla p \) is determined in \( G^2(\Omega) \) via the variational problem

\[
(\nabla p, \nabla \psi) = (u, \nabla \psi) \quad \text{for all} \quad \nabla \psi \in G^2(\Omega)
\]

using the Lemma of Lax-Milgram. Obviously, \( \|\nabla p\|_2 \leq \|u\|_2 \) and \( u_0 := u - \nabla p \perp \nabla p \) leading to the \textit{a priori} estimate

\[
\|u_0\|_2 + \|\nabla p\|_2 \leq 2\|u\|_2. \tag{1.3}
\]

Note that the constant \( C = 2 \) in (1.3) is independent of the domain.

In a recent paper, the authors proved the existence of the Helmholtz projection for general unbounded domains \( \Omega \subset \mathbb{R}^3 \) of uniform \( C^2 \)-class (cf. \textsc{Definition 1.1} below) by replacing the space \( L^q \) by

\[
\tilde{L}^q(\Omega) = \begin{cases} 
L^q(\Omega) \cap L^2(\Omega), & 2 \leq q < \infty \\
L^q(\Omega) + L^2(\Omega), & 1 < q < 2 
\end{cases}
\]

We may extend this definition to general unbounded domains \( \Omega \subset \mathbb{R}^n, n \geq 2 \), and equip \( \tilde{L}^q(\Omega) \) with the norm \( \|u\|_{\tilde{L}^q(\Omega)} = \max(\|u\|_q, \|u\|_2) \) if \( q \geq 2 \), and

\[
\|u\|_{\tilde{L}^q(\Omega)} = \inf \{ \|u_1\|_q + \|u_2\|_2 : u = u_1 + u_2, \ u_1 \in L^q, \ u_2 \in L^2 \}
\]

\[
= \sup \left\{ \frac{|(u_1 + u_2, f)|}{\|f\|_{L^q \cap L^2}} : 0 \neq f \in L^q \cap L^2 \right\}
\]

if \( 1 < q < 2 \) and where \( q' = q/(q - 1) \). Note that

\[
(\tilde{L}^q(\Omega)')' = \tilde{L}^{q'}(\Omega),
\]

see [1]. By analogy, we define the spaces

\[
\tilde{L}^q(\Omega) = \begin{cases} 
L^q(\Omega) \cap L^2(\Omega), & 2 \leq q < \infty \\
L^q(\Omega) + L^2(\Omega), & 1 < q < 2 
\end{cases}, \quad \tilde{G}^q(\Omega) = \begin{cases} 
G^q(\Omega) \cap G^2(\Omega), & 2 \leq q < \infty \\
G^q(\Omega) + G^2(\Omega), & 1 < q < 2 
\end{cases}
\]

For more properties of the intersection and sum of such compatible pairs of Banach spaces we refer to [6].

\textsc{Definition 1.1}. A domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \), is called a uniform \( C^1 \)-domain of type \( (\alpha, \beta, K) \) (where \( \alpha > 0, \beta > 0, K > 0 \)) if for each \( x_0 \in \partial \Omega \) we can choose a Cartesian coordinate system with origin at \( x_0 \) and coordinates \( y = (y', y_n), y' = (y_1, \ldots, y_{n-1}) \), and a \( C^1 \)-function \( h(y'), |y'| \leq \alpha \), with \( C^1 \)-norm \( \|h\|_{C^1} \leq K \) such that the neighborhood

\[
U_{\alpha,\beta,h}(x_0) := \{(y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}
\]

of \( x_0 \) satisfies

\[
U_{\alpha,\beta,h}(x_0) := \{(y', y_n) : h(y') - \beta < y_n < h(y'), |y'| < \alpha\} = \Omega \cap U_{\alpha,\beta,h}(x_0),
\]

and

\[
\partial \Omega \cap U_{\alpha,\beta,h}(x_0) = \{(y', h(y')) : |y'| < \alpha\}.
\]
Then our main theorem reads as follows:

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform $C^1$–domain of type $(\alpha, \beta, K)$ and let $q \in (1, \infty)$. Then each $u \in \tilde{L}^q(\Omega)$ has a unique decomposition

$$u = u_0 + \nabla p, \quad u_0 \in \tilde{L}_g^q(\Omega), \quad \nabla p \in \tilde{G}^q(\Omega),$$

satisfying the estimate

$$\|u_0\|_{\tilde{L}^q} + \|\nabla p\|_{L^q} \leq c\|u\|_{\tilde{L}^q}, \quad c = c(\alpha, \beta, K, q) > 0.$$  \hspace{1cm} (1.4)

In particular, the Helmholtz projection $\tilde{P}_q$ defined by $\tilde{P}_q u = u_0$ is a bounded linear projection on $\tilde{L}^q(\Omega)$ with range $\tilde{L}_g^q(\Omega)$ and kernel $\tilde{G}^q(\Omega)$ and satisfies $(\tilde{P}_q)' = \tilde{P}_q^\perp$.

**Corollary 1.3.** Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform $C^1$–domain of type $(\alpha, \beta, K)$.

1. $\tilde{L}_g^q(\Omega) = C_{0,\alpha}(\Omega)^\perp \|\cdot\|_{L^q}$.

2. The following isomorphisms hold:

$$(\tilde{L}_g^q(\Omega))' \cong \tilde{L}_g^q(\Omega), \quad (\tilde{G}^q(\Omega))' \cong \tilde{G}^q(\Omega).$$

3. The annihilator identities

$$(\tilde{L}_g^q(\Omega))^\perp = \tilde{G}^q(\Omega), \quad (\tilde{G}^q(\Omega))^\perp = \tilde{L}_g^q(\Omega)$$

hold.

Besides the spaces $\tilde{L}_g^q$ and $\tilde{G}^q$ we consider the spaces

$$\tilde{L}_g^q(\Omega) = \{ u \in \tilde{L}^q(\Omega)^n : \text{div} u = 0 \text{ in } \Omega, u \cdot N = 0 \text{ on } \partial \Omega \}$$

and

$$\tilde{G}^q(\Omega) = \overline{\nabla C_{0,\alpha}^\infty(\Omega)}^\|_{L^q},$$

the closure in $\tilde{G}^q(\Omega)$ of its subspace $\nabla C_{0,\alpha}^\infty(\Omega)$; here $\tilde{L}_g^q(\Omega)$ is defined in the sense of distributions, i.e., $\langle u, \nabla \varphi \rangle = 0$ for all $\varphi \in C_{0,\alpha}^\infty(\Omega)$. Hence by definition

$$\tilde{L}_g^q(\Omega) = \tilde{G}^q(\Omega)^\perp$$

and, due to reflexivity, $\tilde{G}^q(\Omega)^\perp = \tilde{L}_g^q(\Omega)$.

As is well known, for bounded or exterior domains, see [10], $\tilde{L}_g^q = \tilde{L}_g^q$ and $\tilde{G}^q = \tilde{G}^q$. However, for an aperture domain, see [3], [5], [8], $\tilde{L}_g^q$ is a closed subspace of $\tilde{L}_g^q$ of codimension 1 if and only if $q > n'$, and $\tilde{G}^q$ is a closed subspace of $\tilde{G}^q$ of codimension 1 if and only if $1 < q < n$. In an arbitrary unbounded domain of uniform $C^1$-type the same phenomena may occur; moreover, the codimensions could equal an arbitrary positive integer or even infinity.

**Corollary 1.4.** Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform $C^1$–domain of type $(\alpha, \beta, K)$.

1. The following isomorphisms hold:

$$(\tilde{L}_g^q(\Omega)/\tilde{L}_g^q(\Omega))' \cong \tilde{G}^q(\Omega)/\tilde{G}^q(\Omega), \quad (\tilde{G}^q(\Omega)/\tilde{G}^q(\Omega))' \cong \tilde{L}_g^q(\Omega)/\tilde{L}_g^q(\Omega).$$
2. The space $\mathcal{L}_d^q(\Omega)$ admits the following direct algebraic and topological decomposition:

$$\mathcal{L}_d^q(\Omega) = \tilde{L}_d^q(\Omega) \oplus (\tilde{L}_d^q(\Omega) \cap \tilde{G}^d(\Omega)).$$

By Corollary 1.4 (1) $\tilde{L}_d^q$ has a finite codimension in $\mathcal{L}_d^q$ if and only if $\tilde{G}^d$ has a finite codimension in $\tilde{G}^d$; in this case the codimensions coincide.

2. Proofs.

2.1. Preliminaries. Concerning Definition 1.1 we introduce further notation and discuss some properties. Obviously, the axes $e_i, i = 1, \ldots, n,$ of the new coordinate system $(y', y_n)$ may be chosen in such a way that $e_1, \ldots, e_{n-1}$ are tangential to $\partial \Omega$ at $x_0.$ Hence at $y' = 0$ we have $h(y') = 0$ and $\nabla' h(y') = 0.$ Since $h \in C^1$, for any given constant $M_0 > 0$, we may choose $\alpha > 0$ sufficiently small such that $\|h\|_{C^1} \leq M_0$ is satisfied.

It is easily shown that there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ of fixed radius $r > 0$ with centers $x_j \in \overline{\Omega}$, such that with suitable functions $h_j \in C^1$ of type $(\alpha, \beta, K)$

$$\mathcal{B}_j \subset U_{\alpha,\beta,h_j}(x_j) \text{ if } x_j \in \partial \Omega, \quad \mathcal{B}_j \subset \Omega \text{ if } x_j \in \Omega. \tag{2.1}$$

Here $j$ runs from 1 to a finite number $N = N(\Omega) \in \mathbb{N}$ if $\Omega$ is bounded, and $j \in \mathbb{N}$ if $\Omega$ is unbounded. The covering $\{B_j\}$ of $\Omega$ may be constructed in such a way that not more than a fixed number $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these balls can have a nonempty intersection. Moreover, there exists a partition of unity $\{\varphi_j\}, \varphi_j \in C_0^\infty(\mathbb{R}^n),$ such that

$$0 \leq \varphi_j \leq 1, \quad \text{supp} \varphi_j \subset B_j, \quad \text{and} \quad \sum_{j=1}^N \varphi_j = 1 \text{ or } \sum_{j=1}^\infty \varphi_j = 1 \text{ on } \Omega. \tag{2.2}$$

The functions $\varphi_j$ may be chosen so that $|\nabla \varphi_j(x)| \leq C$ uniformly in $j$ and $x \in \Omega$ with $C = C(\alpha, \beta, K)$.

If $\Omega$ is unbounded, then $\Omega$ can be represented as the union of an increasing sequence of bounded domains $\Omega_k \subset \Omega, k \in \mathbb{N},$

$$\ldots \subset \Omega_k \subset \Omega_{k+1} \subset \ldots, \quad \Omega = \bigcup_{k=1}^\infty \Omega_k, \tag{2.3}$$

each $\Omega_k$ is of the same type $(\alpha', \beta', K').$ Without loss of generality we assume that $\alpha = \alpha', \beta = \beta', K = K'$. Using the partition of unity $\{\varphi_j\}$ the construction of the Helmholtz decomposition will be based on well known results for certain bounded and unbounded domains. For this reason, we introduce for $h \in C^1_0(\mathbb{R}^{n-1})$ satisfying $h(0) = 0, \nabla h(0) = 0$ and supp $h \subset B_r'(0) \subset \mathbb{R}^{n-1}, 0 < r = r(\alpha, \beta, K) < \alpha$, the bounded domain

$$H = H_{\alpha,\beta,h,r} = \{y \in \mathbb{R}^n : h(y') - \beta < y_n < h(y'), |y'| < \alpha\} \cap B_r(0);$$

here we assume that $B_r(0) \subset \{y : |y_n - h(y')| < \beta, |y'| < \alpha\}$.

On $H$ we consider the classical Sobolev spaces $W^{1,q}(H)$ and $W^{1,q}_0(H)$, the dual space $W^{-1,q}(H) = (W^{1,q}_0(H))'$ and the space

$$L_d^q(H) = \{u \in L^q(H) : \int_H u \, dx = 0\}$$

of $L^q$-functions with vanishing mean on $H$. 

Lemma 2.1. Let $1 < q < \infty$ and $H = H_{\alpha,\beta,h,r}$.
1. Assume that $\|\nabla h\|_\infty \leq M_0$ for a sufficiently small constant $M_0 = M_0(q, n) > 0$, and let $u \in L^q(H)^n$ admit the Helmholtz decomposition $u = u_0 + \nabla p$ with $u_0 \in L_0^q(H)$, $p \in W^{1,q}(H)$ and supp $u_0$, supp $p \subset B_r(0)$. Then there exists a constant $C = C(\alpha, \beta, K, q) > 0$ such that

$$\|u_0\|_q + \|\nabla p\|_q \leq C\|u\|_q. \quad (2.4)$$

2. There exists a bounded linear operator $R : L_0^q(H) \to W_0^{1,q}(H)^n$ such that div $\circ R = \text{id}$ on $L_0^q(H)$ and a constant $C = C(\alpha, \beta, K, q) > 0$ such that

$$\|Rf\|_{W^{1,q}} \leq C\|f\|_q \quad \text{for all } f \in L_0^q(H). \quad (2.5)$$

3. There exists $C = C(\alpha, \beta, K, q) > 0$ such that for every $p \in L_0^q(H)$

$$\|p\|_q \leq C\|\nabla p\|_{W^{-1,q}} = C\sup \left\{|\langle p, \nabla v \rangle| - \frac{\|\nabla v\|_q}{q} : 0 \neq v \in W_0^{1,q'}(H)\right\}. \quad (2.6)$$

Proof.

1. Since supp $u_0$, supp $p \subset B_r(0)$ and since $h$ has compact support, the decomposition $u = u_0 + \nabla p$ on $H$ may be considered as a Helmholtz decomposition in the bent half space

$$H_h = \{y \in \mathbb{R}^n : y_n < h(y'), y' \in \mathbb{R}^{n-1}\}.$$ 

Then [10, Lemma 3.8 a)] yields (2.4) provided that $\|\nabla h\|_\infty \leq M_0$ is sufficiently small.

2. It is well known that there exists a bounded linear operator $R : L_0^q(H) \to W_0^{1,q}(H)^n$ such that $u = Rf$ solves the divergence problem $\text{div } u = f$. Moreover, the estimate (2.5) holds with $C = C(\alpha, \beta, K, q) > 0$, see [8, III, Theorem 3.1].

3. The dual map $R' : W^{-1,q}(H)^n \to L_0^q(H)$ of the map $R$ in 2., replacing $q$ by $q'$, is continuous with bound $C = C(\alpha, \beta, K, q) > 0$. Given $p \in L_0^q(H)$, we get that $\nabla p \in W^{-1,q}(H)^n$ using the definition $\langle \nabla p, v \rangle = -\langle p, \text{div } v \rangle$ for $v \in W_0^{1,q}(H)$. Then for all $f \in L_0^q(H)$,

$$(f, R'(\nabla p)) = \langle Rf, \nabla p \rangle = -(\text{div } Rf, p) = -(f, p).$$

Hence $R'(\nabla p) = -p$, yielding (2.6).

2.2. The case $\Omega$ bounded, $q \geq 2$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded uniform $C^1$-domain of type $(\alpha, \beta, K)$. Then each $u \in L^q(\Omega)^n$, $2 \leq q < \infty$, has a unique decomposition $u = u_0 + \nabla p$, $u_0 \in L_0^q(\Omega)$, $\nabla p \in G^q(\Omega)$, satisfying (1.1) with constant $c = c(q, \Omega) > 0$ depending somehow on $\Omega$, see [7], [10].

Given the partition of unity $\{\varphi_j\}_{j=1}^N$, the balls $B_j$ and the sets $U_{\alpha,\beta,h_j}(x_j)$, $U_{\alpha,\beta,h_j}^-(x_j)$, see Definition 1.1 and Subsection 2.1, we define the sets

$$U_j = U_{\alpha,\beta,h_j}(x_j) \cap B_j \text{ if } x_j \in \partial \Omega \quad \text{and} \quad U_j = B_j \text{ if } x_j \in \Omega,$$

$1 \leq j \leq N$. We may assume that in both cases Lemma 2.1 applies to the domain $H = U_j$ (in Lemma 2.1 1. the smallness assumption is satisfied if $x_j \in \partial \Omega$, whereas the case
\(x_j \in \Omega\) is related to the Helmholtz decomposition in the whole space. Moreover, at most \(N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}\) of these sets will have a nonempty intersection. Multiplying \(u = u_0 + \nabla p\) with \(\varphi_j\) we get that
\[
\varphi_j u = \varphi_j u_0 + \nabla (\varphi_j(p - M_j)) - (\nabla \varphi_j)(p - M_j)
\]
where \(M_j = \frac{1}{|U_j|} \int_{U_j} p \, dx\) yielding \(p - M_j \in L^q_0(U_j)\). Moreover, using the operator \(R = R_j\) in \(U_j\), see Lemma 2.1 (2), we find \(w_j = R_j(u_0 \cdot \nabla \varphi_j) \in W^{1,q}_0(U_j)\) such that \(\text{div} \, w_j = u_0 \cdot \nabla \varphi_j \in U_j\) and \(\varphi_j u_0 - w_j \in L^q(U_j)\). Then
\[
\varphi_j u + (\nabla \varphi_j)(p - M_j) - w_j = (\varphi_j u_0 - w_j) + \nabla (\varphi_j(p - M_j))
\]
is the Helmholtz decomposition of the left-hand side \(\varphi_j u + (\nabla \varphi_j)(p - M_j) - w_j\) in \(U_j\). To estimate \(w_j\) and \(\varphi_j u\) let \(s := \max\left\{\frac{mu}{n + q}, 2\right\} \in \{2, q\}, s' = s/(s - 1)\). Then the Sobolev embeddings \(W^{1,s}_0(U_j) \hookrightarrow L^q(U_j)\) and \(W^{1,q}_0(U_j) \hookrightarrow L^{s'}(U_j)\) hold with embedding constants depending on \(\alpha, \beta, K\) and \(q, s\) only. Hence, by Lemma 2.1 2. (with \(q\) replaced by \(s\))
\[
\|w_j\|_{L^q(U_j)} \leq c\|w_j\|_{W^{1,q}(U_j)} \leq C\|u_0\|_{L^s(U_j)},
\]
and by Lemma 2.1 3.
\[
\|u_0\|_{W^{-1,s}(U_j)} = \sup \left\{\frac{|(u_0, v)|}{\|\nabla v\|_{L^{s'}(U_j)}} : 0 \neq v \in W^{1,q}_0(U_j)\right\} \leq C\|u_0\|_{L^s(U_j)},
\]
where \(c = c(\alpha, \beta, K) > 0\) and \(C = C(\alpha, \beta, K) > 0\). By (2.9) we conclude that
\[
\|p - M_j\|_{L^q(U_j)} \leq c\|\nabla p\|_{W^{-1,s}(U_j)} \leq c(\|u\|_{W^{-1,s}(U_j)} + \|u_0\|_{W^{-1,s}(U_j)})
\]
\[
\leq C(\|u\|_{L^s(U_j)} + \|u_0\|_{L^s(U_j)})
\]
with constants \(c, C > 0\) depending only on \(\alpha, \beta, K\).

Now Lemma 2.1 1. and (2.7) imply the estimate
\[
\|\varphi_j u_0 - w_j\|_{L^s(U_j)} + \|\nabla (\varphi_j(p - M_j))\|_{L^q(U_j)} \leq c\|\varphi_j u + (\nabla \varphi_j)(p - M_j)\|_{L^q(U_j)},
\]
which may be simplified by virtue of (2.8), (2.10) to the inequality
\[
\|\varphi_j u_0\|_{L^s(U_j)} + \|\varphi_j \nabla p\|_{L^q(U_j)} \leq C(\|u\|_{L^s(U_j)} + \|u_0\|_{L^s(U_j)})
\]
with constants \(c, C > 0\) depending only on \(\alpha, \beta, K\). Taking the \(q\)th power in (2.11), summing over \(j = 1, \ldots, N\) and exploiting the crucial property of the number \(N_0\) we are led to the estimate
\[
\|u_0\|_{L^q(U_j)}^q + \|\nabla p\|_{L^q(U_j)}^q \leq \int_{\Omega} \left(\left(\sum_j \varphi_j|u_0|\right)^q + \left(\sum_j \varphi_j|\nabla p|\right)^q\right) \, dx
\]
\[
\leq \int_{\Omega} N_0^\frac{q}{s'} \left(\sum_j |\varphi_j u_0|^q + \sum_j |\varphi_j \nabla p|^q\right) \, dx
\]
\[
\leq C N_0^\frac{q}{s'} \left(\sum_j \|u\|_{L^s(U_j)}^q + \sum_j \|u_0\|_{L^s(U_j)}^q\right).
\]
Finally, the assertion \( C \) of uniform boundedness \( \|u\|_{L^s(\Omega)} \) so that (2.12) may be simplified to the estimate
\[
\|u_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leq C \left( \|u\|_{L^q(\Omega)} + \|u_0\|_{L^q(\Omega)} \right)
\]  
where \( C = C(\alpha, \beta, K) > 0 \). To get rid of the term \( \|u_0\|_{L^q(\Omega)} \) in the case when \( s > 2 \) we use the elementary interpolation inequality
\[
\|u_0\|_{L^q(\Omega)} \leq \alpha \left( \frac{1}{2} \right)^{1/\alpha} \|u_0\|_{L^2(\Omega)} + (1 - \alpha) \varepsilon^{1/(1-\alpha)} \|u_0\|_{L^p(\Omega)}, \quad \varepsilon > 0,
\]
where \( \alpha \in (0,1) \) is defined by \( \frac{1}{2} = \frac{\alpha}{2} + \frac{1-\alpha}{q} \). Choosing \( \varepsilon > 0 \) sufficiently small, the new term \( \|u_0\|_{L^q(\Omega)} \) on the right-hand side of (2.13) may be absorbed by the same term on the left-hand side so that (2.13) leads to the inequality
\[
\|u_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leq C \left( \|u\|_{L^q(\Omega)} + \|u_0\|_{L^2(\Omega)} \right)
\]  
with \( C = C(\alpha, \beta, K) > 0 \). Finally we use the \( L^2 \)-estimate (1.3) for the term \( \|u_0\|_{L^2(\Omega)} \) and add (1.3) to (2.14). This proves the estimate
\[
\|u_0\|_{L^q \cap L^2} + \|\nabla p\|_{L^q \cap L^2} \leq C \|u\|_{L^q \cap L^2}
\]  
for every \( q \geq 2 \).

2.3. The case \( \Omega \) bounded, \( 1 < q < 2 \). For \( u \in L^q + L^2 \) there exist \( u_1 \in L^q \), \( u_2 \in L^2 \) satisfying \( u = u_1 + u_2 \) and \( \|u\|_{L^q + L^2} = \|u_1\|_{L^q} + \|u_2\|_{L^2} \). Define \( u_0 \) and \( \nabla p \) by
\[
\begin{align*}
\begin{array}{c}
u_p u_1 + P_2 u_2 \in L^q + L^2; \\
\nabla p = (I - P_q) u_1 + (I - P_2) u_2 \in G^q + G^2
\end{array}
\end{align*}
\]
yielding \( u = u_0 + \nabla p \). Then, using duality arguments and (2.15) for \( q' > 2 \),
\[
\|u_0\|_{L^q + L^2} = \sup \left\{ \frac{\langle P_q u_1 + P_2 u_2, v \rangle}{\|v\|_{L^{q'} \cap L^2}} : 0 \neq v \in L^{q'} \cap L^2 \right\}
\]
\[
= \sup \left\{ \|u_1 + u_2, P_q v\|_{L^{q'} \cap L^2} : 0 \neq v \in L^{q'} \cap L^2 \right\}
\]
\[
\leq \sup \left\{ \left[ \|u_1\|_q + \|u_2\|_2 \right] \max \left( \|P_q v\|_{q'}, \|P_2 v\|_2 \right) \right\} : 0 \neq v \in L^{q'} \cap L^2 \right\}
\]
\[
\leq C \|u\|_{L^q + L^2}
\]
with the same constant \( C = C(\alpha, \beta, K) \) as in (2.15) (with \( q' \) instead of \( q \)). It follows that
\[
\|u_0\|_{L^q + L^2} + \|\nabla p\|_{L^q + L^2} \leq C \|u\|_{L^q + L^2}, \quad \text{i.e., (1.4) for } q \in (1,2) \).
\]

Summarizing both cases we proved the existence of a bounded linear projection \( \tilde{P}_q \) on \( \tilde{L}^q \) for a bounded domain \( \tilde{\Omega} \subset \mathbb{R}^n \) of uniform \( C^1 \)-type \( (\alpha, \beta, K) \) such that \( \tilde{P}_q u = P_q u \) for all \( u \in \tilde{L}^q = L^q \). Moreover, \( \nabla p = (I - \tilde{P}_q) u = (I - P_q) u \in \tilde{G}^q = G^q \). The crucial property of \( \tilde{P}_q \) is the fact that its operator norm on \( \tilde{L}^q \) is bounded by a constant \( C = C(\alpha, \beta, K) > 0 \). Finally, the assertion \( (\tilde{P}_q)' = \tilde{P}_q' \) follows from standard duality arguments.
2.4. The case Ω unbounded. Let Ω ⊂ ℝ^n be an unbounded domain of uniform C^1-type (α, β, K). Given u ∈ L^q(Ω)^n, 1 < q < ∞, define u_k = u|_{Ω_k}, k ∈ ℕ, where Ω_k ⊂ Ω is the bounded domain introduced in §2.1; note that Ω_k ⊂ Ω again is of uniform C^1-type (α, β, K). Since obviously u_k ∈ L^q(Ω_k)^n, there exists a unique Helmholtz decomposition

u_k = u_{k,0} + ∇p_k with u_{k,0} ∈ L^q_0(Ω_k), ∇p_k ∈ G^q(Ω_k),

satisfying the estimate

\[ \|u_{k,0}\|_{L^q(Ω_k)} + \|\nabla p_k\|_{L^q(Ω_k)} \leq C \|u_k\|_{L^q(Ω_k)} \leq C \|u\|_{L^q(Ω)} \tag{2.16} \]

with a constant C = C(α, β, K) independent of k ∈ ℕ. Extending u_{k,0} and ∇p_k by 0 from Ω_k to Ω we get bounded sequences in L^q(Ω)^n. Since L^q(Ω) is reflexive, there exist – suppressing the notation of subsequences – weak limits

\[ u_0 = (w- \lim_{k \to \infty} u_{k,0} \in L^q(Ω)^n, \quad Q = (w- \lim_{k \to \infty} \nabla p_k \in L^q(Ω)^n, \tag{2.17} \]

satisfying u = u_0 + Q and the estimate \( \|u_0\|_{L^q(Ω)} + \|Q\|_{L^q(Ω)} \leq C \|u\|_{L^q(Ω)} \). Since u_{k,0} ∈ L^q_0(Ω_k) ⊂ L^q(Ω) and since L^q(Ω) is closed with respect to weak convergence, u_0 ∈ L^q_0(Ω). Moreover, de Rham’s argument, see [11], [12], implies that there exists p ∈ L^1_{loc}(Ω) such that Q = ∇p ∈ G^q(Ω). Hence the pair (u_0, ∇p) determines a Helmholtz decomposition of u in L^q(Ω)^n. The uniqueness of the Helmholtz decomposition is proved by a classical duality argument and the weak convergence properties (2.17). Now the existence of the Helmholtz projection \( P_q \) on \( L^q(Ω)^n \) with range \( L^q_0(Ω) \) and kernel \( G^q(Ω) \) is proved. Moreover, the assertion \( (\tilde{P}_q)' = \tilde{P}_q' \) follows from standard duality arguments.

**Proof of Corollary 1.3.**

1. Note that obviously \( C_{0,σ}^∞(Ω) \) is dense in \( L^q(Ω) \), 1 < q < ∞. Now let u = u_0 ∈ L^q_0(Ω). By the proof above, cf. (2.17), the sequence \((u_{k,0})\) converges weakly in \( L^q(Ω)^n \) towards \( \tilde{P}_q u = u \). By Mazur’s theorem there exists a sequence of convex combinations of the elements \((u_{k,0})\), say \((v_m)\), converging strongly in \( L^q_0(Ω) \) to u. Each element \( v_m \) has its support in some bounded domain \( Ω_{k(m)} \) yielding \( v_m \in L^q_0(Ω_{k(m)}) \). Since \( C_{0,σ}^∞(Ω_{k(m)}) \) is dense in \( L^q_0(Ω_{k(m)}) \) and since for a bounded domain the norms in \( L^q \) and \( L^q_0 \) are equivalent, we conclude that \( (v_m) \) converges to u in \( L^q_0(Ω) \) as \( m \to \infty \); hence \( u \in C_{0,σ}^∞(Ω) \).

The assertions \( (\tilde{L}_q(Ω))' = \tilde{L}_q(Ω) \) and \( (\tilde{P}_q)' = \tilde{P}_q' \) follow from standard duality arguments.

2., 3. All claims are easily proved by duality arguments. \( \square \)

**Proof of Corollary 1.4.**

1. By Corollary 1.3 2., 3. both assertions are special cases of the following general result and of the reflexivity of the space \( L^q \), 1 < q < ∞:

Let \( X_0 \) be a Banach space with dual space \( Y_0 = (X_0)' \) and let \( X_1, X_2 \) and \( Y_1, Y_2 \) be closed subspaces of \( X_0 \) and \( Y_0 \), respectively, such that

\[ X_2 \subset X_1 \subset X_0, \quad Y_2 \subset Y_1 \subset Y_0, \quad X_2^⊥ = Y_1, \quad X_1^⊥ = Y_2. \]

Then

\[ (X_1/X_2)' \cong Y_1/Y_2. \]

For the proof of this abstract result first consider arbitrary equivalence classes \( \overline{y}_1 = y_1 + Y_2 \in Y_1/Y_2 \) and \( \overline{x}_1 = x_1 + X_2 \in X_1/X_2 \). Then \( \langle \langle \overline{y}_1, \overline{x}_1 \rangle \rangle := \langle y_1, x_1 \rangle \) is well-defined and defines an injective map \( J \) from \( Y_1/Y_2 \) into \( (X_1/X_2)' \). Next, given any
f ∈ (X₁/X₂)', define f₁ ∈ X₁' by ⟨f₁, x₁⟩ := ⟨⟨f, x₁⟩⟩ and use Hahn-Banach’s theorem to extend f₁ ∈ X₁' to an element f₀ ∈ Y₁. Note that f₀ ∈ Y₁, but that the map f ↦ f₀ is not necessarily linear. Then define J := f₀ + Y₂ ∈ Y₁/Y₂. We note that the map (X₁/X₂)' ↦ Y₁/Y₂, f ↦ J, is linear (!) and bounded. Since it is easily seen that this map is the inverse of the map J constructed in the first part of the proof, the isomorphism is found.

2. By Theorem 1.2 ˜Lqσ ∩ (˜Lqσ ∩ ˜Gq) = {0}. Each u ∈ ˜Lqσ has a unique decomposition u = u₀ + ∇p, u₀ ∈ ˜Lqσ, ∇p ∈ ˜Gq. Then ∇p = u − u₀ ∈ ˜Lqσ proving the algebraic decomposition of ˜Lqσ as stated. Moreover, by Theorem 1.2, this decomposition is also a topological one.

REFERENCES