Proceedings of Equadiff-11
2005, pp. 77–85
ISBN 978-80-227-2624-5

# THE HELMHOLTZ DECOMPOSITION IN ARBITRARY UNBOUNDED DOMAINS – A THEORY BEYOND $L^2$

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**Abstract.** It is well known that the usual $L^q$-theory of the Stokes operator valid for bounded or exterior domains cannot be extended to arbitrary unbounded domains $\Omega \subset \mathbb{R}^n$ when $q \neq 2$. One reason is given by the Helmholtz projection which fails to exist for certain unbounded smooth planar domains unless $q = 2$. However, as recently shown [6], the Helmholtz projection does exist for general unbounded domains in $\mathbb{R}^3$ if we replace the space $L^q, 1 < q < \infty$, by $L^2 \cap L^q$ for $q > 2$ and by $L^q + L^2$ for $1 < q < 2$. In this paper, we generalize this new approach from the three-dimensional case to the $n$-dimensional case, $n \geq 2$.

**Key words.** Helmholtz decomposition, Helmholtz projection, general unbounded domains, domains of uniform $C^1$-type, intersection spaces, sum spaces

**AMS subject classifications.** 35Q30, 76D05

**1. Introduction.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and let $1 < q < \infty$. Then the classical Helmholtz projection $P_q$ on $L^q(\Omega)^n$ defines a topological and algebraic decomposition of $L^q(\Omega)^n$ into the direct sum of the solenoidal subspace

$$L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q} = \mathcal{R}(P_q),$$

where $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}$, and the space of gradients

$$G^q(\Omega) = \{\nabla p \in L^q(\Omega)^n : p \in L^q_{\text{loc}}(\Omega)\} = \operatorname{Ker}(P_q).$$

Hence every vector field $u \in L^q$ (here $L^q$ stands for $L^q(\Omega)^n$) has a unique decomposition $u = u_0 + \nabla p$ where $u_0 = P_q u \in L^q_\sigma$ and

$$\|u_0\|_q + \|\nabla p\|_q \leq c\|u\|_q \tag{1.1}$$

with a constant $c = c(q, \Omega) > 0$. The existence of $P_q$ is well known for several classes of domains with boundary of class $C^1$, namely for bounded domains, for exterior domains, aperture domains, layers, tubes, half spaces and perturbations of them, see e.g. [3], [4], [5], [7], [8], [10]. However, the decomposition

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G^q(\Omega), \quad 1 < q < \infty, \tag{1.2}$$

no longer holds for infinite cones in $\mathbb{R}^2$ with "smoothed vertex" at the origin and of opening angle larger than $\pi$ when $q \neq 2$, see [2], [9].

On the other hand, an $L^2$-theory works for every bounded and unbounded domain without any assumptions on the boundary. Actually, the decomposition $u = u_0 + \nabla p$ can be found by solving the weak Neumann problem

$$\Delta p = \operatorname{div} u \quad \text{in} \quad \Omega, \quad \frac{\partial p}{\partial N} = u \cdot N \quad \text{on} \quad \partial\Omega,$$

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where $N$ denotes the exterior normal unit vector on $\partial\Omega$; i.e., $\nabla p$ is determined in $G^2(\Omega)$ via the variational problem

$$(\nabla p, \nabla\psi) = (u, \nabla\psi) \quad \text{for all} \quad \nabla\psi \in G^2(\Omega)$$

using the Lemma of Lax-Milgram. Obviously, $\|\nabla p\|_2 \le \|u\|_2$ and $u_0 := u - \nabla p \perp \nabla p$ leading to the *a priori* estimate

$$\|u_0\|_2 + \|\nabla p\|_2 \le 2\|u\|_2. \tag{1.3}$$

Note that the constant $C = 2$ in (1.3) is independent of the domain.

In a recent paper, the authors proved the existence of the Helmholtz projection for general unbounded domains $\Omega \subset \mathbb{R}^3$ of uniform $C^2$-class (cf. Definition 1.1 below) by replacing the space $L^q$ by

$$\tilde{L}^q(\Omega) = \begin{cases} L^q(\Omega) \cap L^2(\Omega), & 2 \le q < \infty \\ L^q(\Omega) + L^2(\Omega), & 1 < q < 2 \end{cases}.$$

We may extend this definition to general unbounded domains $\Omega \subset \mathbb{R}^n$, $n \ge 2$, and equip $\tilde{L}^q(\Omega)$ with the norm $\|u\|_{\tilde{L}^q(\Omega)} = \max(\|u\|_q, \|u\|_2)$ if $q \ge 2$, and

$$\begin{aligned} \|u\|_{\tilde{L}^q(\Omega)} &= \inf\{\|u_1\|_q + \|u_2\|_2 \,:\, u = u_1 + u_2,\ u_1 \in L^q,\ u_2 \in L^2\} \\ &= \sup\Big\{\frac{|\langle u_1 + u_2, f\rangle|}{\|f\|_{L^{q'} \cap L^2}} \,:\, 0 \ne f \in L^{q'} \cap L^2\Big\} \end{aligned}$$

if $1 < q < 2$ and where $q' = q/(q-1)$. Note that

$$\big(\tilde{L}^q(\Omega)\big)' \cong \tilde{L}^{q'}(\Omega),$$

see [1]. By analogy, we define the spaces

$$\tilde{L}^q_\sigma(\Omega) = \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \end{cases}, \quad \tilde{G}^q(\Omega) = \begin{cases} G^q(\Omega) \cap G^2(\Omega), & 2 \le q < \infty \\ G^q(\Omega) + G^2(\Omega), & 1 < q < 2 \end{cases}.$$

For more properties of the intersection and sum of such compatible pairs of Banach spaces we refer to [6].

Definition 1.1. A domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is called a uniform $C^1$-domain of type $(\alpha, \beta, K)$ (where $\alpha > 0$, $\beta > 0$, $K > 0$) if for each $x_0 \in \partial\Omega$ we can choose a Cartesian coordinate system with origin at $x_0$ and coordinates $y = (y', y_n)$, $y' = (y_1, \ldots, y_{n-1})$, and a $C^1$-function $h(y')$, $|y'| \le \alpha$, with $C^1$-norm $\|h\|_{C^1} \le K$ such that the neighborhood

$$U_{\alpha,\beta,h}(x_0) := \{(y', y_n) \in \mathbb{R}^n :\ h(y') - \beta < y_n < h(y') + \beta,\ |y'| < \alpha\}$$

of $x_0$ satisfies

$$U^-_{\alpha,\beta,h}(x_0) := \{(y', y_n) :\ h(y') - \beta < y_n < h(y'),\ |y'| < \alpha\} = \Omega \cap U_{\alpha,\beta,h}(x_0),$$

and

$$\partial\Omega \cap U_{\alpha,\beta,h}(x_0) = \{(y', h(y')) :\ |y'| < \alpha\}.$$

Then our main theorem reads as follows:

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform $C^1-$domain of type $(\alpha, \beta, K)$ and let $q \in (1, \infty)$. Then each $u \in \tilde{L}^q(\Omega)$ has a unique decomposition*

$$u = u_0 + \nabla p, \quad u_0 \in \tilde{L}^q_\sigma(\Omega), \, \nabla p \in \tilde{G}^q(\Omega),$$

*satisfying the estimate*

$$\|u_0\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \leq c\|u\|_{\tilde{L}^q}, \quad c = c(\alpha, \beta, K, q) > 0. \tag{1.4}$$

*In particular, the Helmholtz projection $\tilde{P}_q$ defined by $\tilde{P}_q u = u_0$ is a bounded linear projection on $\tilde{L}^q(\Omega)$ with range $\tilde{L}^q_\sigma(\Omega)$ and kernel $\tilde{G}^q(\Omega)$ and satisfies $\big(\tilde{P}_q\big)' = \tilde{P}_{q'}$.*

COROLLARY 1.3. *Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform $C^1-$domain of type $(\alpha, \beta, K)$.*

1. $\tilde{L}^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^q}}$.

2. *The following isomorphisms hold:*

$$\big(\tilde{L}^q_\sigma(\Omega)\big)' \cong \tilde{L}^{q'}_\sigma(\Omega), \quad \big(\tilde{G}^q(\Omega)\big)' \cong \tilde{G}^{q'}(\Omega).$$

3. *The annihilator identities*

$$\big(\tilde{L}^q_\sigma(\Omega)\big)^\perp = \tilde{G}^{q'}(\Omega), \quad \big(\tilde{G}^q(\Omega)\big)^\perp = \tilde{L}^{q'}_\sigma(\Omega)$$

*hold.*

Besides the spaces $\tilde{L}^q_\sigma$ and $\tilde{G}^q$ we consider the spaces

$$\tilde{\mathcal{L}}^q_\sigma(\Omega) = \big\{u \in \tilde{L}^q(\Omega)^n : \, \operatorname{div} u = 0 \text{ in } \Omega, \, u \cdot N = 0 \text{ on } \partial\Omega\big\}$$

and

$$\tilde{\mathcal{G}}^q(\Omega) = \overline{\nabla C^\infty_0(\overline{\Omega})}^{\|\cdot\|_{\tilde{L}^q}},$$

the closure in $\tilde{G}^q(\Omega)$ of its subspace $\nabla C^\infty_0(\overline{\Omega})$; here $\tilde{\mathcal{L}}^q_\sigma(\Omega)$ is defined in the sense of distributions, i.e., $\langle u, \nabla\varphi\rangle = 0$ for all $\varphi \in C^\infty_0(\overline{\Omega})$. Hence by definition

$$\tilde{\mathcal{L}}^q_\sigma(\Omega) = \tilde{\mathcal{G}}^{q'}(\Omega)^\perp$$

and, due to reflexivity, $\tilde{\mathcal{G}}^q(\Omega)^\perp = \tilde{\mathcal{L}}^{q'}_\sigma(\Omega)$.

As is well known, for bounded or exterior domains, see [10], $\tilde{\mathcal{L}}^q_\sigma = \tilde{L}^q_\sigma$ and $\tilde{\mathcal{G}}^q = \tilde{G}^q$. However, for an aperture domain, see [3], [5], [8], $\tilde{L}^q_\sigma$ is a closed subspace of $\tilde{\mathcal{L}}^q_\sigma$ of codimension 1 if and only if $q > n'$, and $\tilde{\mathcal{G}}^q$ is a closed subspace of $\tilde{G}^q$ of codimension 1 if and only if $1 < q < n$. In an arbitrary unbounded domain of uniform $C^1$-type the same phenomena may occur; moreover, the codimensions could equal an arbitrary positive integer or even infinity.

COROLLARY 1.4. *Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform $C^1$-domain of type $(\alpha, \beta, K)$.*

1. *The following isomorphisms hold:*

$$\big(\tilde{\mathcal{L}}^q_\sigma(\Omega)/\tilde{L}^q_\sigma(\Omega)\big)' \cong \tilde{G}^{q'}(\Omega)/\tilde{\mathcal{G}}^{q'}(\Omega), \quad \big(\tilde{G}^q(\Omega)/\tilde{\mathcal{G}}^q(\Omega)\big)' \cong \tilde{\mathcal{L}}^{q'}_\sigma(\Omega)/\tilde{L}^{q'}_\sigma(\Omega).$$

2. *The space* $\tilde{\mathcal{L}}^q_\sigma(\Omega)$ *admits the following direct algebraic and topological decomposition:*

$$\tilde{\mathcal{L}}^q_\sigma(\Omega) = \tilde{L}^q_\sigma(\Omega) \oplus \big(\tilde{\mathcal{L}}^q_\sigma(\Omega) \cap \tilde{G}^q(\Omega)\big).$$

By Corollary 1.4 (1) $\tilde{L}^q_\sigma$ has a *finite* codimension in $\tilde{\mathcal{L}}^q_\sigma$ if and only if $\tilde{\mathcal{G}}^{q'}$ has a *finite* codimension in $\tilde{G}^{q'}$; in this case the codimensions coincide.

## 2. Proofs.

**2.1. Preliminaries.** Concerning Definition 1.1 we introduce further notation and discuss some properties. Obviously, the axes $e_i$, $i = 1, \dots, n$, of the new coordinate system $(y', y_n)$ may be chosen in such a way that $e_1, \dots, e_{n-1}$ are tangential to $\partial\Omega$ at $x_0$. Hence at $y' = 0$ we have $h(y') = 0$ and $\nabla' h(y') = 0$. Since $h \in C^1$, for any given constant $M_0 > 0$, we may choose $\alpha > 0$ sufficiently small such that $\|h\|_{C^1} \le M_0$ is satisfied.

It is easily shown that there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ of fixed radius $r > 0$ with centers $x_j \in \overline{\Omega}$, such that with suitable functions $h_j \in C^1$ of type $(\alpha, \beta, K)$

$$\overline{B}_j \subset U_{\alpha,\beta,h_j}(x_j) \ \text{ if } x_j \in \partial\Omega, \quad \overline{B}_j \subset \Omega \ \text{ if } x_j \in \Omega. \tag{2.1}$$

Here $j$ runs from 1 to a finite number $N = N(\Omega) \in \mathbb{N}$ if $\Omega$ is bounded, and $j \in \mathbb{N}$ if $\Omega$ is unbounded. The covering $\{B_j\}$ of $\Omega$ may be constructed in such a way that not more than a fixed number $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these balls can have a nonempty intersection. Moreover, there exists a partition of unity $\{\varphi_j\}$, $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, such that

$$0 \le \varphi_j \le 1, \ \operatorname{supp}\varphi_j \subset B_j, \quad \text{and} \quad \sum_{j=1}^N \varphi_j = 1 \ \text{ or } \ \sum_{j=1}^\infty \varphi_j = 1 \ \text{ on } \Omega. \tag{2.2}$$

The functions $\varphi_j$ may be chosen so that $|\nabla\varphi_j(x)| \le C$ uniformly in $j$ and $x \in \Omega$ with $C = C(\alpha, \beta, K)$.

If $\Omega$ is unbounded, then $\Omega$ can be represented as the union of an increasing sequence of bounded domains $\Omega_k \subset \Omega$, $k \in \mathbb{N}$,

$$\ldots \subset \Omega_k \subset \Omega_{k+1} \subset \ldots, \quad \Omega = \bigcup_{k=1}^\infty \Omega_k, \tag{2.3}$$

each $\Omega_k$ is of the same type $(\alpha', \beta', K')$. Without loss of generality we assume that $\alpha = \alpha'$, $\beta = \beta'$, $K = K'$.

Using the partition of unity $\{\varphi_j\}$ the construction of the Helmholtz decomposition will be based on well known results for certain bounded and unbounded domains. For this reason, we introduce for $h \in C_0^1(\mathbb{R}^{n-1})$ satisfying $h(0) = 0$, $\nabla' h(0) = 0$ and $\operatorname{supp} h \subset B'_r(0) \subset \mathbb{R}^{n-1}$, $0 < r = r(\alpha, \beta, K) < \alpha$, the bounded domain

$$H = H_{\alpha,\beta,h;r} = \{y \in \mathbb{R}^n : \ h(y') - \beta < y_n < h(y'), \ |y'| < \alpha\} \cap B_r(0)\,;$$

here we assume that $\overline{B_r(0)} \subset \{y : \ |y_n - h(y')| < \beta, |y'| < \alpha\}$.

On $H$ we consider the classical Sobolev spaces $W^{1,q}(H)$ and $W_0^{1,q}(H)$, the dual space $W^{-1,q}(H) = \big(W_0^{1,q'}(H)\big)'$ and the space

$$L_0^q(H) = \Big\{u \in L^q(H) : \int_H u \,\mathrm{d}x = 0\Big\}$$

of $L^q$-functions with vanishing mean on $H$.

LEMMA 2.1. *Let $1 < q < \infty$ and $H = H_{\alpha,\beta,h;r}$.*

1. *Assume that $\|\nabla' h\|_\infty \le M_0$ for a sufficiently small constant $M_0 = M_0(q,n) > 0$, and let $u \in L^q(H)^n$ admit the Helmholtz decomposition $u = u_0 + \nabla p$ with $u_0 \in L^q_\sigma(H)$, $p \in W^{1,q}(H)$ and* $\operatorname{supp} u_0, \operatorname{supp} p \subset B_r(0)$. *Then there exists a constant $C = C(\alpha,\beta,K,q) > 0$ such that*

$$\|u_0\|_q + \|\nabla p\|_q \le C\|u\|_q. \tag{2.4}$$

2. *There exists a bounded linear operator*

$$R : L^q_0(H) \to W^{1,q}_0(H)^n$$

*such that* $\operatorname{div} \circ R = \mathrm{id}$ *on $L^q_0(H)$ and a constant $C = C(\alpha,\beta,K,q) > 0$ such that*

$$\|Rf\|_{W^{1,q}} \le C\|f\|_q \quad \textit{for all} \quad f \in L^q_0(H). \tag{2.5}$$

3. *There exists $C = C(\alpha,\beta,K,q) > 0$ such that for every $p \in L^q_0(H)$*

$$\|p\|_q \le C\|\nabla p\|_{W^{-1,q}} = C \sup\Big\{\frac{|\langle p, \operatorname{div} v\rangle|}{\|\nabla v\|_{q'}} : 0 \ne v \in W^{1,q'}_0(H)\Big\}. \tag{2.6}$$

*Proof.*

1. Since $\operatorname{supp} u_0$, $\operatorname{supp} p \subset B_r(0)$ and since $h$ has compact support, the decomposition $u = u_0 + \nabla p$ on $H$ may be considered as a Helmholtz decomposition in the bent half space

$$H_h = \big\{y \in \mathbb{R}^n : y_n < h(y'),\, y' \in \mathbb{R}^{n-1}\big\}.$$

Then [10, Lemma 3.8 a)] yields (2.4) provided that $\|\nabla' h\|_\infty \le M_0$ is sufficiently small.

2. It is well known that there exists a bounded linear operator $R : L^q_0(H) \to W^{1,q}_0(H)^n$ such that $u = Rf$ solves the divergence problem $\operatorname{div} u = f$. Moreover, the estimate (2.5) holds with $C = C(\alpha,\beta,K,q) > 0$, see [8, III, Theorem 3.1].

3. The dual map $R' : W^{-1,q}(H)^n \to L^q_0(H)$ of the map $R$ in 2., replacing $q$ by $q'$, is continuous with bound $C = C(\alpha,\beta,K,q) > 0$. Given $p \in L^q_0(H)$, we get that $\nabla p \in W^{-1,q}(H)^n$ using the definition $\langle \nabla p, v\rangle = -(p, \operatorname{div})$ for $v \in W^{1,q'}_0(H)$. Then for all $f \in L^{q'}_0(H)$,

$$(f, R'(\nabla p)) = \langle Rf, \nabla p\rangle = -(\operatorname{div} Rf, p) = -(f, p).$$

Hence $R'(\nabla p) = -p$, yielding (2.6).

□

**2.2. The case $\Omega$ bounded,** $q \ge 2$**.** Assume that $\Omega \subset \mathbb{R}^n$ is a bounded uniform $C^1$-domain of type $(\alpha,\beta,K)$. Then each $u \in L^q(\Omega)^n$, $2 \le q < \infty$, has a unique decomposition $u = u_0 + \nabla p$, $u_0 \in L^q_\sigma(\Omega)$, $\nabla p \in G^q(\Omega)$, satisfying (1.1) with constant $c = c(q,\Omega) > 0$ depending somehow on $\Omega$, see [7], [10].

Given the partition of unity $\{\varphi_j\}_{j=1}^N$, the balls $B_j$ and the sets $U_{\alpha,\beta,h_j}(x_j)$, $U^-_{\alpha,\beta,h_j}(x_j)$, see DEFINITION 1.1 and Subsection 2.1, we define the sets

$$U_j = U^-_{\alpha,\beta,h_j}(x_j) \cap B_j \;\text{ if }\; x_j \in \partial\Omega \quad\text{and}\quad U_j = B_j \;\text{ if }\; x_j \in \Omega,$$

$1 \le j \le N$. We may assume that in both cases LEMMA 2.1 applies to the domain $H = U_j$ (in LEMMA 2.1 1. the smallness assumption is satisfied if $x_j \in \partial\Omega$, whereas the case

$x_j \in \Omega$ is related to the Helmholtz decomposition in the whole space). Moreover, at most $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these sets will have a nonempty intersection. Multiplying $u = u_0 + \nabla p$ with $\varphi_j$ we get that

$$\varphi_j u = \varphi_j u_0 + \nabla\big(\varphi_j(p - M_j)\big) - (\nabla\varphi_j)(p - M_j)$$

where $M_j = \frac{1}{|U_j|}\int_{U_j} p \,\mathrm{d}x$ yielding $p - M_j \in L^q_0(U_j)$. Moreover, using the operator $R = R_j$ in $U_j$, see Lemma 2.1 (2), we find $w_j = R_j(u_0 \cdot \nabla\varphi_j) \in W^{1,q}_0(U_j)$ such that $\operatorname{div} w_j = u_0 \cdot \nabla\varphi_j$ in $U_j$ and $\varphi_j u_0 - w_j \in L^q_\sigma(U_j)$. Then

$$\varphi_j u + (\nabla\varphi_j)(p - M_j) - w_j = (\varphi_j u_0 - w_j) + \nabla\big(\varphi_j(p - M_j)\big) \tag{2.7}$$

is the Helmholtz decomposition of the left-hand side $\varphi_j u + (\nabla\varphi_j)(p - M_j) - w_j$ in $U_j$. To estimate $\varphi_j u$ and $\varphi_j \nabla p$ let $s := \max\big(\frac{nq}{n+q}, 2\big) \in [2, q)$, $s' = s/(s-1)$. Then the Sobolev embeddings $W^{1,s}_0(U_j) \hookrightarrow L^q(U_j)$ and $W^{1,q'}_0(U_j) \hookrightarrow L^{s'}(U_j)$ hold with embedding constants depending on $\alpha, \beta, K$ and $q, s$ only. Hence, by Lemma 2.1 2. (with $q$ replaced by $s$)

$$\|w_j\|_{L^q(U_j)} \le c\|w_j\|_{W^{1,s}(U_j)} \le C\|u_0\|_{L^s(U_j)}, \tag{2.8}$$

and by Lemma 2.1 3.

$$\|u_0\|_{W^{-1,q}(U_j)} = \sup\Big\{\frac{|(u_0, v)|}{\|\nabla v\|_{L^{q'}(U_j)}} : 0 \ne v \in W^{1,q'}_0(U_j)\Big\} \le C\|u_0\|_{L^s(U_j)}, \tag{2.9}$$

where $c = c(\alpha, \beta, K) > 0$ and $C = C(\alpha, \beta, K) > 0$. By (2.9) we conclude that

$$\begin{aligned}\|p - M_j\|_{L^q(U_j)} &\le c\|\nabla p\|_{W^{-1,q}(U_j)} \le c\big(\|u\|_{W^{-1,q}(U_j)} + \|u_0\|_{W^{-1,q}(U_j)}\big)\\ &\le C\big(\|u\|_{L^q(U_j)} + \|u_0\|_{L^s(U_j)}\big)\end{aligned} \tag{2.10}$$

with constants $c, C > 0$ depending only on $\alpha$, $\beta$, $K$.

Now Lemma 2.1 1. and (2.7) imply the estimate

$$\|\varphi_j u_0 - w_j\|_{L^q(U_j)} + \|\nabla\big(\varphi_j(p - M_j)\big)\|_{L^q(U_j)} \le c\|\varphi_j u + (\nabla\varphi_j)(p - M_j)\|_{L^q(U_j)},$$

which may be simplified by virtue of (2.8), (2.10) to the inequality

$$\|\varphi_j u_0\|_{L^q(U_j)} + \|\varphi_j \nabla p\|_{L^q(U_j)} \le C\big(\|u\|_{L^q(U_j)} + \|u_0\|_{L^s(U_j)}\big) \tag{2.11}$$

with constants $c, C > 0$ depending only on $\alpha$, $\beta$, $K$. Taking the $q$th power in (2.11), summing over $j = 1, \dots, N$ and exploiting the crucial property of the number $N_0$ we are led to the estimate

$$\begin{aligned}\|u_0\|^q_{L^q(\Omega)} + \|\nabla p\|^q_{L^q(\Omega)} &\le \int_\Omega \Big(\Big(\sum_j \varphi_j|u_0|\Big)^q + \Big(\sum_j \varphi_j|\nabla p|\Big)^q\Big)\,\mathrm{d}x\\ &\le \int_\Omega N_0^{\frac{q}{q'}}\Big(\sum_j |\varphi_j u_0|^q + \sum_j |\varphi_j \nabla p|^q\Big)\,\mathrm{d}x\\ &\le C N_0^{\frac{q}{q'}}\Big(\sum_j \|u\|^q_{L^q(U_j)} + \sum_j \|u_0\|^q_{L^s(U_j)}\Big).\end{aligned} \tag{2.12}$$

The last sum on the right-hand side may be estimated by the reverse Hölder inequality $\sum_j |a_j|^q \le \big(\sum_j |a_j|^s\big)^{q/s}$. Using again the property of the number $N_0$ and taking the $q$th root, (2.12) may be simplified to the estimate

$$\|u_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \le C\big(\|u\|_{L^q(\Omega)} + \|u_0\|_{L^s(\Omega)}\big) \tag{2.13}$$

where $C = C(\alpha, \beta, K) > 0$. To get rid of the term $\|u_0\|_{L^s(\Omega)}$ in the case when $s > 2$ we use the elementary interpolation inequality

$$\|u_0\|_{L^s(\Omega)} \le \alpha \left(\frac{1}{\varepsilon}\right)^{1/\alpha} \|u_0\|_{L^2(\Omega)} + (1-\alpha)\varepsilon^{1/(1-\alpha)}\|u_0\|_{L^q(\Omega)}, \quad \varepsilon > 0,$$

where $\alpha \in (0,1)$ is defined by $\frac{1}{s} = \frac{\alpha}{2} + \frac{1-\alpha}{q}$. Choosing $\varepsilon > 0$ sufficiently small, the new term $\|u_0\|_{L^q(\Omega)}$ on the right-hand side of (2.13) may be absorbed by the same term on the left-hand side so that (2.13) leads to the inequality

$$\|u_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \le C\big(\|u\|_{L^q(\Omega)} + \|u_0\|_{L^2(\Omega)}\big) \tag{2.14}$$

with $C = C(\alpha, \beta, K) > 0$. Finally we use the $L^2$-estimate (1.3) for the term $\|u_0\|_{L^2(\Omega)}$ and add (1.3) to (2.14). This proves the estimate

$$\|u_0\|_{L^q \cap L^2} + \|\nabla p\|_{L^q \cap L^2} \le C\|u\|_{L^q \cap L^2} \tag{2.15}$$

for every $q \ge 2$.

**2.3. The case $\Omega$ bounded,** $1 < q < 2$. For $u \in L^q + L^2$ there exist $u_1 \in L^q$, $u_2 \in L^2$ satisfying $u = u_1 + u_2$ and $\|u\|_{L^q+L^2} = \|u_1\|_{L^q} + \|u_2\|_{L^2}$. Define $u_0$ and $\nabla p$ by

$$u_0 = P_q u_1 + P_2 u_2 \in L^q_\sigma + L^2_\sigma, \quad \nabla p = (I - P_q)u_1 + (I - P_2)u_2 \in G^q + G^2$$

yielding $u = u_0 + \nabla p$. Then, using duality arguments and (2.15) for $q' > 2$,

$$\begin{aligned}
\|u_0\|_{L^q+L^2} &= \sup\Big\{\frac{|\langle P_q u_1 + P_2 u_2, v\rangle|}{\|v\|_{L^{q'} \cap L^2}} : \ 0 \ne v \in L^{q'} \cap L^2\Big\} \\
&= \sup\Big\{\frac{\|\langle u_1 + u_2, P_{q'} v\rangle|}{\|v\|_{L^{q'} \cap L^2}} : \ 0 \ne v \in L^{q'} \cap L^2\Big\} \\
&\le \sup\Big\{\frac{(\|u_1\|_q + \|u_2\|_2)\max\big(\|P_{q'} v\|_{q'}, \|P_2 v\|_2\big)}{\|v\|_{L^{q'} \cap L^2}} : \ 0 \ne v \in L^{q'} \cap L^2\Big\} \\
&\le C\|u\|_{L^q+L^2}
\end{aligned}$$

with the same constant $C = C(\alpha, \beta, K)$ as in (2.15) (with $q'$ instead of $q$). It follows that $\|u_0\|_{L^q+L^2} + \|\nabla p\|_{L^q+L^2} \le C\|u\|_{L^q+L^2}$, i.e., (1.4) for $q \in (1,2)$.

Summarizing both cases we proved the existence of a bounded linear projection $\tilde{P}_q$ on $\tilde{L}^q$ for a bounded domain $\Omega \subset \mathbb{R}^n$ of uniform $C^1$-type $(\alpha, \beta, K)$ such that $\tilde{P}_q u = P_q u$ for all $u \in \tilde{L}^q = L^q$. Moreover, $\nabla p = (I - \tilde{P}_q)u = (I - P_q)u \in \tilde{G}^q = G^q$. The crucial property of $\tilde{P}_q$ is the fact that its operator norm on $\tilde{L}^q$ is bounded by a constant $C = C(\alpha, \beta, K) > 0$. Finally, the assertion $\big(\tilde{P}_q\big)' = \tilde{P}_{q'}$ follows from standard duality arguments.

**2.4. The case $\Omega$ unbounded.** Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain of uniform $C^1$-type $(\alpha, \beta, K)$. Given $u \in \tilde{L}^q(\Omega)^n$, $1 < q < \infty$, define $u_k = u|_{\Omega_k}$, $k \in \mathbb{N}$, where $\Omega_k \subset \Omega$ is the bounded domain introduced in §2.1; note that $\Omega_k \subset \Omega$ again is of uniform $C^1$-type $(\alpha, \beta, K)$. Since obviously $u_k \in \tilde{L}^q(\Omega_k)^n$, there exists a unique Helmholtz decomposition $u_k = u_{k,0} + \nabla p_k$ with $u_{k,0} \in \tilde{L}^q_\sigma(\Omega_k)$, $\nabla p_k \in \tilde{G}^q(\Omega_k)$, satisfying the estimate

$$\|u_{k,0}\|_{\tilde{L}^q(\Omega_k)} + \|\nabla p_k\|_{\tilde{L}^q(\Omega_k)} \leq C\|u_k\|_{\tilde{L}^q(\Omega_k)} \leq C\|u\|_{\tilde{L}^q(\Omega)} \tag{2.16}$$

with a constant $C = C(\alpha, \beta, K)$ independent of $k \in \mathbb{N}$. Extending $u_{k,0}$ and $\nabla p_k$ by 0 from $\Omega_k$ to $\Omega$ we get bounded sequences in $\tilde{L}^q(\Omega)^n$. Since $\tilde{L}^q(\Omega)$ is reflexive, there exist – suppressing the notation of subsequences – weak limits

$$u_0 = (\mathrm{w-}) \lim_{k\to\infty} u_{k,0} \in \tilde{L}^q(\Omega)^n, \quad Q = (\mathrm{w-}) \lim_{k\to\infty} \nabla p_k \in \tilde{L}^q(\Omega)^n, \tag{2.17}$$

satisfying $u = u_0 + Q$ and the estimate $\|u_0\|_{\tilde{L}^q(\Omega)} + \|Q\|_{\tilde{L}^q(\Omega)} \leq C\|u\|_{\tilde{L}^q(\Omega)}$. Since $u_{k,0} \in \tilde{L}^q_\sigma(\Omega_k) \subset \tilde{L}^q_\sigma(\Omega)$ and since $\tilde{L}^q_\sigma(\Omega)$ is closed with respect to weak convergence, $u_0 \in \tilde{L}^q_\sigma(\Omega)$. Moreover, de Rham's argument, see [11], [12], implies that there exists $p \in L^1_{\mathrm{loc}}(\Omega)$ such that $Q = \nabla p \in \tilde{G}^q(\Omega)$. Hence the pair $(u_0, \nabla p)$ determines a Helmholtz decomposition of $u$ in $\tilde{L}^q(\Omega)^n$. The uniqueness of the Helmholtz decomposition is proved by a classical duality argument and the weak convergence properties (2.17). Now the existence of the Helmholtz projection $\tilde{P}_q$ on $\tilde{L}^q(\Omega)^n$ with range $\tilde{L}^q_\sigma(\Omega)$ and kernel $\tilde{G}^q(\Omega)$ is proved. Moreover, the assertion $\big(\tilde{P}_q\big)' = \tilde{P}_{q'}$ follows from standard duality arguments.

*Proof of Corollary* 1.3.

1. Note that obviously $\overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^q}} \subset \tilde{L}^q_\sigma(\Omega)$, $1 < q < \infty$. Now let $u = u_0 \in \tilde{L}^q_\sigma(\Omega)$. By the proof above, cf. (2.17), the sequence $(u_{k,0})$ converges weakly in $\tilde{L}^q(\Omega)^n$ towards $\tilde{P}_q u = u$. By Mazur's theorem there exists a sequence of convex combinations of the elements $(u_{k,0})$, say $(v_m)$, converging strongly in $\tilde{L}^q_\sigma(\Omega)$ to $u$. Each element $v_m$ has its support in some bounded domain $\Omega_{k(m)}$ yielding $v_m \in L^q_\sigma(\Omega_{k(m)})$. Since $C^\infty_{0,\sigma}(\Omega_{k(m)})$ is dense in $L^q_\sigma(\Omega_{k(m)})$ and since for a bounded domain the norms in $L^q$ and $\tilde{L}^q$ are equivalent, we conclude that $(v_m)$ converges to $u$ in $\tilde{L}^q_\sigma(\Omega)$ as $m \to \infty$; hence $u \in \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^q}}$.
The assertions $\big(\tilde{L}_q(\Omega)\big)' = \tilde{L}_{q'}(\Omega)$ and $\big(\tilde{P}_q\big)' = \tilde{P}_{q'}$ follow from standard duality arguments.

2., 3. All claims are easily proved by duality arguments. □

*Proof of Corollary* 1.4.

1. By Corollary 1.3 2., 3. both assertions are special cases of the following general result and of the reflexivity of the space $\tilde{L}^q$, $1 < q < \infty$:
Let $X_0$ be a Banach space with dual space $Y_0 = (X_0)'$ and let $X_1, X_2$ and $Y_1, Y_2$ be closed subspaces of $X_0$ and $Y_0$, respectively, such that

$$X_2 \subset X_1 \subset X_0, \quad Y_2 \subset Y_1 \subset Y_0, \quad X_2^\perp = Y_1, \quad X_1^\perp = Y_2.$$

Then

$$\big(X_1/X_2\big)' \cong Y_1/Y_2.$$

For the proof of this abstract result first consider arbitrary equivalence classes $\overline{y}_1 = y_1 + Y_2 \in Y_1/Y_2$ and $\overline{x}_1 = x_1 + X_2 \in X_1/X_2$. Then $\langle\langle \overline{y}_1, \overline{x}_1 \rangle\rangle := \langle y_1, x_1 \rangle$ is well-defined and defines an injective map $J$ from $Y_1/Y_2$ into $\big(X_1/X_2\big)'$. Next, given any

$f \in \big(X_1/X_2\big)'$, define $f_1 \in X_1'$ by $\langle f_1, x_1\rangle := \langle\langle f, \overline{x}_1\rangle\rangle$ and use Hahn-Banach's theorem to extend $f_1 \in X_1'$ to an element $f_0 \in X_0'$. Note that $f_0 \in Y_1$, but that the map $f \mapsto f_0$ is not necessarily linear. Then define $\overline{f} := f_0 + Y_2 \in Y_1/Y_2$. We note that the map $\big(X_1/X_2\big)' \to Y_1/Y_2,\ f \mapsto \overline{f}$, is linear (!) and bounded. Since it is easily seen that this map is the inverse of the map $J$ constructed in the first part of the proof, the isomorphism is found.

2. By THEOREM 1.2 $\tilde{L}^q_\sigma \cap \big(\tilde{\mathcal{L}}^q_\sigma \cap \tilde{G}^q\big) = \{0\}$. Each $u \in \tilde{\mathcal{L}}^q_\sigma$ has a unique decomposition $u = u_0 + \nabla p,\ u_0 \in \tilde{L}^q_\sigma, \nabla p \in \tilde{G}^q$. Then $\nabla p = u - u_0 \in \tilde{\mathcal{L}}^q_\sigma$ proving the algebraic decomposition of $\tilde{\mathcal{L}}^q_\sigma$ as stated. Moreover, by THEOREM 1.2, this decomposition is also a topological one. □

## REFERENCES


[1] J. Bergh, J. Löfström, *Interpolation Spaces,* Springer-Verlag, New York, 1976.

[2] M. E. Bogovski, *Decomposition of $L_p(\Omega, \mathbb{R}^n)$ into the direct sum of subspaces of solenoidal and potential vector fields,* Soviet Math. Dokl., **33** (1986), 161–165.

[3] R. Farwig, *The weak Neumann problem and the Helmholtz decomposition in general aperture domains,* in Progress in Partial Differential Equations: the Metz Surveys 2, M. Chipot, ed., Pitman Research Notes in Mathematics **296** (1993), 86–96.

[4] R. Farwig, *Weighted $L^q$–Helmholtz decompositions in infinite cylinders and in infinite layers,* Adv. Differential Equations **8** (2003), 357–384.

[5] R. Farwig and H., Sohr, *Helmholtz decomposition and Stokes resolvent system for aperture domains in $L^q$–spaces,* Analysis **16** (1996), 1–26.

[6] R. Farwig, H. Kozono and H. Sohr, *An $L^q$–approach to Stokes and Navier-Stokes equations in general domains,* Acta Math., **195** (2005), 21–53.

[7] D. Fujiwara and H. Morimoto, *An $L_r$–theorem of the Helmholtz decomposition of vector fields,* J. Fac. Sci. Univ. Tokyo, Sec. Math., **24** (1977), 685–700.

[8] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I, Linearized Steady Problems,* Springer-Verlag, New York, 1994.

[9] V. N. Maslennikova and M. E. Bogovski, *Elliptic boundary value problems in unbounded domains with noncompact and nonsmooth boundaries,* Rend. Sem. Mat. Fis. Milano, **LVI** (1986), 125–138.

[10] C. G. Simader and H. Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in $L^q$-spaces for bounded and exterior domains,* Ser. Adv. Math. Appl. Sci., **11**, Mathematical problems relating to the Navier-Stokes equations, World Scientific Publ., Singapore, 1992.

[11] H. Sohr, *The Navier-Stokes Equations,* Birkhäuser Verlag, Basel, 2001.

[12] R. Temam, *Navier-Stokes Equations,* North-Holland, Amsterdam, 1977.