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Proceedings of Equadiff-11 2005, pp. 77–85

THE HELMHOLTZ DECOMPOSITION IN ARBITRARY UNBOUNDED DOMAINS – A THEORY BEYOND L^2

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Abstract. It is well known that the usual L^q -theory of the Stokes operator valid for bounded or exterior domains cannot be extended to arbitrary unbounded domains $\Omega \subset \mathbb{R}^n$ when $q \neq 2$. One reason is given by the Helmholtz projection which fails to exist for certain unbounded smooth planar domains unless q = 2. However, as recently shown [6], the Helmholtz projection does exist for general unbounded domains in \mathbb{R}^3 if we replace the space L^q , $1 < q < \infty$, by $L^2 \cap L^q$ for q > 2 and by $L^q + L^2$ for 1 < q < 2. In this paper, we generalize this new approach from the three-dimensional case to the *n*-dimensional case, $n \geq 2$.

Key words. Helmholtz decomposition, Helmholtz projection, general unbounded domains, domains of uniform C^1 -type, intersection spaces, sum spaces

AMS subject classifications. 35Q30, 76D05

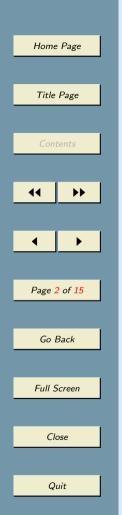
1. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and let $1 < q < \infty$. Then the classical Helmholtz projection P_q on $L^q(\Omega)^n$ defines a topological and algebraic decomposition of $L^q(\Omega)^n$ into the direct sum of the solenoidal subspace

$$L^{q}_{\sigma}(\Omega) = \overline{C^{\infty}_{0,\sigma}(\Omega)}^{\|\cdot\|_{q}} = \mathcal{R}(P_{q}),$$

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where $C_{0,\sigma}^{\infty}(\Omega) = \{ u \in C_0^{\infty}(\Omega)^n : \text{div } u = 0 \}$, and the space of gradients

$$G^q(\Omega) = \{ \nabla p \in L^q(\Omega)^n : p \in L^q_{\text{loc}}(\Omega) \} = \text{Ker}(P_q).$$

Hence every vector field $u \in L^q$ (here L^q stands for $L^q(\Omega)^n$) has a unique decomposition $u = u_0 + \nabla p$ where $u_0 = P_q u \in L^q_\sigma$ and

$$\|u_0\|_q + \|\nabla p\|_q \le c \|u\|_q \tag{1.1}$$

with a constant $c = c(q, \Omega) > 0$. The existence of P_q is well known for several classes of domains with boundary of class C^1 , namely for bounded domains, for exterior domains, aperture domains, layers, tubes, half spaces and perturbations of them, see e.g. [3], [4], [5], [7], [8], [10]. However, the decomposition

$$L^{q}(\Omega)^{n} = L^{q}_{\sigma}(\Omega) \oplus G^{q}(\Omega), \quad 1 < q < \infty,$$
(1.2)

no longer holds for infinite cones in \mathbb{R}^2 with "smoothed vertex" at the origin and of opening angle larger than π when $q \neq 2$, see [2], [9].

On the other hand, an L^2 -theory works for every bounded and unbounded domain without any assumptions on the boundary. Actually, the decomposition $u = u_0 + \nabla p$ can be found by solving the weak Neumann problem

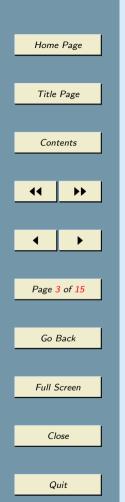
$$\Delta p = \operatorname{div} u \quad \text{in} \quad \Omega, \quad \frac{\partial p}{\partial N} = u \cdot N \quad \text{on} \quad \partial \Omega,$$

where N denotes the exterior normal unit vector on $\partial\Omega$; i.e., ∇p is determined in $G^2(\Omega)$ via the variational problem

$$(\nabla p, \nabla \psi) = (u, \nabla \psi) \text{ for all } \nabla \psi \in G^2(\Omega)$$

using the Lemma of Lax-Milgram. Obviously, $\|\nabla p\|_2 \leq \|u\|_2$ and $u_0 := u - \nabla p \perp \nabla p$ leading to the *a priori* estimate

$$||u_0||_2 + ||\nabla p||_2 \le 2||u||_2.$$
(1.3)



Note that the constant C = 2 in (1.3) is independent of the domain.

In a recent paper, the authors proved the existence of the Helmholtz projection for general unbounded domains $\Omega \subset \mathbb{R}^3$ of uniform C^2 -class (cf. DEFINITION 1.1 below) by replacing the space L^q by

$$\tilde{L}^{q}(\Omega) = \begin{cases} L^{q}(\Omega) \cap L^{2}(\Omega), & 2 \le q < \infty \\ L^{q}(\Omega) + L^{2}(\Omega), & 1 < q < 2 \end{cases}$$

.

We may extend this definition to general unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and equip $\tilde{L}^q(\Omega)$ with the norm $\|u\|_{\tilde{L}^q(\Omega)} = \max(\|u\|_q, \|u\|_2)$ if $q \geq 2$, and

$$\begin{aligned} \|u\|_{\tilde{L}^{q}(\Omega)} &= \inf \left\{ \|u_{1}\|_{q} + \|u_{2}\|_{2} : u = u_{1} + u_{2}, u_{1} \in L^{q}, u_{2} \in L^{2} \right\} \\ &= \sup \left\{ \frac{|\langle u_{1} + u_{2}, f \rangle|}{\|f\|_{L^{q'} \cap L^{2}}} : 0 \neq f \in L^{q'} \cap L^{2} \right\} \end{aligned}$$

if 1 < q < 2 and where q' = q/(q-1). Note that

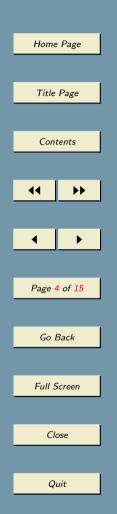
$$\left(\tilde{L}^q(\Omega)\right)' \cong \tilde{L}^{q'}(\Omega),$$

see [1]. By analogy, we define the spaces

$$\tilde{L}^q_{\sigma}(\Omega) = \begin{cases} L^q_{\sigma}(\Omega) \cap L^2_{\sigma}(\Omega), & 2 \le q < \infty \\ L^q_{\sigma}(\Omega) + L^2_{\sigma}(\Omega), & 1 < q < 2 \end{cases}, \quad \tilde{G}^q(\Omega) = \begin{cases} G^q(\Omega) \cap G^2(\Omega), & 2 \le q < \infty \\ G^q(\Omega) + G^2(\Omega), & 1 < q < 2 \end{cases}$$

For more properties of the intersection and sum of such compatible pairs of Banach spaces we refer to [6].

DEFINITION 1.1.A domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is called a uniform C^1 -domain of type (α, β, K) (where $\alpha > 0, \beta > 0, K > 0$) if for each $x_0 \in \partial \Omega$ we can choose a Cartesian coordinate



system with origin at x_0 and coordinates $y = (y', y_n), y' = (y_1, \ldots, y_{n-1})$, and a C^1 -function $h(y'), |y'| \leq \alpha$, with C^1 -norm $||h||_{C^1} \leq K$ such that the neighborhood

$$U_{\alpha,\beta,h}(x_0) := \{ (y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha \}$$

of x_0 satisfies

$$U_{\alpha,\beta,h}^{-}(x_0) := \{ (y', y_n) : h(y') - \beta < y_n < h(y'), |y'| < \alpha \} = \Omega \cap U_{\alpha,\beta,h}(x_0),$$

and

$$\partial \Omega \cap U_{\alpha,\beta,h}(x_0) = \{ (y', h(y')) : |y'| < \alpha \}.$$

Then our main theorem reads as follows:

THEOREM 1.2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform C^1 -domain of type (α, β, K) and let $q \in (1, \infty)$. Then each $u \in \tilde{L}^q(\Omega)$ has a unique decomposition

$$u = u_0 + \nabla p, \quad u_0 \in \tilde{L}^q_{\sigma}(\Omega), \, \nabla p \in \tilde{G}^q(\Omega),$$

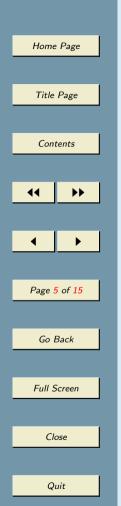
satisfying the estimate

$$\|u_0\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \le c \|u\|_{\tilde{L}^q}, \quad c = c(\alpha, \beta, K, q) > 0.$$
(1.4)

In particular, the Helmholtz projection \tilde{P}_q defined by $\tilde{P}_q u = u_0$ is a bounded linear projection on $\tilde{L}^q(\Omega)$ with range $\tilde{L}^q_{\sigma}(\Omega)$ and kernel $\tilde{G}^q(\Omega)$ and satisfies $(\tilde{P}_q)' = \tilde{P}_{q'}$.

COROLLARY 1.3. Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a uniform C^1 -domain of type (α, β, K) .

1.
$$\tilde{L}^{q}_{\sigma}(\Omega) = \overline{C^{\infty}_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^{q}}}$$



2. The following isomorphisms hold:

$$\left(\tilde{L}^{q}_{\sigma}(\Omega)\right)' \cong \tilde{L}^{q'}_{\sigma}(\Omega), \quad \left(\tilde{G}^{q}(\Omega)\right)' \cong \tilde{G}^{q'}(\Omega).$$

3. The annihilator identities

$$\left(\tilde{L}^{q}_{\sigma}(\Omega)\right)^{\perp} = \tilde{G}^{q'}(\Omega), \quad \left(\tilde{G}^{q}(\Omega)\right)^{\perp} = \tilde{L}^{q'}_{\sigma}(\Omega)$$

hold.

Besides the spaces \tilde{L}^q_{σ} and \tilde{G}^q we consider the spaces

$$\tilde{\mathcal{L}}^q_{\sigma}(\Omega) = \left\{ u \in \tilde{L}^q(\Omega)^n : \text{ div } u = 0 \text{ in } \Omega, \ u \cdot N = 0 \text{ on } \partial \Omega \right\}$$

and

$$\tilde{\mathcal{G}}^q(\Omega) = \overline{\nabla C_0^\infty(\overline{\Omega})}^{\|\cdot\|_{\tilde{L}^q}},$$

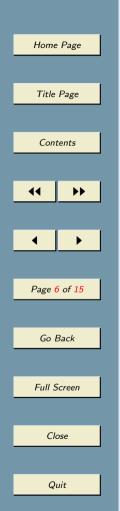
the closure in $\tilde{G}^q(\Omega)$ of its subspace $\nabla C_0^{\infty}(\overline{\Omega})$; here $\tilde{\mathcal{L}}_{\sigma}^q(\Omega)$ is defined in the sense of distributions, i.e., $\langle u, \nabla \varphi \rangle = 0$ for all $\varphi \in C_0^{\infty}(\overline{\Omega})$. Hence by definition

$$\tilde{\mathcal{L}}^q_{\sigma}(\Omega) = \tilde{\mathcal{G}}^{q'}(\Omega)^{\perp}$$

and, due to reflexivity, $\tilde{\mathcal{G}}^q(\Omega)^{\perp} = \tilde{\mathcal{L}}^{q'}_{\sigma}(\Omega)$.

As is well known, for bounded or exterior domains, see [10], $\tilde{\mathcal{L}}_{\sigma}^{q} = \tilde{L}_{\sigma}^{q}$ and $\tilde{\mathcal{G}}^{q} = \tilde{G}^{q}$. However, for an aperture domain, see [3], [5], [8], \tilde{L}_{σ}^{q} is a closed subspace of $\tilde{\mathcal{L}}_{\sigma}^{q}$ of codimension 1 if and only if q > n', and $\tilde{\mathcal{G}}^{q}$ is a closed subspace of \tilde{G}^{q} of codimension 1 if and only if 1 < q < n. In an arbitrary unbounded domain of uniform C^{1} -type the same phenomena may occur; moreover, the codimensions could equal an arbitrary positive integer or even infinity.

COROLLARY 1.4. Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a uniform C^1 -domain of type (α, β, K) .



1. The following isomorphisms hold:

 $\left(\tilde{\mathcal{L}}^{q}_{\sigma}(\Omega)/\tilde{\mathcal{L}}^{q}_{\sigma}(\Omega)\right)' \cong \tilde{G}^{q'}(\Omega)/\tilde{\mathcal{G}}^{q'}(\Omega), \quad \left(\tilde{G}^{q}(\Omega)/\tilde{\mathcal{G}}^{q}(\Omega)\right)' \cong \tilde{\mathcal{L}}^{q'}_{\sigma}(\Omega)/\tilde{\mathcal{L}}^{q'}_{\sigma}(\Omega).$

2. The space $\tilde{\mathcal{L}}^q_{\sigma}(\Omega)$ admits the following direct algebraic and topological decomposition: $\tilde{\mathcal{L}}^q_{\sigma}(\Omega) = \tilde{\mathcal{L}}^q_{\sigma}(\Omega) \oplus \left(\tilde{\mathcal{L}}^q_{\sigma}(\Omega) \cap \tilde{G}^q(\Omega)\right).$

By Corollary 1.4 (1) \tilde{L}_{σ}^{q} has a *finite* codimension in $\tilde{\mathcal{L}}_{\sigma}^{q}$ if and only if $\tilde{\mathcal{G}}^{q'}$ has a *finite* codimension in $\tilde{G}^{q'}$; in this case the codimensions coincide.

2. Proofs.

2.1. Preliminaries. Concerning DEFINITION 1.1 we introduce further notation and discuss some properties. Obviously, the axes e_i , i = 1, ..., n, of the new coordinate system (y', y_n) may be chosen in such a way that e_1, \ldots, e_{n-1} are tangential to $\partial\Omega$ at x_0 . Hence at y' = 0 we have h(y') = 0 and $\nabla' h(y') = 0$. Since $h \in C^1$, for any given constant $M_0 > 0$, we may choose $\alpha > 0$ sufficiently small such that $||h||_{C^1} \leq M_0$ is satisfied.

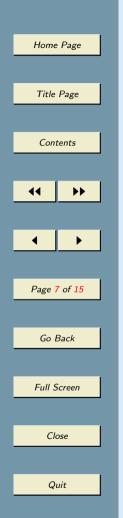
It is easily shown that there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ of fixed radius r > 0 with centers $x_j \in \overline{\Omega}$, such that with suitable functions $h_j \in C^1$ of type (α, β, K)

$$\overline{B}_j \subset U_{\alpha,\beta,h_j}(x_j) \text{ if } x_j \in \partial\Omega, \quad \overline{B}_j \subset \Omega \text{ if } x_j \in \Omega.$$

$$(2.1)$$

Here j runs from 1 to a finite number $N = N(\Omega) \in \mathbb{N}$ if Ω is bounded, and $j \in \mathbb{N}$ if Ω is unbounded. The covering $\{B_j\}$ of Ω may be constructed in such a way that not more than a fixed number $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these balls can have a nonempty intersection. Moreover, there exists a partition of unity $\{\varphi_j\}, \varphi_j \in C_0^{\infty}(\mathbb{R}^n)$, such that

$$0 \le \varphi_j \le 1$$
, $\operatorname{supp} \varphi_j \subset B_j$, and $\sum_{j=1}^N \varphi_j = 1$ or $\sum_{j=1}^\infty \varphi_j = 1$ on Ω . (2.2)



The functions φ_j may be chosen so that $|\nabla \varphi_j(x)| \leq C$ uniformly in j and $x \in \Omega$ with $C = C(\alpha, \beta, K)$.

If Ω is unbounded, then Ω can be represented as the union of an increasing sequence of bounded domains $\Omega_k \subset \Omega, k \in \mathbb{N}$,

$$\ldots \subset \Omega_k \subset \Omega_{k+1} \subset \ldots, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k, \tag{2.3}$$

each Ω_k is of the same type (α', β', K') . Without loss of generality we assume that $\alpha = \alpha', \beta = \beta', K = K'$.

Using the partition of unity $\{\varphi_j\}$ the construction of the Helmholtz decomposition will be based on well known results for certain bounded and unbounded domains. For this reason, we introduce for $h \in C_0^1(\mathbb{R}^{n-1})$ satisfying h(0) = 0, $\nabla' h(0) = 0$ and $\operatorname{supp} h \subset B'_r(0) \subset \mathbb{R}^{n-1}$, $0 < r = r(\alpha, \beta, K) < \alpha$, the bounded domain

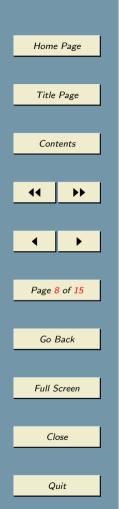
$$H = H_{\alpha,\beta,h;r} = \{ y \in \mathbb{R}^n : h(y') - \beta < y_n < h(y'), |y'| < \alpha \} \cap B_r(0) ;$$

here we assume that $\overline{B_r(0)} \subset \{y : |y_n - h(y')| < \beta, |y'| < \alpha\}.$

On H we consider the classical Sobolev spaces $W^{1,q}(H)$ and $W_0^{1,q}(H)$, the dual space $W^{-1,q}(H) = (W_0^{1,q'}(H))'$ and the space

$$L^q_0(H) = \left\{ u \in L^q(H) : \ \int_H u \ \mathrm{d}x = 0 \right\}$$

of L^q -functions with vanishing mean on H.



LEMMA 2.1. Let $1 < q < \infty$ and $H = H_{\alpha,\beta,h;r}$.

1. Assume that $\|\nabla' h\|_{\infty} \leq M_0$ for a sufficiently small constant $M_0 = M_0(q, n) > 0$, and let $u \in L^q(H)^n$ admit the Helmholtz decomposition $u = u_0 + \nabla p$ with $u_0 \in L^q_{\sigma}(H)$, $p \in W^{1,q}(H)$ and $\operatorname{supp} u_0$, $\operatorname{supp} p \subset B_r(0)$. Then there exists a constant $C = C(\alpha, \beta, K, q) > 0$ such that

$$||u_0||_q + ||\nabla p||_q \le C||u||_q.$$
(2.4)

2. There exists a bounded linear operator

 $R: L^q_0(H) \to W^{1,q}_0(H)^n$

such that div $\circ R = id$ on $L^q_0(H)$ and a constant $C = C(\alpha, \beta, K, q) > 0$ such that

$$||Rf||_{W^{1,q}} \le C ||f||_q \quad for \ all \quad f \in L^q_0(H).$$
(2.5)

3. There exists
$$C = C(\alpha, \beta, K, q) > 0$$
 such that for every $p \in L^q_0(H)$

$$\|p\|_{q} \le C \|\nabla p\|_{W^{-1,q}} = C \sup \Big\{ \frac{|\langle p, \operatorname{div} v \rangle|}{\|\nabla v\|_{q'}} : \ 0 \neq v \in W_{0}^{1,q'}(H) \Big\}.$$
(2.6)

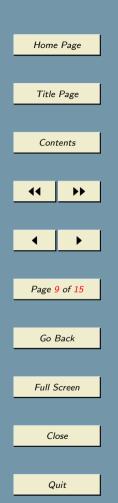
Proof.

1. Since supp u_0 , supp $p \subset B_r(0)$ and since h has compact support, the decomposition $u = u_0 + \nabla p$ on H may be considered as a Helmholtz decomposition in the bent half space

$$H_h = \{ y \in \mathbb{R}^n : y_n < h(y'), y' \in \mathbb{R}^{n-1} \}.$$

Then [10, Lemma 3.8 a)] yields (2.4) provided that $\|\nabla' h\|_{\infty} \leq M_0$ is sufficiently small.

2. It is well known that there exists a bounded linear operator $R: L_0^q(H) \to W_0^{1,q}(H)^n$ such that u = Rf solves the divergence problem div u = f. Moreover, the estimate (2.5) holds with $C = C(\alpha, \beta, K, q) > 0$, see [8, III, Theorem 3.1].



3. The dual map $R': W^{-1,q}(H)^n \to L_0^q(H)$ of the map R in 2., replacing q by q', is continuous with bound $C = C(\alpha, \beta, K, q) > 0$. Given $p \in L_0^q(H)$, we get that $\nabla p \in W^{-1,q}(H)^n$ using the definition $\langle \nabla p, v \rangle = -(p, \operatorname{div})$ for $v \in W_0^{1,q'}(H)$. Then for all $f \in L_0^{q'}(H)$,

$$\langle f, R'(\nabla p) \rangle = \langle Rf, \nabla p \rangle = -(\operatorname{div} Rf, p) = -(f, p).$$

Hence $R'(\nabla p) = -p$, yielding (2.6).

2.2. The case Ω bounded, $q \geq 2$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded uniform C^1 -domain of type (α, β, K) . Then each $u \in L^q(\Omega)^n$, $2 \leq q < \infty$, has a unique decomposition $u = u_0 + \nabla p$, $u_0 \in L^q_{\sigma}(\Omega)$, $\nabla p \in G^q(\Omega)$, satisfying (1.1) with constant $c = c(q, \Omega) > 0$ depending somehow on Ω , see [7], [10].

Given the partition of unity $\{\varphi_j\}_{j=1}^N$, the balls B_j and the sets $U_{\alpha,\beta,h_j}(x_j), U_{\alpha,\beta,h_j}^-(x_j)$, see DEFINITION 1.1 and Subsection 2.1, we define the sets

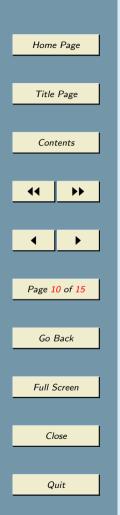
$$U_j = U_{\alpha,\beta,h_j}^-(x_j) \cap B_j$$
 if $x_j \in \partial \Omega$ and $U_j = B_j$ if $x_j \in \Omega$.

 $1 \leq j \leq N$. We may assume that in both cases LEMMA 2.1 applies to the domain $H = U_j$ (in LEMMA 2.1 1. the smallness assumption is satisfied if $x_j \in \partial \Omega$, whereas the case $x_j \in \Omega$ is related to the Helmholtz decomposition in the whole space). Moreover, at most $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these sets will have a nonempty intersection. Multiplying $u = u_0 + \nabla p$ with φ_j we get that

$$\varphi_j u = \varphi_j u_0 + \nabla \big(\varphi_j (p - M_j) \big) - (\nabla \varphi_j) (p - M_j)$$

where $M_j = \frac{1}{|U_j|} \int_{U_j} p \, dx$ yielding $p - M_j \in L^q_0(U_j)$. Moreover, using the operator $R = R_j$ in U_j , see LEMMA 2.1 (2), we find $w_j = R_j(u_0 \cdot \nabla \varphi_j) \in W^{1,q}_0(U_j)$ such that div $w_j = u_0 \cdot \nabla \varphi_j$ in U_j and $\varphi_j u_0 - w_j \in L^q_\sigma(U_j)$. Then

$$\varphi_j u + (\nabla \varphi_j)(p - M_j) - w_j = (\varphi_j u_0 - w_j) + \nabla \big(\varphi_j (p - M_j)\big)$$
(2.7)



is the Helmholtz decomposition of the left-hand side $\varphi_j u + (\nabla \varphi_j)(p - M_j) - w_j$ in U_j . To estimate $\varphi_j u$ and $\varphi_j \nabla p$ let $s := \max\left(\frac{nq}{n+q}, 2\right) \in [2, q), s' = s/(s-1)$. Then the Sobolev embeddings $W_0^{1,s}(U_j) \hookrightarrow L^q(U_j)$ and $W_0^{1,q'}(U_j) \hookrightarrow L^{s'}(U_j)$ hold with embedding constants depending on α, β, K and q, s only. Hence, by LEMMA 2.1 2. (with q replaced by s)

$$\|w_j\|_{L^q(U_j)} \le c \|w_j\|_{W^{1,s}(U_j)} \le C \|u_0\|_{L^s(U_j)},$$
(2.8)

and by LEMMA 2.1 3.

$$\|u_0\|_{W^{-1,q}(U_j)} = \sup\left\{\frac{|(u_0,v)|}{\|\nabla v\|_{L^{q'}(U_j)}}: \ 0 \neq v \in W^{1,q'}_0(U_j)\right\} \le C \|u_0\|_{L^s(U_j)},$$
(2.9)

where $c = c(\alpha, \beta, K) > 0$ and $C = C(\alpha, \beta, K) > 0$. By (2.9) we conclude that

$$\|p - M_j\|_{L^q(U_j)} \le c \|\nabla p\|_{W^{-1,q}(U_j)} \le c (\|u\|_{W^{-1,q}(U_j)} + \|u_0\|_{W^{-1,q}(U_j)})$$

$$\le C (\|u\|_{L^q(U_j)} + \|u_0\|_{L^s(U_j)})$$
(2.10)

with constants c, C > 0 depending only on α , β , K. Now LEMMA 2.1 1. and (2.7) imply the estimate

$$\|\varphi_{j}u_{0} - w_{j}\|_{L^{q}(U_{j})} + \|\nabla(\varphi_{j}(p - M_{j}))\|_{L^{q}(U_{j})} \le c\|\varphi_{j}u + (\nabla\varphi_{j})(p - M_{j})\|_{L^{q}(U_{j})},$$

which may be simplified by virtue of (2.8), (2.10) to the inequality

$$\|\varphi_{j}u_{0}\|_{L^{q}(U_{j})} + \|\varphi_{j}\nabla p\|_{L^{q}(U_{j})} \le C\left(\|u\|_{L^{q}(U_{j})} + \|u_{0}\|_{L^{s}(U_{j})}\right)$$
(2.11)

with constants c, C > 0 depending only on α, β, K . Taking the *q*th power in (2.11), summing over $j = 1, \ldots, N$ and exploiting the crucial property of the number N_0 we are led to the

estimate

$$\begin{aligned} \|u_0\|_{L^q(\Omega)}^q + \|\nabla p\|_{L^q(\Omega)}^q &\leq \int_{\Omega} \left(\left(\sum_j \varphi_j |u_0|\right)^q + \left(\sum_j \varphi_j |\nabla p|\right)^q \right) \,\mathrm{d}x \\ &\leq \int_{\Omega} N_0^{\frac{q}{q'}} \left(\sum_j |\varphi_j u_0|^q + \sum_j |\varphi_j \nabla p|^q \right) \,\mathrm{d}x \\ &\leq C N_0^{\frac{q}{q'}} \left(\sum_j \|u\|_{L^q(U_j)}^q + \sum_j \|u_0\|_{L^s(U_j)}^q \right). \end{aligned}$$
(2.12)

The last sum on the right-hand side may be estimated by the reverse Hölder inequality $\sum_{j} |a_{j}|^{q} \leq \left(\sum_{j} |a_{j}|^{s}\right)^{q/s}$. Using again the property of the number N_{0} and taking the *q*th root, (2.12) may be simplified to the estimate

$$\|u_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \le C(\|u\|_{L^q(\Omega)} + \|u_0\|_{L^s(\Omega)})$$
(2.13)

where $C = C(\alpha, \beta, K) > 0$. To get rid of the term $||u_0||_{L^s(\Omega)}$ in the case when s > 2 we use the elementary interpolation inequality

$$\|u_0\|_{L^s(\Omega)} \le \alpha \left(\frac{1}{\varepsilon}\right)^{1/\alpha} \|u_0\|_{L^2(\Omega)} + (1-\alpha)\varepsilon^{1/(1-\alpha)} \|u_0\|_{L^q(\Omega)}, \quad \varepsilon > 0,$$

where $\alpha \in (0,1)$ is defined by $\frac{1}{s} = \frac{\alpha}{2} + \frac{1-\alpha}{q}$. Choosing $\varepsilon > 0$ sufficiently small, the new term $||u_0||_{L^q(\Omega)}$ on the right-hand side of (2.13) may be absorbed by the same term on the left-hand side so that (2.13) leads to the inequality

$$\|u_0\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \le C(\|u\|_{L^q(\Omega)} + \|u_0\|_{L^2(\Omega)})$$
(2.14)

with $C = C(\alpha, \beta, K) > 0$. Finally we use the L^2 -estimate (1.3) for the term $||u_0||_{L^2(\Omega)}$ and add (1.3) to (2.14). This proves the estimate

$$\|u_0\|_{L^q \cap L^2} + \|\nabla p\|_{L^q \cap L^2} \le C \|u\|_{L^q \cap L^2}$$
(2.15)

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for every $q \geq 2$.

 $\|$

2.3. The case Ω bounded, 1 < q < 2. For $u \in L^q + L^2$ there exist $u_1 \in L^q$, $u_2 \in L^2$ satisfying $u = u_1 + u_2$ and $||u||_{L^q + L^2} = ||u_1||_{L^q} + ||u_2||_{L^2}$. Define u_0 and ∇p by

$$u_0 = P_q u_1 + P_2 u_2 \in L^q_\sigma + L^2_\sigma, \quad \nabla p = (I - P_q) u_1 + (I - P_2) u_2 \in G^q + G^2$$

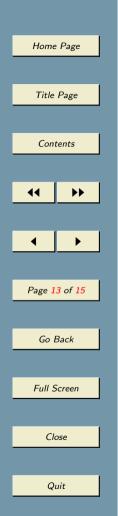
yielding $u = u_0 + \nabla p$. Then, using duality arguments and (2.15) for q' > 2,

$$\begin{split} u_0 \|_{L^q + L^2} &= \sup \left\{ \frac{|\langle P_q u_1 + P_2 u_2, v \rangle|}{\|v\|_{L^{q'} \cap L^2}} : \ 0 \neq v \in L^{q'} \cap L^2 \right\} \\ &= \sup \left\{ \frac{\|\langle u_1 + u_2, P_{q'} v \rangle|}{\|v\|_{L^{q'} \cap L^2}} : \ 0 \neq v \in L^{q'} \cap L^2 \right\} \\ &\leq \sup \left\{ \frac{(\|u_1\|_q + \|u_2\|_2) \max\left(\|P_{q'} v\|_{q'}, \|P_2 v\|_2\right)}{\|v\|_{L^{q'} \cap L^2}} : \ 0 \neq v \in L^{q'} \cap L^2 \right\} \\ &\leq C \|u\|_{L^q + L^2} \end{split}$$

with the same constant $C = C(\alpha, \beta, K)$ as in (2.15) (with q' instead of q). It follows that $||u_0||_{L^q+L^2} + ||\nabla p||_{L^q+L^2} \le C||u||_{L^q+L^2}$, i.e., (1.4) for $q \in (1, 2)$.

Summarizing both cases we proved the existence of a bounded linear projection \tilde{P}_q on \tilde{L}^q for a bounded domain $\Omega \subset \mathbb{R}^n$ of uniform C^1 -type (α, β, K) such that $\tilde{P}_q u = P_q u$ for all $u \in \tilde{L}^q = L^q$. Moreover, $\nabla p = (I - \tilde{P}_q)u = (I - P_q)u \in \tilde{G}^q = G^q$. The crucial property of \tilde{P}_q is the fact that its operator norm on \tilde{L}^q is bounded by a constant $C = C(\alpha, \beta, K) > 0$. Finally, the assertion $(\tilde{P}_q)' = \tilde{P}_{q'}$ follows from standard duality arguments.

2.4. The case Ω unbounded. Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain of uniform C^1 -type (α, β, K) . Given $u \in \tilde{L}^q(\Omega)^n$, $1 < q < \infty$, define $u_k = u|_{\Omega_k}$, $k \in \mathbb{N}$, where $\Omega_k \subset \Omega$ is the bounded domain introduced in §2.1; note that $\Omega_k \subset \Omega$ again is of uniform C^1 -type



 (α, β, K) . Since obviously $u_k \in \tilde{L}^q(\Omega_k)^n$, there exists a unique Helmholtz decomposition $u_k = u_{k,0} + \nabla p_k$ with $u_{k,0} \in \tilde{L}^q_\sigma(\Omega_k)$, $\nabla p_k \in \tilde{G}^q(\Omega_k)$, satisfying the estimate

$$\|u_{k,0}\|_{\tilde{L}^{q}(\Omega_{k})} + \|\nabla p_{k}\|_{\tilde{L}^{q}(\Omega_{k})} \le C\|u_{k}\|_{\tilde{L}^{q}(\Omega_{k})} \le C\|u\|_{\tilde{L}^{q}(\Omega)}$$
(2.16)

with a constant $C = C(\alpha, \beta, K)$ independent of $k \in \mathbb{N}$. Extending $u_{k,0}$ and ∇p_k by 0 from Ω_k to Ω we get bounded sequences in $\tilde{L}^q(\Omega)^n$. Since $\tilde{L}^q(\Omega)$ is reflexive, there exist – suppressing the notation of subsequences – weak limits

$$u_0 = (\mathbf{w}) \lim_{k \to \infty} u_{k,0} \in \tilde{L}^q(\Omega)^n, \quad Q = (\mathbf{w}) \lim_{k \to \infty} \nabla p_k \in \tilde{L}^q(\Omega)^n, \tag{2.17}$$

satisfying $u = u_0 + Q$ and the estimate $||u_0||_{\tilde{L}^q(\Omega)} + ||Q||_{\tilde{L}^q(\Omega)} \leq C||u||_{\tilde{L}^q(\Omega)}$. Since $u_{k,0} \in \tilde{L}^q_{\sigma}(\Omega_k) \subset \tilde{L}^q_{\sigma}(\Omega)$ and since $\tilde{L}^q_{\sigma}(\Omega)$ is closed with respect to weak convergence, $u_0 \in \tilde{L}^q_{\sigma}(\Omega)$. Moreover, de Rham's argument, see [11], [12], implies that there exists $p \in L^1_{loc}(\Omega)$ such that $Q = \nabla p \in \tilde{G}^q(\Omega)$. Hence the pair $(u_0, \nabla p)$ determines a Helmholtz decomposition of u in $\tilde{L}^q(\Omega)^n$. The uniqueness of the Helmholtz decomposition is proved by a classical duality argument and the weak convergence properties (2.17). Now the existence of the Helmholtz projection \tilde{P}_q on $\tilde{L}^q(\Omega)^n$ with range $\tilde{L}^q_{\sigma}(\Omega)$ and kernel $\tilde{G}^q(\Omega)$ is proved. Moreover, the assertion $(\tilde{P}_q)' = \tilde{P}_{q'}$ follows from standard duality arguments.

Proof of Corollary 1.3.

1. Note that obviously $\overline{C_{0,\sigma}^{\infty}(\Omega)}^{\|\cdot\|_{\tilde{L}^q}} \subset \tilde{L}_{\sigma}^q(\Omega), \ 1 < q < \infty$. Now let $u = u_0 \in \tilde{L}_{\sigma}^q(\Omega)$. By the proof above, cf. (2.17), the sequence $(u_{k,0})$ converges weakly in $\tilde{L}^q(\Omega)^n$ towards $\tilde{P}_q u = u$. By Mazur's theorem there exists a sequence of convex combinations of the elements $(u_{k,0})$, say (v_m) , converging strongly in $\tilde{L}_{\sigma}^q(\Omega)$ to u. Each element v_m has its support in some bounded domain $\Omega_{k(m)}$ yielding $v_m \in L_{\sigma}^q(\Omega_{k(m)})$. Since $C_{0,\sigma}^{\infty}(\Omega_{k(m)})$ is dense in $L_{\sigma}^q(\Omega_{k(m)})$ and since for a bounded domain the norms in L^q and \tilde{L}^q are equivalent, we conclude that (v_m) converges to u in $\tilde{L}_{\sigma}^q(\Omega)$ as $m \to \infty$; hence $u \in \overline{C_{0,\sigma}^{\infty}(\Omega)}^{\|\cdot\|_{\tilde{L}^q}}$. The assertions $(\tilde{L}_q(\Omega))' = \tilde{L}_{q'}(\Omega)$ and $(\tilde{P}_q)' = \tilde{P}_{q'}$ follow from standard duality arguments.

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2., 3. All claims are easily proved by duality arguments.

Proof of Corollary 1.4.

1. By Corollary 1.3 2., 3. both assertions are special cases of the following general result and of the reflexivity of the space \tilde{L}^q , $1 < q < \infty$:

Let X_0 be a Banach space with dual space $Y_0 = (X_0)'$ and let X_1, X_2 and Y_1, Y_2 be closed subspaces of X_0 and Y_0 , respectively, such that

$$X_2 \subset X_1 \subset X_0, \quad Y_2 \subset Y_1 \subset Y_0, \quad X_2^{\perp} = Y_1, \quad X_1^{\perp} = Y_2.$$

Then

 $\left(X_1/X_2\right)' \cong Y_1/Y_2.$

For the proof of this abstract result first consider arbitrary equivalence classes $\overline{y}_1 = y_1 + Y_2 \in Y_1/Y_2$ and $\overline{x}_1 = x_1 + X_2 \in X_1/X_2$. Then $\langle \langle \overline{y}_1, \overline{x}_1 \rangle \rangle := \langle y_1, x_1 \rangle$ is well-defined and defines an injective map J from Y_1/Y_2 into $(X_1/X_2)'$. Next, given any $f \in (X_1/X_2)'$, define $f_1 \in X'_1$ by $\langle f_1, x_1 \rangle := \langle \langle f, \overline{x}_1 \rangle \rangle$ and use Hahn-Banach's theorem to extend $f_1 \in X'_1$ to an element $f_0 \in X'_0$. Note that $f_0 \in Y_1$, but that the map $f \mapsto f_0$ is not necessarily linear. Then define $\overline{f} := f_0 + Y_2 \in Y_1/Y_2$. We note that the map $(X_1/X_2)' \to Y_1/Y_2$, $f \mapsto \overline{f}$, is linear (!) and bounded. Since it is easily seen that this map is the inverse of the map J constructed in the first part of the proof, the isomorphism is found.

2. By THEOREM 1.2 $\tilde{L}_{\sigma}^{q} \cap (\tilde{\mathcal{L}}_{\sigma}^{q} \cap \tilde{G}^{q}) = \{0\}$. Each $u \in \tilde{\mathcal{L}}_{\sigma}^{q}$ has a unique decomposition $u = u_0 + \nabla p, u_0 \in \tilde{\mathcal{L}}_{\sigma}^{q}, \nabla p \in \tilde{G}^{q}$. Then $\nabla p = u - u_0 \in \tilde{\mathcal{L}}_{\sigma}^{q}$ proving the algebraic decomposition of $\tilde{\mathcal{L}}_{\sigma}^{q}$ as stated. Moreover, by THEOREM 1.2, this decomposition is also a topological one.

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