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CARATHÉODORY SOLUTIONS TO QUASI-LINEAR HYPERBOLIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAYS

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Abstract. The paper (based on the article [3] under the same title) addresses the existence of a generalized solution and continuous dependence upon initial data for hyperbolic functional differential systems with state dependent delays. The method used in this paper is based on the bicharacteristics theory and on the Banach fixed point theorem. The formulation includes retarded argument, integral and hereditary Volterra terms.

Key words. Hyperbolic systems, Equations with state dependent delays, Generalized Solutions.

AMS subject classifications. 15A15, 15A09, 15A23

Let $C(U,V)$ be the class of all continuous functions defined on U and taking values in V. Let $\mathbb{R}_+ = [0,+\infty)$, $B = [-b_0, 0] \times [-b, b]$, $E_a = [-b_0, a] \times \mathbb{R}^n$, where $b_0 \in \mathbb{R}_+, b \in \mathbb{R}^n_+$, $a \geq 0$, $M_{n\times k}$ means the set of all $n \times k$ real matrices, and let $\Omega = [0,a] \times \mathbb{R}^n \times C(B, \mathbb{R}^k)$. Assume that

$$A : \Omega \to M_{k\times k}, \qquad A = [A_{ij}]_{i,j=1,\dots,k},$$

$$\rho : \Omega \to M_{k\times n}, \qquad \rho = [\rho_{ij}]_{i=1,\dots,k, j=1,\dots,n},$$

$$f : \Omega \to M_{k\times 1}, \qquad f = (f_1, \dots, f_k)^T,$$

and

$$z_{(t,x)} : B \to \mathbb{R}^k, \qquad z_{(t,x)}(\tau, y) = z(t+\tau, x+y),$$

where $(\tau, y) \in B$. The symbol

$$z_{\zeta(t,x,z_{(t,x)})}$$

means the restriction of function z to the set

$$[\zeta_0(t) - b_0, \zeta_0(t)] \times [\zeta_\star(t,x,z_{(t,x)}) - b, \zeta_\star(t,x,z_{(t,x)}) + b],$$

where

$$\zeta(t,x,w) = (\zeta_0(t), \zeta_\star(t,x,w)), \qquad \zeta_0 : [0,a] \to \mathbb{R}, \qquad \zeta_\star : \Omega \to \mathbb{R}^n,$$

and this restriction is shifted to the set B. Let

$$\psi(t,x,w) = (\psi_0(t), \psi_\star(t,x,w)),$$

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and

$$\theta(t,x,w) = (\theta_0(t), \theta_\star(t,x,w)).$$

We will consider the system

$$\sum_{j=1}^{k} A_{ij}(t,x,z_{\theta(t,x,z_{(t,x)})})\left[D_t z_j(t,x) + \sum_{l=1}^{n} \rho_{il}(t,x,z_{\psi(t,x,z_{(t,x)})}) D_{x_l} z_j(t,x)\right] = f_i(t,x,z_{\psi(t,x,z_{(t,x)})}), \tag{1}$$

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0, \tag{2}$$

where φ is a given initial function and the symbol D_t means the partial derivative $\frac{\partial}{\partial t}$.

The function

$$z \in C(E_c, \mathbb{R}^k), \qquad c \in (0,a],$$

is a solution of (1), (2), if

(i) derivatives $D_t z_i$, $D_x z_i$, $i = 1,\dots,k$, exist almost everywhere on $[0,c] \times \mathbb{R}^n$;

(ii) z satisfies (1) almost everywhere on $[0,c] \times \mathbb{R}^n$;

(iii) condition (2) holds.

We define different norms:

$$\|U\| = \max\left\{\sum_{j=1}^{n} |u_{ij}| : 1 \le i \le k\right\},$$

where

$$U \in M_{k\times n}, \quad u_i = (u_{i1},\dots,u_{in}), \quad i = 1,\dots,k,$$

and

$$\|\eta\| = \max\{|\eta_i| : 1 \le i \le k\},$$

where

$$\eta \in R^n, \quad \eta = (\eta_1,\dots,\eta_k).$$

Let $C_{*.L}(B,\mathbb{R}^k)$ be the class of all functions $w \in C(B,\mathbb{R}^k)$, such that
$\|w\|_L = \sup\{\|w(t,r) - w(\bar t,\bar r)\|(|t-\bar t| + \|r-\bar r\|)^{-1} :\ (t,r),(\bar t,\bar r) \in B, t \ne \bar t, x \ne \bar x\} < +\infty.$
For $w \in C(B,\mathbb{R}^k)$ we denote by $\|w\|_*$ the supremum norm of w. We define

$$\|w\|_{*.L} = \|w\|_* + \|w\|_L, \qquad w \in C_{*.L}(B,\mathbb{R}^k).$$

We write

$$C(B,\mathbb{R}^k;\kappa) = \left\{w \in C(B,\mathbb{R}^k) : \|w\|_* \le \kappa\right\},$$

$$C_{*.L}(B,\mathbb{R}^k;\kappa) = \left\{w \in C_{*.L}(B,\mathbb{R}^k) : \|w\|_{*.L} \le \kappa\right\}.$$

where $\kappa \in \mathbb{R}_+$. We put

$$\|z\|_t = \sup\{\|z(\tau,y)\| : (\tau,y) \in (0,t] \times \mathbb{R}^n\},$$

where $z \in C(E_c,\mathbb{R}^k)$, $t \in [0,c]$, and $c \in (0,a]$.

We denote by $J[\lambda]$ the set of all functions $\phi \in C(E_0, \mathbb{R}^n)$, such that

(i) $\|\phi(t,x)\| \le \lambda_0 \quad \text{for} \quad (t,x) \in E_0$;

(ii) $\|\phi(t,x) - \phi(\bar{t},\bar{x})\| \le \lambda_1|t-\bar{t}| + \lambda_2\|x-\bar{x}\| \quad \text{on} \quad E_0$,

where $\lambda_0,\, \lambda_1,\, \lambda_2 \in \mathbb{R}_+$, and $\lambda = \lambda_0 + \lambda_1 + \lambda_2$.

Let $K_{\varphi.c}[d]$, where $\varphi \in J[\lambda]$, and $c \in (0,a]$, be the class of all functions $z \in C(E_c, \mathbb{R}^k)$, such that

(i) $z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0$;

(ii) for (t,x), $(\bar{t},\bar{x}) \in [0,c] \times \mathbb{R}^n$ we have

$$\|z(t,x)\| \le d_0,$$
$$\|z(t,x) - z(\bar{t},\bar{x})\| \le d_1|t-\bar{t}| + d_2\|x-\bar{x}\|,$$

where $d_i \in \mathbb{R}_+$, and $d_i \ge \lambda_i$ for $i = 0,\, 1,\, 2$, and $d = d_0 + d_1 + d_2$.

We denote by $\mathcal{P}$ the set of all nondecreasing functions $\gamma \in C(\mathbb{R}_+, \mathbb{R}_+)$, such that $\gamma(0) = 0$.

Assumption $H[\rho]$**.** Suppose that

(i) $\rho(\cdot,x,w) : [0,a] \to M_{k\times n}$ is measurable for $(x,w) \in \mathbb{R}^n \times C(B,\mathbb{R}^k)$, and $\rho(t,\cdot) : \mathbb{R}^n \times C(B,\mathbb{R}^k) \to M_{k\times n}$ is continuous for almost all $t \in [0,a]$;

(ii) there exist $\alpha_0, \alpha_1 \in \mathcal{P}$, such that

$$\|\rho(t,x,w)\| \le \alpha_0(\kappa)$$
$$\|\rho(t,x,w) - \rho(t,\bar{x},\bar{w})\| \le \alpha_1(\kappa)\Big[\|x-\bar{x}\| + \|w-\bar{w}\|_*\Big]$$

for (x,w), $(\bar{x},\bar{w}) \in \mathbb{R}^n \times C(B,\mathbb{R}^k;\kappa)$, $t \in [0,a]$.

Assumption $H[\psi]$**.** Suppose that

(i) $\psi_\star(\cdot,x,w) : [0,a] \to \mathbb{R}^n$ is measurable for $(x,w) \in \mathbb{R}^n \times C(B,\mathbb{R}^k)$, and $\psi_\star(t,\cdot) : \mathbb{R}^n \times C(B,\mathbb{R}^k) \to \mathbb{R}^n$ is continuous for almost all $t \in [0,a]$;

(ii) $\psi_0 \in L([0,a],\mathbb{R})$, $-b_0 \le \psi_0(t) - t \le 0$ for almost all $t \in [0,a]$, and there exists $\beta \in \mathcal{P}$, such that

$$\|\psi_\star(t,x,w) - \psi_\star(t,\bar{x},\bar{w})\| \le \beta(\kappa)\Big[\|x-\bar{x}\| + \|w-\bar{w}\|_*\Big]$$

for (x,w), $(\bar{x},\bar{w}) \in \mathbb{R}^n \times C_{0.L}(B,\mathbb{R}^k;\kappa)$ almost everywhere on $[0,a]$.

Let $\varphi \in J[\lambda]$, $c \in (0,a]$, and $z \in K_{\varphi.c}[d]$. Consider the Cauchy problem

$$\eta'(\tau) = \rho_i(\tau,\eta(\tau),z_{\psi(\tau,\eta(\tau),z_{(\tau,\eta(\tau))})}), \qquad \eta(t) = x, \tag{3}$$

where $(t,x) \in [0,c] \times \mathbb{R}^n$, $i = 1,\dots,k$.

Let $g_i[z](\cdot,t,x)$ be the Carathéodory solution of (3). The function $g_i[z]$ is the i-th bicharacteristic of system (1) corresponding to $z \in K_{\varphi.c}[d]$.

LEMMA 1 (proved in [3]). *Suppose that Assumptions $H[\rho]$, $H[\psi]$ are satisfied, and $\varphi, \bar{\varphi} \in J[\lambda]$, $z \in K_{\varphi.c}[d]$, $\bar{z} \in K_{\bar{\varphi}.c}[d]$, where $c \in (0,a]$, Then for $i = 1,\ldots,k$ bicharacteristics $g_i[z](\cdot,t,x)$, and $g_i[\bar{z}](\cdot,t,x)$ are defined on $[0,c]$, and they are unique. Moreover we have the estimates*

$$\|g_i[z](\tau,t,x) - g_i[z](\tau,\bar{t},\bar{x})\| \le \Lambda(t,\tau)\left[\alpha_0(d_0)|t-\bar{t}| + \|x-\bar{x}\|\right] \tag{4}$$

for $(t,x),(\bar{t},\bar{x}) \in [0,c]\times\mathbb{R}^n$, $\tau \in [0,c]$,

$$\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \le \alpha_1(d_0)\delta_1(d)\Lambda(t,\tau)\left|\int_t^\tau \|z-\bar{z}\|_s\,\mathrm{d}s\right| \tag{5}$$

for $(\tau,t,x) \in [0,c]\times[0,c]\times\mathbb{R}^n$, where

$$\delta_1(d) = 1 + d_2\beta(d), \quad \delta_2(d) = 1 + d_2\beta(d) + d_2^2\beta(d),$$
$$\Lambda(t,\tau) = \exp(\alpha_1(d_0)\delta_2(d)|t-\tau|)\,.$$

Assumption $H[A,\theta]$. Suppose that

(i) $A \in C(\Omega, M_{k\times k})$, and there is $\nu > 0$, such that $\det A(t,x,w) \ge \nu$ for $(t,x,w) \in \Omega$;

(ii) the following estimates hold

$$\|A(t,x,w)\| \le \alpha_0(\kappa)$$
$$\|A(t,x,w) - A(t,\bar{x},\bar{w})\| \le \alpha_1(\kappa)\left[\|x-\bar{x}\| + \|w-\bar{w}\|_*\right],$$

for (x,w), $(\bar{x},\bar{w}) \in \mathbb{R}^n \times C(B,\mathbb{R}^k;\kappa)$ $t \in [0,a]$;

(iii) there exists $\bar{\beta} \in \mathcal{P}$, such that

$$\|\theta(t,x,w) - \theta(\bar{t},\bar{x},\bar{w})\| \le \bar{\beta}(\kappa)\left[|t-\bar{t}| + \|x-\bar{x}\| + \|w-\bar{w}\|_*\right],$$

for (x,w), $(\bar{x},\bar{w}) \in \mathbb{R}^n \times C_{*.L}(B,\mathbb{R}^k;\kappa)$ $t \in [0,a]$ and

$$-b_0 \le \theta_0(t) - t \le 0 \quad \text{on} \quad [0,a].$$

Assumption $H[f]$**.** Suppose that

(i) $f(\cdot,x,w) : [0,a] \to \mathbb{R}^k$ is measurable on $\mathbb{R}^n \times C(B,\mathbb{R}^k)$, and $f(t,\cdot) : \mathbb{R}^n \times C(B,\mathbb{R}^k) \to \mathbb{R}^k$ is continuous for almost all $t \in [0,a]$;

(ii) the following estimations hold

$$\|f(t,x,w)\| \le \alpha_0(\kappa),$$
$$\|f(t,x,w) - f(t,\bar{x},\bar{w})\| \le \alpha_1(\kappa)\left[\|x-\bar{x}\| + \|w-\bar{w}\|_*\right]$$

for (x,w), $(\bar{x},\bar{w}) \in \mathbb{R}^n \times C(B,\mathbb{R}^k;\kappa)$, $t \in [0,a]$.

We denote by $U * V$ the vector

$$\omega = \Big(\omega_1, \dots, \omega_n\Big),$$

as follows

$$\omega_i = \sum_{j=1}^{k} u_{ij} v_{ji}, \quad i = 1, \dots, k,$$

where $U = [u_{ij}]_{i,j=1,\dots,k} \in M_{k\times k}$, $V = [v_{ij}]_{i,j=1,\dots,k} \in M_{k\times k}$.

Suppose that $\varphi \in J[\lambda]$, $c \in [0, a]$, $z \in K_{\varphi.c}[d]$, and $g_i[z]$ is a solution of (3). We denote

$$\begin{aligned}
A[g,z](s,t,x) &= \Big[A_{ij}(s, g_i[z](s,t,x), z_{\theta(s,g_i[z](s,t,x),z_{(s,g_i[z](s,t,x))})})\Big]_{i,j=1,\dots,k},\\
\varphi[g](s,t,x) &= \Big[\varphi_i(0, g_j[z](s,t,x))\Big]_{i,j=1,\dots,k},\\
Z[g,z](s,t,x) &= \Big[z_i(s, g_j[z](s,t,x))\Big]_{i,j=1,\dots,k},\\
f[g,z](s,t,x) &= \Big(f_i(s, g_i[z](s,t,x), z_{\psi(s,g_i[z](s,t,x),z_{(s,g_i[z](s,t,x))})})\Big)^T_{i=1,\dots,k}.
\end{aligned}$$

Assumption $H[d]$. Suppose that
(i) $d_0 > \lambda_0$,
(ii) $d_1 > \alpha_0(d_0)\bar{\alpha}_1(d_0)\,[1 + \lambda_2\alpha_0(d_0)\Lambda(c,0)]$, where $c \in (0, a]$,
(iii) $d_2 > \lambda_2\,[1 + \bar{\alpha}_1(d_0)\alpha_0(d_0)]$.

For $z \in K_{\varphi.c}[d]$ we define the operator

$$\begin{aligned}
F[z](t,x) = {} & A^{-1}(t,x,z_{\theta(t,x,z_{(t,x)})})\,\{A[g,z](0,t,x) * \varphi[g](0,t,x)\}\\
&+ A^{-1}(t,x,z_{\theta(t,x,z_{(t,x)})}) \int_0^t \{D_\tau A[g,z](\tau,t,x) * Z[g,z](\tau,t,x)\\
&+ f[g,z](\tau,t,x)\}\,\mathrm{d}\tau,
\end{aligned} \tag{6}$$

for $t \in [0,c] \times \mathbb{R}^n$, $c \in (0,a]$, and

$$F[z](t,x) = \varphi(t,x) \quad \text{on} \quad E_0. \tag{7}$$

Lemma 2 (proved in [3]). *Suppose that Assumptions $H[\rho]$, $H[\psi]$, $H[f]$, $H[A,\theta]$, $H[d]$ are satisfied and φ, $\bar{\varphi} \in J[\lambda]$. Then there exists $c \in (0,a]$, such that*

$$F : K_{\varphi.c}[d] \to K_{\varphi.c}[d].$$

Theorem 3 (proved in [3]). *Suppose that Assumptions $H[\rho]$, $H[\psi]$, $H[f]$, $H[A,\theta]$, $H[d]$ are satisfied. Then for each $\varphi \in J[\lambda]$ there exists $c \in (0,a]$, such that problem* (1), (2) *has a solution $u \in K_{\varphi.c}[d]$, and this solution is unique in the class $K_{\varphi.c}[d]$. If $\bar{\varphi} \in J[\lambda]$,* *and if $\bar{u}$ is a solution of system* (1) *with the initial condition $z(t,x)=\bar{\varphi}(t,x)$ on E_0, then there exists $M_c \in \mathbb{R}_+$, such that*

$$\|u-\bar{u}\|_t \le M_c \|\varphi-\bar{\varphi}\|_* \qquad \text{on} \quad [0,c]. \tag{8}$$

Some special cases. Now we list below some examples of systems which can be derived from system (1). Assume that

$$\begin{aligned} &\tilde{A}:\tilde{\Omega}\to M_{k\times k}, && \tilde{A}=[\tilde{A}_{ij}]_{i,j=1,\dots,k},\\ &\tilde{\rho}:\tilde{\Omega}\to M_{k\times n}, && \tilde{\rho}=[\tilde{\rho}_{ij}]_{i=1,\dots,k,j=1,\dots,n},\\ &\tilde{f}:\tilde{\Omega}\to M_{k\times 1}, && \tilde{f}=(\tilde{f}_1,\dots,\tilde{f}_k)^T, \end{aligned}$$

where $\tilde{\Omega}=[0,a]\times\mathbb{R}^n\times\mathbb{R}^k$.

Case I. Consider the operators

$$\begin{aligned} &A:\Omega\to M_{k\times k}, && A=[A_{ij}]_{i,j=1,\dots,k},\\ &\rho:\Omega\to M_{k\times n}, && \rho=[\rho_{ij}]_{i=1,\dots,k,j=1,\dots,n},\\ &f:\Omega\to M_{k\times 1}, && f=(f_1,\dots,f_k)^T, \end{aligned}$$

given by formulas

$$A(t,x,w)=\tilde{A}(t,x,w(0,0)),$$

$$\rho(t,x,w)=\tilde{\rho}(t,x,w(0,0)), \quad f(t,x,w)=\tilde{f}(t,x,w(0,0)).$$

Let

$$\psi(t,x,w)=\tilde{\psi}(t,x,w(0,0)), \quad \theta(t,x,w)=\tilde{\theta}(t,x,w(0,0)),$$

and

$$\tilde{\theta}(t,x,w(0,0))=\tilde{\psi}(t,x,w(0,0))=(\gamma(t),\Phi(t,x,)),$$

where $\gamma:[0,a]\to\mathbb{R}$, $\Phi:[0,a]\times\mathbb{R}^n\to\mathbb{R}^n$, and

$$-b_0 \le \gamma(t)-t \le 0.$$

Then system (1) reduces to the system

$$\begin{aligned} &\sum_{j=1}^{k} A_{ij}(t,x,z(\gamma(t),\Phi(t,x)))\\ &\cdot\left[D_t z_j(t,x)+\sum_{l=1}^{n}\rho_{il}(t,x,z(\gamma(t),\Phi(t,x)))D_{x_l}z_j(t,x)\right]=f_i(t,x,z(\gamma(t),\Phi(t,x))), \end{aligned} \tag{9}$$

where $i=1,\dots,k$.

Case II. Suppose that functions A, ρ, f are given by following formulas

$$A(t,x,w)=\tilde{A}(t,x,\int_B w(\tau,y)\,\mathrm{d}\tau),$$

$$\rho(t,x,w)=\tilde{\rho}(t,x,\int_B w(\tau,y)\,\mathrm{d}\tau),$$

and

$$f(t,x,w)=\tilde f(t,x,\int_B w(\tau,y)\,\mathrm{d}\tau).$$

Put

$$\begin{aligned}\Gamma_\psi[t,x,z]=\big\{(\tau,y)\in\mathbb{R}^{1+n}:\ &\psi_0(t)-b_0\le\tau\le\psi_0(t),\\ &\psi_\star(t,x,z_{(t,x)})-b\le y\le\psi_\star(t,x,z_{(t,x)})+b\big\},\end{aligned}$$

and

$$\begin{aligned}\Gamma_\theta[t,x,z]=\big\{(\tau,y)\in\mathbb{R}^{1+n}:\ &\theta_0(t)-b_0\le\tau\le\theta_0(t),\\ &\theta_\star(t,x,z_{(t,x)})-b\le y\le\theta_\star(t,x,z_{(t,x)})+b\big\}.\end{aligned}$$

Then system (1) reduces to the differential-integral system

$$\begin{aligned}&\sum_{j=1}^{k}\tilde A_{ij}(t,x,\int_{\Gamma_\theta[t,x,z]}z(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y)\\ &\quad\cdot\left[D_tz_j(t,x)+\sum_{l=1}^{n}\tilde\rho_{il}(t,x,\int_{\Gamma_\psi[t,x,z]}z(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y)D_{x_l}z_j(t,x)\right]\\ &=\tilde f_i(t,x,\int_{\Gamma_\psi[t,x,z]}z(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}y),\end{aligned}$$

where $i=1,\dots,k$.

There are a lot of papers concerning the theory of solutions of equations (1) and some particular cases of these equations for given functions A, ρ, f, ψ, and θ (see for example [1]–[5]).

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