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INTERIOR REGULARITY FOR WEAK SOLUTIONS OF NONLINEAR SECOND ORDER ELLIPTIC SYSTEMS*

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Abstract. Let $div(\mathcal{A}(Du)) = 0$ be a nonlinear elliptic system with C^1-matrix of coefficients. In our contribution we study the regularity of a weak solution belonging to $W^{2,1}(\Omega)$, where Ω is bounded domain in R^n, $n \geq 3$. For $d > 0$ denote Ω_d any subdomain of Ω with Lipschitz boundary such that $dist(x, \partial\Omega) > 2d$ for all $x \in \Omega_d$. We formulate the conditions connecting $||Du||_{L^2(\Omega)}$, coefficient of ellipticity ν, upper bound of derivatives of coefficients M and their modulus of continuity ω, guaranteeing $Du \in C^{0,\alpha}(\Omega_d)$.

Key words. Nonlinear elliptic systems, weak solutions, regularity

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1. Introduction. Let $\Omega \subset \mathcal{R}^n, n \geq 3$ be a bounded domain. Consider the system

$$\mathrm{div}(\mathcal{A}(Du)) = 0 \tag{1.1}$$

The detailed form of (1.1) sounds like $D_\alpha(\mathcal{A}_i^\alpha(Du)) = 0$, $i = 1, 2, \ldots, N$, where Einstein summation convention is used for $\alpha \in \{1, 2, \ldots, n\}$.

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Suppose that the matrix $\mathcal{A} = \{\mathcal{A}_i^\alpha\}$ belongs to C^1. *Regularity* of the weak solution $u \in W^{2,1}(\Omega)$ of (1.1) on $\Omega^* \subset\subset \Omega$ is defined as Hölder continuity of Du on $\bar{\Omega}^*$.

Denote

$$A_{ij}^{\alpha\beta}(p) = \frac{\partial \mathcal{A}_i^\alpha(p)}{\partial p_j^\beta}, \quad i,j = 1,\dots,N, \quad \alpha,\beta = 1,\dots,n \tag{1.2}$$

and write $A(p) = \{A_{ij}^{\alpha\beta}(p)\}$ with $|A(p)|$ for its Euclidean norm.

We suppose

$$\exists M > 0 \ \forall p \in \mathcal{R}^{nN} : \quad |A(p)| \le M, \tag{1.3}$$

$$\exists \nu > 0 \ \forall p \in \mathcal{R}^{nN} \ \forall \xi \in \mathcal{R}^{nN} : \quad (A(p)\xi, \xi) \ge \nu \xi|^2, \tag{1.4}$$

There is a function $\omega : \langle 0,\infty) \to \langle 0,\infty)$, $\omega(0) = 0$, ω nondecreasing, continuous, concave and bounded, such that

$$\forall p,q \in \mathcal{R}^{nN} : \quad |A(p) - A(q)| \le \omega(|p-q|). \tag{1.5}$$

Let further for $d > 0$ Ω_d be any subdomain of Ω with Lipschitz boundary such that $\operatorname{dist}(x, \partial\Omega) > 2d$ for all $x \in \Omega_d$.

In what follows we establish for fixed d the conditions on ν, M, ω and $\|Du\|_{L^2(\Omega)}$ guaranteeing that Du is Hölder continuous on $\bar{\Omega}_d$.

2. Basic estimate for weak solution. Algebraic lemma. In this section we prepare needed estimates and formulate an algebraic lemma following the procedure known from the deduction of partial regularity results. (See e.g. Giaquinta [1])

Let u be a weak solution of (1.1), $x \in \Omega_d, R \in (0,d\rangle$. Denote $A_0 = A((Du)_R)$, $\tilde{A} = \int_0^1 [A((Du)_R) - A((Du)_R + t(Du - (Du)_R)]\,\mathrm{d}t$, where $(Du)_R = \frac{1}{|B_R|}\int_{B_R} Du(x)\,\mathrm{d}x$, $B_R = \{y \in \mathcal{R}^n\,;\, |y - x| < R\}$.

Using this notation we can rewrite (1.1) as

$$\mathrm{div}(A_0 Du) = \mathrm{div}[(A_0 - \tilde{A})(Du - (Du)_R)] \qquad \text{on } B_R \tag{2.1}$$

Split $u = v + w$ in a way that

$$\mathrm{div}(A_0 Dv) = 0 \qquad \text{on } B_R,\ v - u \in W_0^{2,1}(B_R), \tag{2.2}$$

$$\mathrm{div}(A_0 Dw) = \mathrm{div}[(A_0 - \tilde{A}(Du - (Du)_R)], \qquad w \in W_0^{2,1}(B_R). \tag{2.3}$$

On the function v we can use Campanato's lemma saying that

$$\exists C > 0\ \forall \rho \in (0, R) : \int_{B_\rho} |Dv - (Dv)_\rho|^2 \le C(\frac{\rho}{R})^{n+2} \int_{B_R} |Dv - (Dv)_R|^2. \tag{2.4}$$

As for w, we can use it in a weak formulation of (2.3) as a test function. By means of ellipticity condition (1.4), Hölder inequality and the estimate (1.5) of $A_0 - \tilde{A}$ by the modulus of continuity ω we obtain

$$\nu^2 \int_{B_R} |Dw|^2 \le \int_{B_R} \omega^2(|Du - (Du)_R|) \cdot |Du - (Du)_R|^2. \tag{2.5}$$

REMARK 1.In our notation we suppress the dependence on $x \in \Omega_d$ so that instead of writing $B_R(x)$ we write B_R etc. This simplification does not make any harm because (2.4), (2.5) work in Ω_d uniformly.

Using (2.4), (2.5) and taking account in the fact that $u = v + w$ we obtain finally the estimate for u. With use of the notation

$$\Phi(\rho) = \int_{B_\rho} |Du - (Du)_\rho|^2 \tag{2.6}$$

it reads as

$$\exists C, D \text{ (depending on M}/\nu) \ \forall x \in \Omega_d \ \forall \rho : \ 0 < \rho < R \le d \tag{2.7}$$

$$\Phi(\rho) \le C \left(\frac{\rho}{R}\right)^{n+2} \Phi(R) + \frac{D}{\nu^2} \int_{B_R} \omega^2(|Du - (Du)_R|) \cdot |Du - (Du)_R|^2. \tag{2.8}$$

For the function Φ denote $U(R) = \frac{1}{R^n}\Phi(R)$.

Lemma 2.1 (Algebraic lemma). *Let $A > 0, d > 0, \beta > 0$ and $\delta \in (n, n+2)$ be given. There exist ε_0, $C^* > 0$ such that for each nonnegative nonincreasing function Φ defined on $\langle 0, 2d)$ satisfying the estimate*

$$\Phi(\rho) \le \left(A \left(\frac{\rho}{R}\right)^{n+2} + \mathcal{B}_1 + \mathcal{B}_2 U(2R) \right) \Phi(2R), \quad 0 < \rho < R \le d \tag{2.9}$$

with $\mathcal{B}_1 < \varepsilon_0$, $\mathcal{B}_2 U^\beta(2R) < \varepsilon_0$ it holds

$$\Phi(\rho) \le C^* \rho^\delta, \qquad 0 < \rho < d. \tag{2.10}$$

If – in accordance with this lemma – we are able to estimate

$$\frac{D}{\nu^2} \int_{B_R} \omega^2(|Du - (Du)_R|) \cdot |Du - (Du)_R|^2 \le (\mathcal{B}_1 + \mathcal{B}_2 U^\beta(2R))\Phi(2R) \tag{2.11}$$

for $0 < R < d$ with $\mathcal{B}_1$, $\mathcal{B}_2 U^\beta(2d)$ sufficiently small, we obtain (2.10) with the function Φ given in (2.6). It can be rewritten as

$$\frac{1}{\rho^\delta} \int_{B_\rho} |Du - (Du)_\rho|^2 \le C, \ x \in \Omega_d, \qquad \rho \in \langle 0, d). \tag{2.12}$$

As this estimate is uniform with respect to x in Ω_d, we conclude that Du belongs to the Campanato space $\mathcal{L}_{2,\delta}(\Omega_d)$. From the theorem of isomorphism between Campanato and Hölder spaces (see e.g. Kufner, John, Fučík [2]) we can conclude that $Du \in C^{0,(\delta-n)/2}(\bar{\Omega}_d)$.

3. Deduction of estimate (2.11). Denote

$$I = \int_{B_R} \omega^2(|Du - (Du)_R|) \cdot |Du - (Du)_R|^2 \tag{3.1}$$

With use of Young inequality for the couple of Young functions

$$\Theta(t) = \frac{t^p}{p}, \quad \Psi(s) = \frac{s^q}{q}, \qquad \text{where } 1 < p \le \frac{n}{n-2}, \quad q = \frac{p}{p-1} \tag{3.2}$$

we can write for any positive ε

$$\begin{aligned} I &\le \int_{B_R} \Theta(\varepsilon|Du - (Du)_R|^2) + \int_{B_R} \Psi(\frac{1}{\varepsilon}\omega^2(|Du - (Du)_R|) \\ &= \frac{\varepsilon^p}{p} \int_{B_R} |Du - (Du)_R|^{2p} + \frac{p-1}{p} \varepsilon^{\frac{p}{1-p}} \int_{B_R} \omega^{\frac{2p}{p-1}}(|Du - (Du)_R|) \\ &= \frac{\varepsilon^p}{p} I_1 + \frac{p-1}{p} \varepsilon^{\frac{2p}{p-1}} I_2. \end{aligned} \tag{3.3}$$

(We followed here the idea of the paper [3], where Young functions of different kind were used.)

Estimate now the first integral. Using successively Hölder inequality, Sobolev embedding theorem and Caccioppoli inequality we obtain

$$\begin{aligned} I_1 &\le c \left(\int_{B_R} |Du - (Du)_R|^{\frac{2n}{n-2}} \right)^{\frac{p(n-2)}{n}} \cdot R^{n(1-p)+2p} \\ &\le c \left(\int_{B_R} |D^2 u| \right)^p \cdot R^{n(1-p)+2p} \\ &\le c \left(\frac{1}{R^2} \int_{B_{2R}} |Du - (Du)_{2R}|^2 \right)^p \cdot R^{n(1-p)+2p} \\ &= c\Phi(2R) \cdot U^{p-1}(2R). \end{aligned} \tag{3.4}$$

As for the second integral I_2, denoting

$$E_t = \{\ x \in B_R;\ |Du(x) - (Du)_R| > t\ \}$$

we have

$$\begin{aligned} I_2 &= \int_{B_R} \omega^{\frac{2p}{p-1}}(|Du - (Du)_R|)\ \mathrm{d}x = \int_0^{+\infty} \frac{d}{\mathrm{d}t}[\omega^{\frac{2p}{p-1}}(t)] \cdot |E_t|\,\mathrm{d}t \\ &\le \sup_{t>0} \frac{d}{\mathrm{d}t}[\omega^{\frac{2p}{p-1}}(t)] \int_{B_R} |Du - (Du)_R| \end{aligned} \tag{3.5}$$

In the last inequality denote

$$\omega_p = \sup_{t>0} \frac{\mathrm{d}}{\mathrm{d}t}[\omega^{\frac{2p}{p-1}}(t)] \tag{3.6}$$

and use Hölder inequality once again. So we get (in case of $U(2R) \neq 0$)

$$I_2 \le c\ \omega_p\ (\int_{B_R} |Du - (Du)_R|^2)^{\frac{1}{2}}\ R^{\frac{n}{2}} \le c\ \omega_p\ \Phi(2R)U^{-\frac{1}{2}}(2R) \tag{3.7}$$

From (3.4), (3.7) and (3.3) we get

$$I \le c\left(\frac{1}{p}\ \varepsilon^p\ U^{p-1}(2R) + \left(1 - \frac{1}{p}\right)\ \varepsilon^{\frac{p}{1-p}}\ \omega_p\ U^{-\frac{1}{2}}(2R)\right)\ \Phi(2R) \tag{3.8}$$

The optimal choice of ε which minimizes the expression on the right hand side of (3.8) is

$$\varepsilon = \omega_p^{\frac{p-1}{p^2}}\ U^{\frac{(1-2p)(p-1)}{2p^2}}(2R).$$

Using this, we have finaly

$$I \leq c\, \omega_p^{\frac{p-1}{p}}\; U^{\frac{p-1}{2p}}(2R)\; \Phi(2R). \tag{3.9}$$

(In case of $U(2R) = 0$ this estimate is trivial.)

Coming back to (2.11) we put there $\mathcal{B}_1 = 0$ and take care of the smallness of the product

$$\frac{1}{\nu^2}\, \omega_p^{\frac{p-1}{p}}\; U^{\frac{p-1}{2p}}(2R). \tag{3.10}$$

From this and from the fact that the expression $U(2d)$ can be estimated by $||Du||^2_{L^2(\Omega)}/d^n$ we conclude that the algebraic lemma can be applied if

$$\frac{1}{\nu^2}\, \omega_p^{\frac{p-1}{p}} \left(\frac{\|Du\|^2_{L^2(\Omega)}}{d^n} \right)^{\frac{p-1}{2p}} \tag{3.11}$$

is sufficiently small.

Remark 2.Coming back to (2.8) we can see that in the case of $\tilde{\omega} = \sup_{t>0} \omega^2(t)$ sufficiently small we can derive the estimate of the type (2.11)which does not depend on d so that we obtain the usual interior regularity in Ω. On the other hand, it is easy to construct an example of modulus of continuity ω for which $\tilde{\omega}$ is big and $\omega_p^{\frac{p-1}{p}}$ is small (for some $p \in (1, \frac{n}{n-2}\rangle)$.

Let $p = \frac{n}{n-2}$, $T > 0, m > 0$. Define $\omega(t) = mt^{\frac{1}{n}}$ for $t \in \langle 0, T\rangle$ and $mT^{\frac{1}{n}}$ for $t > T$. For this choice of ω we calculate $\tilde{\omega} = m^2 T^{\frac{2}{n}}$, meanwhile $\omega^{\frac{1}{n}}_{\frac{n}{n-2}} = m^2$.

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