

## ASYMPTOTIC EQUIVALENCE OF SYSTEMS OF DIFFERENCE EQUATIONS

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**Abstract.** The relation between sets of solutions of the first order linear system of difference equations and of its perturbation is studied. Asymptotic equivalence is proved using Tychonoff fixed point theorem.

**Key words.** System of difference equations, asymptotic equivalence, Tychonoff fixed point theorem.

**AMS subject classifications.** 39A11

Consider the system of difference equations

$$\Delta x_n = Ax_n + f(n, x_n), \quad (1)$$

where  $A$  is a  $k \times k$  matrix,  $k \in \mathbb{N}$ ,  $x_n \in \mathbb{R}^k$ ,  $n \in \mathbb{N}_0$  and  $f: \mathbb{N}_0 \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ ,  $f(n, x)$  is continuous in  $x$  for any  $n \in \mathbb{N}_0$  and  $\Delta$  denotes the forward difference operator, i.e.,  $\Delta x_n = x_{n+1} - x_n$ . Along with (1) we also consider the corresponding linear system

$$\Delta y_n = Ay_n. \quad (2)$$

Here  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

In the sequel let us denote  $\mathbb{N}(a) = \{a, a+1, \dots\}$  for  $a \in \mathbb{N}_0$  and  $\mathbf{x} = \{x_n\}$ ,  $\mathbf{y} = \{y_n\}$ ,  $\mathbf{z} = \{z_n\}$ . Further, let us denote  $|\cdot|$  a norm on  $\mathbb{R}^k$  or  $\mathbb{R}^{k \times k}$  such that  $|Dx| \leq |D| \cdot |x|$  for any matrix  $D$  and any colon  $x$ .

In this contribution asymptotic relationship between solutions of the systems (1) and (2) will be investigated. The approach is inspired by the result for systems of ordinary differential equations studied in [8]. A similar topic for difference systems can be found in [4, 5, 7].

**DEFINITION 1.** The systems (1) and (2) are said to be *asymptotically equivalent* if to each solution  $\mathbf{x}$ ,  $n \in \mathbb{N}(a)$ , of (1) there exists a solution  $\mathbf{y}$ ,  $n \in \mathbb{N}(b)$ , of (2) such that

$$\lim_{n \rightarrow \infty} |x_n - y_n| = 0 \quad (3)$$

and conversely to each solution  $\mathbf{y}$ ,  $n \in \mathbb{N}(a)$ , of (2) there exists a solution  $\mathbf{x}$ ,  $n \in \mathbb{N}(b)$ , of (1) such that (3) holds.

If (3) holds only for some subsets of all solutions of (1) and (2), we will speak about asymptotic equivalence between these sets.

First we will examine the special case of (1) – a non-homogeneous linear system

$$\Delta z_n = Az_n + b_n, \quad (4)$$

where  $b_n \in \mathbb{R}^k$ ,  $n \in \mathbb{N}_0$ .

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We will suppose that the matrix  $A + I = B$  is nonsingular, i.e.,  $\det B \neq 0$ . This guarantees that each solution of (2) and (4) can be extended on  $\mathbb{N}_0$ . Denote  $Y_n$ ,  $n \in \mathbb{N}_0$ ,  $Y_0 = I$ , the fundamental matrix of (2). Thus  $\Delta Y_n = AY_n$  or equivalently  $Y_{n+1} = (A + I)Y_n$ , from which we get  $Y_n = (A + I)^n = B^n$ ,  $n \in \mathbb{N}_0$ .

As every solution  $z_n$  of (4) can be expressed like  $z_n = y_n + \widehat{z}_n$ , where  $\widehat{z}_n$  is a fixed partial solution of (4) and  $y_n$  is an appropriate solution of (2), the proof of the next theorem is evident.

**THEOREM 1.** *The systems (2) and (4) are asymptotically equivalent if and only if the system (4) possesses a solution  $z_n$  such that  $\lim_{n \rightarrow \infty} z_n = 0$ .*

Let us remind the variation of constant formula for (4). We look for a solution of (4) in the form  $z_n = Y_n c_n$ , where  $c_n \in \mathbb{R}^k$  is an appropriate sequence. After substituting it to (4) we get

$$\begin{aligned} \Delta Y_n c_n + Y_{n+1} \Delta c_n &= AY_n c_n + b_n, \\ c_{n+1} &= c_n + Y_{n+1}^{-1} b_n, \end{aligned}$$

and therefore

$$c_n = c_0 + \sum_{i=1}^n Y_i^{-1} b_{i-1}.$$

Choosing  $c_0 = 0$  we obtain that the system (4) has the solution

$$z_n = Y_n c_n = Y_n \sum_{i=1}^n Y_i^{-1} b_{i-1} = \sum_{i=1}^n B^{n-i} b_{i-1}. \quad (5)$$

If the series  $\sum_{i=1}^{\infty} Y_i^{-1} b_{i-1}$  converges, then it is possible to adjust  $z_n$ :

$$z_n = Y_n \left( \sum_{i=1}^{\infty} Y_i^{-1} b_{i-1} - \sum_{i=n+1}^{\infty} Y_i^{-1} b_{i-1} \right) = Y_n \sum_{i=1}^{\infty} Y_i^{-1} b_{i-1} - Y_n \sum_{i=n+1}^{\infty} Y_i^{-1} b_{i-1}.$$

As  $Y_n \sum_{i=1}^{\infty} Y_i^{-1} b_{i-1}$  is a solution of (2), we obtain that (4) has also a solution

$$z_n = -Y_n \sum_{i=n+1}^{\infty} Y_i^{-1} b_{i-1} = -\sum_{i=n+1}^{\infty} B^{n-i} b_{i-1}.$$

**ASSUMPTION.** Without loss of generality we can suppose that  $B$  has the Jordan canonical form.

Denote

$$(0 <) \mu_1 < \mu_2 < \cdots < \mu_s = \lambda$$

different absolute values of eigenvalues  $\lambda_i(B)$ ,  $i = 1, \dots, t$ . Let  $m_i$  be a maximal order of blocks that correspond to the eigenvalues with the absolute value  $\mu_i$ . Let  $m = m_s$ . Further let us put

$$p = \begin{cases} m_j & \text{if } \mu_j = 1, \\ 1 & \text{if no } \mu_j \text{ equals } 1. \end{cases}$$

Assume  $B = \text{diag}(B_1, B_2)$ , where

$$\begin{aligned} |\lambda_j(B_1)| &\leq \alpha = \max_j |\lambda_j(B_1)| < 1, & m^* &= m_i \text{ if } \mu_i = \alpha, \\ |\lambda_j(B_2)| &\geq 1 & \text{for any } j. \end{aligned}$$

Then  $Y_n = (A + I)^n = B^n = \text{diag}\{B_1^n, 0\} + \text{diag}\{0, B_2^n\}$ .

For  $r \in \mathbb{N}$  consider a Jordan  $r$ -dimensional block

$$J = \begin{pmatrix} \mu & 1 & 0 & \dots & 0 \\ 0 & \mu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}.$$

For a sufficiently smooth function  $g$  defined at  $\mu$  it can be defined (see [1])

$$g(J) = \begin{pmatrix} g(\mu) & g'(\mu) & \frac{g''(\mu)}{2!} & \dots & \frac{g^{(r-1)}(\mu)}{(r-1)!} \\ 0 & g(\mu) & g'(\mu) & \dots & \frac{g^{(r-2)}(\mu)}{(r-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & g(\mu) \end{pmatrix}.$$

Assume now that  $\mu \neq 0$ . Then especially for  $g(\mu) = \mu^n$ ,  $n \in \mathbb{Z}$  we have

$$J^n = \begin{pmatrix} \mu^n & n\mu^{n-1} & \dots & n(n-1)\dots(n-r+2) \frac{\mu^{n-r+1}}{(r-1)!} \\ 0 & \mu^n & \dots & n(n-1)\dots(n-r+1) \frac{\mu^{n-r+2}}{(r-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mu^n \end{pmatrix}. \quad (6)$$

From this it follows the existence of constants  $K > 0$  and  $L > 0$  such that

$$|B_1^n| \leq K \cdot n^{m^*-1} \alpha^n \quad \text{for } n \geq 1, \quad (7)$$

$$|B_2^{-n}| \leq L \cdot (n+p-2)^{p-1} \quad \text{for } n \geq 1. \quad (8)$$

LEMMA 1. Let  $0 < q < 1$  and  $g_n \geq 0$  for  $n \in \mathbb{N}$ . If

$$\sum_{i=1}^{\infty} g_i < \infty,$$

then

$$\lim_{n \rightarrow \infty} q^n \sum_{i=1}^n q^{-i} g_i = 0.$$

*Proof.* Assume  $\sum_{i=1}^{\infty} g_i < \infty$ . Denote  $[x]$  an integer part of  $x$ ,  $x \in \mathbb{R}$ . For any  $\varepsilon > 0$  we can find  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$  there is

$$q^{n-[n/2]} \sum_{i=1}^{[n/2]} g_i < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{[n/2]+1}^{\infty} g_i < \frac{\varepsilon}{2}.$$

Then

$$q^n \sum_{i=1}^n q^{-i} g_i = q^n \sum_{i=1}^{\lfloor n/2 \rfloor} q^{-i} g_i + q^n \sum_{i=\lfloor n/2 \rfloor+1}^n q^{-i} g_i \leq q^{n-\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor} g_i + \sum_{i=\lfloor n/2 \rfloor+1}^n g_i < \varepsilon. \quad \square$$

THEOREM 2. *Let*

$$\sum_{i=1}^{\infty} (i+p-2)^{p-1} |b_{i-1}| < \infty. \quad (9)$$

Then the equation (4) has a solution  $z_n$  such that  $\lim_{n \rightarrow \infty} z_n = 0$ .

*Proof.* Denote  $\widehat{B}_1 = \text{diag}\{B_1, O_2\}$ ,  $\widehat{B}_2 = \text{diag}\{O_1, B_2\}$ ,  $\widehat{B}_1^{-1} = \text{diag}\{B_1^{-1}, O_2\}$  and  $\widehat{B}_2^{-1} = \text{diag}\{O_1, B_2^{-1}\}$ , where  $O_i$  is a zero matrix of the same dimension as  $B_i$ ,  $i = 1, 2$ . Further  $Y_n = \widehat{B}_1^n + \widehat{B}_2^n$  holds.

From (5) we see that any solution  $z_n$  of (4) can be written in the form

$$z_n = Y_n z_0 + \sum_{i=1}^n B^{n-i} b_{i-1}.$$

Therefore

$$z_n = Y_n z_0 + \widehat{B}_1^n \sum_{i=1}^n \widehat{B}_1^{-i} b_{i-1} + \widehat{B}_2^n \sum_{i=1}^n \widehat{B}_2^{-i} b_{i-1}.$$

The inequalities (8) and (9) imply that the series  $\sum_{i=1}^{\infty} \widehat{B}_2^{-i} b_{i-1}$  absolutely converges, so we can express  $z_n$  as follows:

$$z_n = \widehat{B}_1^n z_0 + \widehat{B}_2^n \left[ z_0 + \sum_{i=1}^{\infty} \widehat{B}_2^{-i} b_{i-1} \right] + \widehat{B}_1^n \sum_{i=1}^n \widehat{B}_1^{-i} b_{i-1} - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} b_{i-1}.$$

Choose  $z_0 + \sum_{i=1}^{\infty} \widehat{B}_2^{-i} b_{i-1} = 0$ . Then

$$z_n = \widehat{B}_1^n z_0 + \widehat{B}_1^n \sum_{i=1}^n \widehat{B}_1^{-i} b_{i-1} - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} b_{i-1} = I_1 + I_2 + I_3. \quad (10)$$

We will show that  $\lim_{n \rightarrow \infty} I_i = 0$ ,  $i = 1, 2, 3$ .

From (7) evidently  $\lim_{n \rightarrow \infty} I_1 = 0$ .

Further we estimate  $I_2$ . We have

$$|I_2| \leq \left| \sum_{i=1}^{n-1} \widehat{B}_1^{n-i} b_{i-1} \right| + |\widehat{B}_1^n b_{n-1}| = |I_{2a}| + |I_{2b}|.$$

If  $1 \leq i \leq n-1$ , then  $n-i \geq 1$  and from (7) we have

$$|\widehat{B}_1^{n-i} b_{i-1}| \leq K(n-i)^{m^*-1} \alpha^{n-i} |b_{i-1}|.$$

As  $\lim_{k \rightarrow \infty} k^{m^*-1}(\sqrt{\alpha})^k = 0$ , because  $\sqrt{\alpha} < 1$ , a constant  $M > 0$  exists such that

$$0 \leq (n-i)^{m^*-1}(\sqrt{\alpha})^{n-i} \leq M$$

for  $n \geq i$ . Therefore

$$\begin{aligned} |I_{2a}| &\leq \sum_{i=1}^{n-1} K(n-i)^{m^*-1} \alpha^{n-i} |b_{i-1}| \leq KM \sum_{i=1}^{n-1} (\sqrt{\alpha})^{n-i} |b_{i-1}| \\ &= KM(\sqrt{\alpha})^n \sum_{i=1}^n (\sqrt{\alpha})^{-i} |b_{i-1}| \rightarrow 0 \end{aligned}$$

for  $n \rightarrow \infty$  by LEMMA 1.

Due to (9) we have  $b_{n-1} \rightarrow \infty$  for  $n \rightarrow \infty$  and

$$|I_{2b}| \leq |\widehat{B}_1^0| \cdot |b_{n-1}| \rightarrow \infty$$

for  $n \rightarrow \infty$ . Thus  $\lim_{n \rightarrow \infty} I_2 = 0$ .

Now we estimate  $I_3$ . From (8) we obtain

$$\begin{aligned} |I_3| &\leq \sum_{i=n+1}^{\infty} |\widehat{B}_2^{-(i-n)}| \cdot |b_{i-1}| \leq \sum_{i=n+1}^{\infty} L(i-n+p-2)^{p-1} |b_{i-1}| \\ &\leq L \sum_{i=n+1}^{\infty} (i+p-2)^{p-1} |b_{i-1}| \rightarrow 0 \end{aligned}$$

for  $n \rightarrow \infty$  by (9). Thus  $\lim_{n \rightarrow \infty} I_3 = 0$ .  $\square$

Consider a linear space  $\ell(n_1)$  of real sequences  $\mathbf{x} = \{x_n\}$ ,  $x_n \in \mathbb{R}^k$ ,  $n \geq n_1$ ,  $n_1 \in \mathbb{N}_0$ , endowed with the topology  $\tau$  induced by the set of seminorms  $P_m(x) = |x_m|$ ,  $m \geq n_1$ . Then  $(\ell(n_1), \tau)$  is a Fréchet space – see [6, p. 37]. If  $\mathbf{x}^r = \{x_n^r\} \in \ell(n_1)$ ,  $r \in \mathbb{N}_0$ , then  $\mathbf{x}^r \rightarrow \mathbf{x}^0$  in  $(\ell(n_1), \tau)$  if and only if  $\lim_{r \rightarrow \infty} x_n^r = x_n^0$  for each  $n \geq n_1$ .

LEMMA 2. A subset  $B \subset \ell(n_1)$  is relatively compact in  $(\ell(n_1), \tau)$  if and only if  $\sup_{\mathbf{x} \in B} \{ |x_n| \} < \infty$  for each  $n \geq n_1$ .

*Proof. Sufficiency:* Consider a sequence  $\{\mathbf{x}^r\} \subset B$ . It is possible to choose its subsequence  $\{\mathbf{y}^r\}$  such that  $\{y_1^r\}$  is Cauchy. Then it is possible to choose a subsequence  $\{\mathbf{z}^r\}$  of  $\{\mathbf{y}^r\}$  such that  $\{z_2^r\}$  is Cauchy etc. The diagonal sequence  $\{\mathbf{y}^1, \mathbf{z}^2, \dots\}$  is chosen from  $\{\mathbf{x}^r\}$  and is Cauchy in  $(\ell(n_1), \tau)$ .

*Necessity:* If  $\sup_{\mathbf{x} \in B} \{ |x_n| \} = \infty$  for some  $n \geq n_1$ , it is possible to find a sequence  $\{\mathbf{x}^r\} \subset B$  such that  $\lim_{r \rightarrow \infty} |x_n^r| = \infty$ . Evidently, for no subsequence  $\{\mathbf{y}^r\}$  of  $\{\mathbf{x}^r\}$  the sequence  $\{|y_n^r|\}$  is Cauchy.  $\square$

THEOREM 3. Assume

$$|f(n, x)| \leq F(n, |x|), \quad (11)$$

where  $F: \mathbb{N}_0 \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$  and  $F(n, u)$  is nondecreasing and continuous in  $u$  for any  $n \in \mathbb{N}_0$ . Let

$$\sum_{i=1}^{\infty} (i+p-2)^{p-1} F(i-1, c) < \infty \quad (12)$$

for any  $c \in \mathbb{R}_0^+$ .

Then the sets of bounded solutions of (1) and (2) are asymptotically equivalent.

*Proof.* Let  $x_n$ ,  $n \in \mathbb{N}(n_0)$ , be a bounded solution of (1). Then there exists  $c \geq 0$  such that  $|x_n| \leq c$  for  $n \in \mathbb{N}(n_0)$ . If  $y_n$  is an arbitrary solution of (2), then  $z_n = x_n - y_n$  is a solution of an equation

$$\Delta z_n = Az_n + f(n, x_n). \quad (13)$$

Conversely, if  $z_n$  is an arbitrary solution of (13), then  $y_n = x_n - z_n$  is a solution of (2).

According to (11) and (12) we have

$$\begin{aligned} \sum_{i=1}^{\infty} (i+p-2)^{p-1} |f(i-1, x_{i-1})| &\leq \sum_{i=1}^{\infty} (i+p-2)^{p-1} F(i-1, |x_{i-1}|) \\ &\leq \sum_{i=1}^{\infty} (i+p-2)^{p-1} F(i-1, c) < \infty. \end{aligned}$$

Then due to THEOREM 2 the equation (13) has a solution  $z_n$  such that  $\lim_{n \rightarrow \infty} z_n = 0$ . Thus  $y_n = x_n - z_n$  is a solution of (2) for which  $\lim_{n \rightarrow \infty} |x_n - y_n| = \lim_{n \rightarrow \infty} |z_n| = 0$ .

Assume now that  $y_n$  is a bounded solution of (2),  $n \in \mathbb{N}_0$ . Consider an equation

$$x_n = y_n + \widehat{B}_1^n \sum_{i=n_0+1}^n \widehat{B}_1^{-i} f(i-1, x_{i-1}) - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1}), \quad (14)$$

$n_0 \in \mathbb{N}_0$ ,  $n \geq n_0$ . It is easy to verify that any solution of (14) is a solution of (1), too. In fact, if we denote  $I = \widehat{I}_1 + \widehat{I}_2 = \text{diag}\{I_1, O_2\} + \text{diag}\{O_1, I_2\}$ , where  $I_i$  is a unit matrix of the same dimension as  $B_i$ ,  $i = 1, 2$ , we obtain

$$\begin{aligned} \Delta x_n &= \Delta y_n + \Delta \widehat{B}_1^n \sum_{i=n_0+1}^n \widehat{B}_1^{-i} f(i-1, x_{i-1}) + \widehat{B}_1^{n+1} \widehat{B}_1^{-(n+1)} f(n, x_n) \\ &\quad - \Delta \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1}) + \widehat{B}_2^{n+1} \widehat{B}_2^{-(n+1)} f(n, x_n) \\ &= Ay_n + A \widehat{B}_1^n \sum_{i=n_0+1}^n \widehat{B}_1^{-i} f(i-1, x_{i-1}) + \widehat{I}_1 f(n, x_n) \\ &\quad - A \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1}) + \widehat{I}_2 f(n, x_n) \\ &= Ax_n + f(n, x_n). \end{aligned}$$

For  $\rho > 0$ , let us denote  $\mathcal{B}_\rho = \{\mathbf{x} \in \ell(n_0) : |x_n| \leq \rho, n \in \mathbb{N}(n_0)\}$ . Evidently  $\mathcal{B}_\rho$  is a convex closed subset of  $\ell(n_0)$ , which is compact by LEMMA 2. We will show that (14) has a solution in  $\mathcal{B}_\rho$ , where  $\rho$  and  $n_0$  will be chosen later.

Define a mapping  $T: \mathcal{B}_\rho \rightarrow \ell(n_0)$  as follows:

$$T(\mathbf{x})_n = y_n + \widehat{B}_1^n \sum_{i=n_0+1}^n \widehat{B}_1^{-i} f(i-1, x_{i-1}) - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1})$$

for  $\mathbf{x} \in \mathcal{B}_\rho$ . By (8) and (11) we have

$$|\widehat{B}_2^{-i} f(i-1, x_{i-1})| \leq L(i+p-2)^{p-1} F(i-1, |x_{i-1}|) \leq L(i+p-2)^{p-1} F(i-1, \rho),$$

which shows (using (12)) that  $\sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} f(i-1, x_{i-1})$  converges and  $T$  is correctly defined.

Let  $|y_n| \leq c_1$ ,  $n \in \mathbb{N}_0$ , and choose  $\rho \geq 2c_1$ . Then

$$|T(\mathbf{x})_n| \leq |y_n| + \sum_{i=n_0+1}^n |\widehat{B}_1^{n-i}| F(i-1, \rho) + \sum_{i=n+1}^{\infty} |\widehat{B}_2^{-(i-n)}| F(i-1, \rho).$$

With respect to (7) there exists a constant  $M > 0$  such that  $|\widehat{B}_1^{n-i}| \leq M$  for  $n_0+1 \leq i \leq n$ .

Using (12) we get that  $\sum_{i=1}^{\infty} F(i-1, \rho) < \infty$ , which implies that

$$\begin{aligned} |T(\mathbf{x})_n| &\leq c_1 + M \sum_{i=n_0+1}^n F(i-1, \rho) + L \sum_{i=n+1}^{\infty} (i-n+p-2)^{p-1} F(i-1, \rho) \\ &\leq c_1 + M \sum_{i=n_0+1}^{\infty} F(i-1, \rho) + L \sum_{i=n+1}^{\infty} (i+p-2)^{p-1} F(i-1, \rho). \end{aligned}$$

Therefore, it is possible to find  $n_0$  large enough such that

$$M \sum_{i=n_0+1}^{\infty} F(i-1, \rho) + L \sum_{i=n+1}^{\infty} (i+p-2)^{p-1} F(i-1, \rho) < c_1.$$

Then  $|T(\mathbf{x})_n| \leq 2c_1 \leq \rho$ . i.e.  $T: \mathcal{B}_\rho \rightarrow \mathcal{B}_\rho$ .

Further we will verify that  $T$  is a continuous operator. Let  $\mathbf{x}^k \rightarrow \mathbf{x}$  for  $k \rightarrow \infty$  in  $\ell(n_0)$ , which means that  $\lim_{k \rightarrow \infty} x_n^k = x_n$ ,  $n \in \mathbb{N}(n_0)$ . Choose a fixed  $n \geq n_0$  and  $n_1 > n$ .

Then

$$\begin{aligned} |T(\mathbf{x}^k)_n - T(\mathbf{x})_n| &= \left| \widehat{B}_1^n \sum_{i=n_0+1}^n \widehat{B}_1^{-i} [f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})] \right. \\ &\quad \left. - \widehat{B}_2^n \sum_{i=n+1}^{\infty} \widehat{B}_2^{-i} [f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})] \right| \\ &\leq \sum_{i=n_0+1}^n |\widehat{B}_1^{n-i}| \cdot |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\quad + \sum_{i=n+1}^{n_1} |\widehat{B}_2^{-(i-n)}| \cdot |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\quad + 2 \sum_{i=n_1+1}^{\infty} |\widehat{B}_2^{-(i-n)}| F(i-1, \rho) \\ &\leq M \sum_{i=n_0+1}^{n_1} |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\quad + L \sum_{i=n+1}^{n_1} (i-n+p-2)^{p-1} |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\quad + 2 \sum_{i=n_1+1}^{\infty} (i-n+p-2)^{p-1} F(i-1, \rho) \end{aligned}$$

$$\begin{aligned} &\leq M \sum_{i=n_0+1}^{n_1} |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\quad + L(n_1 + p - 2)^{p-1} \sum_{i=n_0+1}^{n_1} |f(i-1, x_{i-1}^k) - f(i-1, x_{i-1})| \\ &\quad + 2 \sum_{i=n_1+1}^{\infty} (i + p - 2)^{p-1} F(i-1, \rho). \end{aligned}$$

Let  $\varepsilon > 0$  be an arbitrary number. From (12) there exists  $n_1$  such that

$$\sum_{i=n_1+1}^{\infty} (i + p - 2)^{p-1} F(i-1, \rho) < \frac{\varepsilon}{4}.$$

Further, as  $\mathbf{x}^k \rightarrow \mathbf{x}$  in  $\ell(n_0)$  and  $f$  is continuous at its second argument we can find  $k_0 \in \mathbb{N}$  such that for  $k \geq k_0$  and  $n_0 \leq i \leq n_1 - 1$  the inequality

$$|f(i, x_i^k) - f(i, x_i)| < \frac{\varepsilon}{2(n_1 - n_0)(M + L(n_1 + p - 2)^{p-1})}$$

holds. Thus  $|T(\mathbf{x}^k)_n - T(\mathbf{x})_n| < \varepsilon$  for  $k \geq k_0$ , i.e.  $\lim_{k \rightarrow \infty} T(\mathbf{x}^k)_n = T(\mathbf{x})_n$ . But this means that  $T(\mathbf{x}^k) \rightarrow T(\mathbf{x})$  as  $k \rightarrow \infty$  in  $\ell(n_0)$ .

From Tychonoff fixed point theorem – see [2, p. 405] or [3, p. 45] – we conclude that  $T$  has a fixed point in  $\mathcal{B}_\rho$ . Thus (14) and also (1) has a bounded solution.

We will show that  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ . Analogously as in the proof of THEOREM 2 for  $I_2$  and  $I_3$  (see (10)) it can be proved that

$$\sum_{i=n_0+1}^n \widehat{B}_1^{n-i} f(i-1, x_{i-1}) - \sum_{i=n_0+1}^{\infty} \widehat{B}_2^{-(i-n)} f(i-1, x_{i-1}) \rightarrow 0$$

as  $n \rightarrow \infty$  ( $b_{i-1}$  is to be replaced by  $F(i-1, \rho)$ ). This proves the theorem.  $\square$

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