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SOME EXISTENCE RESULT TO ELLIPTIC EQUATIONS WITH SEMILINEAR COEFFICIENTS*

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Abstract. For the quasilinear elliptic equation

$$-\sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x,u)u = f(x,u,\nabla u)$$

on a bounded smooth domain Ω in R^N with $c(x,r) \geqslant \alpha_0$ and $|f(x,r,\xi)| \leqslant C_0 + h(|r|)|\xi|^\theta, 0 \leqslant \theta < 2$, we note that every solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), p \geqslant \frac{2N}{N+2}$, is L^∞ -bounded by $\frac{C_0}{\alpha_0}$. Consequently, the existence of such solution is irrelevant to $a_{ij}(x,r)$ on $|r| \geqslant \frac{C_0}{\alpha_0}$. It is then shown that if the oscillation $a_{ij}(x,r)$ with respect to r are sufficiently small for $|r| \leqslant \frac{C_0}{\alpha_0}$, then there exists a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), 1 \leqslant p < \infty$.

Key words. Quasilinear elliptic problem, strong solution, $W^{2,p}$ estimate

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1. Introduction. For a bounded domain Ω in $\mathbb{R}^N$, $N \geqslant 3$, which is $C^{1,1}$ diffeomorphic to a ball in $\mathbb{R}^N$, let L_v, L, D_v, and D are elliptic operators defined by

$$\begin{aligned} L_v u &= -\sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x,v)u, \\ Lu &= L_u u, \\ D_v u &= -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x,v)\frac{\partial u}{\partial x_j} + c(x,v)u, \\ Du &= D_u u, \end{aligned}$$

where the coefficients a_{ij}, c, and $\frac{\partial a_{ij}}{\partial x_i}$, $\frac{\partial a_{ij}}{\partial r}$ are bounded Carathéodory functions, $c \geqslant \alpha_0 > 0$ and $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2$ for some constants α_0 and λ.

Let $f(x,r,\xi)$ be a Carathéodory function satisfying $|f(x,r,\xi)| \leqslant C_0 + h(|r|)|\xi|^2$, where $h(|r|)$ is a locally bounded function. It is shown in [1] that there exists a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to the equation

$$Du = f(x,u,\nabla u) \qquad \text{in } \Omega. \tag{1.1}$$

Moreover, every solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (1.1) is L^∞ -bounded by $r_0 = \frac{C_0}{\alpha_0}$. In pursuit of strong solutions, we note that (1.1) can be reformulated as

$$Lu = f(x,u,\nabla u) + \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial r}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{1.2}$$

It then suffices to examine the existence of solutions to (1.2) with $a_{ij}(x,r)$ replaced by

$$b_{ij}(x,r) = \begin{cases} a_{ij}(x,r), & \text{if } |r| < r_0 \\ a_{ij}(x,r_0), & \text{if } |r| \geqslant r_0 \end{cases} \tag{1.3}$$

For the main purpose of this paper, we shall study the existence of solutions $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to the equation

$$Lu = f(x, u, \nabla u) \qquad \text{in } \Omega, \tag{1.4}$$

where

$$|f(x, r, \xi)| \leqslant C_0 + h(|r|)|\xi|^{\theta}, \qquad 0 \leqslant \theta < 2. \tag{1.5}$$

Recall that in the linear case when $a_{ij} = a_{ij}(x)$ and $f = f(x) \in L^p(\Omega)$, one has the $W^{2,p}$ estimate from [2] that

$$\|u\|_{W^{2,p}(\Omega)} \leqslant C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$$

for every solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (1.4). This estimate remains valid if the oscillation $a_{ij}(x, r)$ with respect to r are sufficiently small [3]. A $W^{2,p}$ estimate was then performed to deduce the existence of solutions to (1.4). In Section 2, we observe that every solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (1.4), $p \geqslant \frac{2N}{N+2}$, is L^{∞} -bounded by $r_0 = \frac{C_0}{\alpha_0}$. Thus, the existence of solution is irrelevent to $a_{ij}(x, r)$ on $|r| \geqslant r_0$. Our main result in Theorem 2.4 shows that if the oscillation of $a_{ij}(x, r)$ with respect to r for $|r| \leqslant r_0$ are sufficiently small, then there exists a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $1 \leqslant p < \infty$.

2. Existence of Strong Solutions. Our main result in this section aims to show the existence of solutions $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (1.4). In the light of [1, p. 45], every solution $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$ to (1.1) is L^{∞}-bounded by r_0, one gets readily that

Lemma 2.1. *Every solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $p \geqslant \frac{2N}{N+2}$, is L^{∞} -bounded by r_0.*

Proof. Let

$$\tilde{f}(x, r, \xi) = f(x, r, \xi) - \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} - \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial r} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{2.1}$$

(1.4) can be rewritten as

$$Du = \tilde{f}(x, u, \nabla u).$$

For every $\varepsilon > 0$, one deduces from (1.5) that $|\tilde{f}(x, r, \xi)| \leqslant C_0 + \varepsilon + h_1(|r|)|\xi|^2$, where $h_1(|r|)$ is a locally bounded function. Notice that a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (1.1) is L^∞-bounded by $r_0 + \frac{\varepsilon}{\alpha_0}$. Therefore, every solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $p \geqslant \frac{2N}{N+2}$, is L^∞-bounded by r_0. □

For a fixed x in $\mathbb{R}^N$, let $a_{ij}(x, r; s)$ denote the oscillation of $a_{ij}(x, r)$ with respect to r for $|r| \leqslant s$, i.e.,

$$\operatorname{osc} a_{ij}(x, r; s) = \sup\{|a_{ij}(x, r_1) - a_{ij}(x, r_2)| : |r_1|, |r_2| \leqslant s\}.$$

Let $\operatorname{osc} a(x, r; s) = \max_{1 \leqslant i,j \leqslant N} \operatorname{osc} a_{ij}(x, r; s)$ and $\operatorname{osc} a(x, r) = \operatorname{osc} a(x, r; +\infty)$.

For operators L_v, we quote the following result from [3, p. 191]

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ which is $C^{1,1}$ diffeomorphic to a ball in $\mathbb{R}^N$, and the coefficients $a_{ij} \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, $|a_{ij}|$, $|c| \leqslant \Lambda$, where Λ is a positive constant, $i, j = 1, \ldots, N$. Assume that $\operatorname{osc} a_{ij}(x, r)$ is sufficiently small with respect to r and uniformly for $x \in \Omega$. Then if $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $L_v u \in L^p(\Omega)$, $1 < p < \infty$. One has the estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leqslant C(\|L_v u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

where C is a constant (independent of v) dependent on N, p, λ, Λ, $\partial\Omega$, and Ω, the diffeomorphism and the moduli of continuity of $a_{ij}(x, r)$ with respect to x in $\bar{\Omega}$.

Denote $\tilde{L}v$ and $\tilde{L}$ the elliptic operators with $a_{ij}(x, r)$ replaced by $b_{ij}(x, r)$, i.e.,

$$\tilde{L}_v u = -\sum_{i,j=1}^{N} b_{ij}(x, v) \frac{\partial^2 u}{\partial x_i \partial x_j} + c(x, v)u,$$

and

$$\tilde{L}u = \tilde{L}_u u.$$

Consider now the equation

$$\tilde{L}u = f(x, u, \nabla u) \tag{2.2}$$

Let f_n be the truncature of f by $\pm n$. For $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, the Dirichlet problem $\tilde{L}_v u = f_n(x, v, \nabla v)$ has a unique solution $u_n \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and by Lemma 2.2

$$\|u_n\|_{W^{2,p}(\Omega)} \leqslant C(\|u_n\|_{L_p(\Omega)} + \|f_n(x, v, \nabla v)\|_{L_p(\Omega)}).$$

An application of the weak maximum principle of A. D. Aleksandrov [2, p. 220] together with the Schauder Fixed Point Theorem implies that there exists a solution $u_n \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (2.2). Moreover, by the constraint (1.5) on f, one has the following estimate from [3].

Lemma 2.3. *The approximating solutions (u_n) are $W^{2,p}$-bounded.*

The existence of solutions can now be deduced from above lemmas.

Theorem 2.4. *Let Ω be a bounded $C^{1,1}$-smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, $a_{ij} \in C^{0,1}(\bar{\boldsymbol{\Omega}} \times \mathbb{R})$, $\frac{\partial a_{ij}}{\partial x_i}$, $\frac{\partial a_{ij}}{\partial r}$, c be bounded Carathodory functions, $c(x, r) \geqslant \alpha_0 > 0$. Assume that $f(x, r, \xi)$ satisfies (1.5) and $\operatorname{osc} a(x, r; r_0)$ is sufficiently small uniformly for $x \in \bar{\Omega}$. Then there exists a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to Equation (1.4).*

Proof. By Lemma2.3 we get the approximating solutions (u_n) which are $W^{2,p}$-bounded. It follows from the compact imbedding $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ that there exists a convergent subsequence in $W_0^{1,p}(\Omega)$, which is still denoted by (u_n), such that $u_n \to u$ a.e., $\nabla u_n \to \nabla u$ a.e. and $u_n \to u$ in $W^{1,p}(\Omega)$. Moreover, since $\|u_n\|_{W^{2,p}(\Omega)} \leqslant M$ and the set $\{v \in$

$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) | \|v\|_{W^{2,p}(\Omega)} \leqslant M\}$ is closed in $W^{1,p}(\Omega)$, the limit u of (u_n) belongs to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Passing to the limit and using Vitali Convergence Theorem, one can show that $Lu_n \to Lu$ in $\mathfrak{D}'(\boldsymbol{\Omega})$ and $f_n(x, u, \nabla u_n) \to f(x, u, \nabla u)$ in $L^1(\boldsymbol{\Omega})$, which proves that u is a solution to (2.2). □

Notice that $\|u\|_{L^\infty} \leqslant r_0$ by Lemma 2.1, hence $b_{ij}(x, u(x)) = a_{ij}(x, u(x))$ a.e. Therefore one concludes that u is in fact a solution to (1.4).

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