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ON THE EXISTENCE OF HOPF BIFURCATION IN AN OPEN ECONOMY MODEL

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Abstract. In the paper a four dimensional open economy model describing the development of output, exchange rate, interest rate and money supply is analyzed. Sufficient conditions for the existence of equilibrium stability and the existence of business cycles are found. Formulae for the calculation of bifurcation coefficients are derived.

Key words. dynamical model, equilibrium, stability, bifurcation, cycles

AMS subject classifications. 37Gxx

1. Introduction. In economic theory there are many macroeconomic models describing the development of output Y in an economy. The majority of them are based on the well-known IS-LM model. One of them is the Schinasi's model (see [5]) describing the development of output Y, interest rate R and money supply L_s in a closed economy. In [8] the Schinasi's model was extended to an open economy model of the kind

$$\begin{aligned}\dot{Y} &= \alpha \left[I(Y,R) + G + X(\rho) - S(Y^D,R) - T(Y) - M(Y,\rho)\right]\\ \dot{\rho} &= \beta\left[M(Y,\rho) + C_x(R) - X(\rho) - C_m(R)\right]\\ \dot{R} &= \gamma\left[L(Y,R) - L_S\right]\\ \dot{L_S} &= G + M(Y,\rho) + C_x(R) - T(Y) - X(\rho) - C_m(R),\end{aligned} \tag{1.1}$$

where Y – output, ρ – exchange rate, R – interest rate, L_S – money supply, I – investments, S – savings, G – government expenditures, T – tax collections, X – – export, M – import, C_x – capital export, C_m – capital import, L – money demand, α, β, γ – positive parameters, t – time and

$$Y^D = Y - T(Y), \qquad \dot{Y} = \frac{dY}{dt}, \qquad \dot{\rho} = \frac{d\rho}{dt}, \qquad \dot{R} = \frac{dR}{dt}, \qquad \dot{L_S} = \frac{dL_S}{dt}.$$

The economic properties of the functions in (1.1) are expressed by the following partial derivatives:

$$\begin{aligned}&\frac{\partial I(Y,R)}{\partial Y} > 0, \quad \frac{\partial I(Y,R)}{\partial R} < 0, \quad \frac{\partial S(Y^D,R)}{\partial Y} > 0, \quad \frac{\partial S(Y^D,R)}{\partial R} > 0,\\ &\frac{\partial T(Y)}{\partial Y} > 0, \quad \frac{\partial X(\rho)}{\partial \rho} > 0, \quad \frac{\partial M(Y,\rho)}{\partial Y} > 0, \quad \frac{\partial M(Y,\rho)}{\partial \rho} < 0,\\ &\frac{\partial C_x(R)}{\partial R} < 0, \quad \frac{\partial C_m(R)}{\partial R} > 0, \quad \frac{\partial L(Y,R)}{\partial Y} > 0, \quad \frac{\partial L(Y,R)}{\partial R} < 0.\end{aligned} \tag{1.2}$$

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The development of Y, ρ, R and L_S in the model (1.1) in a neighborhood of its equilibrium was studied under fixed exchange rate regime in [6] and under flexible exchange rate regime in [2] from the point of equilibrium stability and the existence of business cycles. In [4] this model was analyzed under assumption that both ρ and R are flexible. In the paper [4] the assertion on the existence of business cycles was formulated under assumption of the existence of the Liapunov bifurcation constant a. In the present paper a formula for the calculation of this Liapunov bifurcation constant a is found supposing that the functions $I(Y,R)$ and $S(Y^D,R)$ are nonlinear with respect to Y and linear with respect to R, and other functions in (1.1) are linear with respect to all their variables except government expenditures G, which are constant. In the end of the paper an example illustrating the achieved results is presented.

2. Analysis of the model (1.1)**.** Using denotation

$$\begin{aligned} B(Y,R) &= I(Y,R) - S(Y^D,R) \\ C(R) &= C_m(R) - C_x(R) \\ D(Y) &= G - T(Y) \\ F(Y,\rho) &= X(\rho) - M(Y,\rho), \end{aligned}$$

the model (1.1) has the form

$$\begin{aligned} \dot{Y} &= \alpha\,[B(Y,R) + D(Y) + F(Y,\rho)] \\ \dot{\rho} &= -\beta\,[F(Y,\rho) + C(R)] \\ \dot{R} &= \gamma\,[L(Y,R) - L_S] \\ \dot{L_S} &= D(Y) - F(Y,\rho) - C(R), \end{aligned} \tag{2.1}$$

where on the base of (1.2)

$$\frac{\partial B(Y,R)}{\partial R} < 0, \quad \frac{\partial C(R)}{\partial R} > 0, \quad \frac{\partial D(Y)}{\partial Y} < 0, \quad \frac{\partial F(Y,\rho)}{\partial Y} < 0, \quad \frac{\partial F(Y,\rho)}{\partial \rho} > 0,$$

and the sign of $\frac{\partial B(Y,R)}{\partial Y}$ can be positive or negative or zero according to the properties of $I(Y,R)$ and $S(Y^D,R)$.

Assume:

1. The functions $I(Y,R)$, $S(Y^D,R)$, $T(Y)$, $C(R)$, $D(Y)$, $F(Y,\rho)$, $L(Y,R)$ are of the kind

$$\begin{aligned} I(Y,R) &= i_0 + i_1\sqrt{Y} - i_3R, & C(R) &= c_0 + c_3R, \\ S(Y^D,R) &= s_0 + s_1(Y^D)^2 + s_3R, & D(Y) &= d_0 - d_1Y, \\ Y^D &= Y - T(Y), & F(Y,\rho) &= f_0 - f_1Y + f_2\rho, \\ T(Y) &= t_0 + t_1Y, & L(Y,R) &= l_0 + l_1Y - l_3R, \end{aligned}$$

with positive coefficients i_1, i_3, s_1, s_3, t_1, c_3, d_1, f_1, f_2, l_1, l_3.

2. The model (2.1) has an isolated equilibrium $E^* = (Y^*, \rho^*, R^*, L_S^*)$, $Y^* > 0$, $\rho^* > 0$, $R^* > 0$, $L_S^* > 0$.

Remark 1. *A sufficient condition for the existence of an isolated equilibrium E^* of the model* (2.1) *was presented in [4].*

Consider an isolated equilibrium $E^* = (Y^*, \rho^*, R^*, L_S^*)$ of (2.1). After the transformation

$$Y_1 = Y - Y^*, \quad \rho_1 = \rho - \rho^*, \quad R_1 = R - R^*, \quad L_{S1} = L_S - L_S^*,$$

the equilibrium E^* goes into the origin $E_1^* = (Y_1^* = 0, \rho_1^* = 0, R_1^* = 0, L_{S1}^* = 0)$ and the model (2.1) takes the form

$$\begin{aligned} \dot{Y}_1 &= \alpha\,[B(Y_1+Y^*, R_1+R^*) + D(Y_1+Y^*) + F(Y_1+Y^*, \rho_1+\rho^*)] \\ \dot{\rho}_1 &= -\beta\,[F(Y_1+Y^*, \rho_1+\rho^*) + C(R_1+R^*)] \\ \dot{R}_1 &= \gamma\,[L(Y_1+Y^*, R_1+R^*) - L_{S1} - L_{S1}^*] \\ \dot{L}_{S1} &= D(Y_1+Y^*) - F(Y_1+Y^*, \rho_1+\rho^*) - C(R_1+R^*). \end{aligned} \tag{2.2}$$

Performing Taylor expansion of the functions on the right-hand side in (2.2) at the equilibrium E_1^* we get the model

$$\begin{aligned} \dot{Y}_1 &= \alpha\,[(b_1-d_1-f_1)Y_1 + f_2\rho_1 - b_3R_1] + \alpha\left[\sum_{k=2}^{4} a_k {Y_1}^k + O(|Y_1|^5)\right] \\ \dot{\rho}_1 &= \beta\,(f_1Y_1 - f_2\rho_1 - c_3R_1) \\ \dot{R}_1 &= \gamma\,(l_1Y_1 - l_3R_1 - L_{S1}) \\ \dot{L}_{S1} &= (f_1-d_1)Y_1 - f_2\rho_1 - c_3R_1, \end{aligned} \tag{2.3}$$

where $b_1 = \frac{\partial B(Y^*,R^*)}{\partial Y}$, $b_3 = i_3 + s_3$, $a_k = \frac{\partial^k B(Y^*,R^*)}{\partial Y^k}$, $k = 2,3,4$.

The linear approximation matrix of (2.3) is

$$\mathbf{A}(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha(b_1-d_1-f_1) & \alpha f_2 & -\alpha b_3 & 0 \\ \beta f_1 & -\beta f_2 & -\beta c_3 & 0 \\ \gamma l_1 & 0 & -\gamma l_3 & -\gamma \\ f_1-d_1 & -f_2 & -c_3 & 0 \end{pmatrix}. \tag{2.4}$$

The characteristic equation of $\mathbf{A}(\alpha,\beta,\gamma)$ is

$$\lambda^4 + a_1(\alpha,\beta,\gamma)\lambda^3 + a_2(\alpha,\beta,\gamma)\lambda^2 + a_3(\alpha,\beta,\gamma)\lambda + a_4(\alpha,\beta,\gamma) = 0, \tag{2.5}$$

where

$$\begin{aligned} a_1 &= \alpha(-b_1+d_1+f_1) + \beta f_2 + \gamma l_3, \\ a_2 &= -\gamma c_3 + \beta\gamma f_2 l_3 + \alpha(\beta f_2(-b_1+d_1) + \gamma(b_3l_1 + l_3(-b_1+d_1+f_1))), \\ a_3 &= \alpha\gamma(\beta f_2(l_1(b_3+c_3) + l_3(-b_1+d_1)) - c_3(-b_1+d_1+f_1) - b_3(f_1-d_1)), \\ a_4 &= \alpha\beta\gamma d_1 f_2(b_3+c_3). \end{aligned}$$

As we are interested in equilibrium stability and in the existence of cycles in the model (2.3) it is suitable to find such values of parameters α, β, γ at which the equation (2.5) has a pair of purely imaginary eigenvalues $\lambda_1 = i\omega, \lambda_2 = -i\omega$, and the rest two eigenvalues λ_3, λ_4 are negative or have negative real parts. We shall call such values of parameters α, β, γ as critical values of the model (2.3) and denote them $\alpha_0, \beta_0, \gamma_0$. Mentioned types of eigenvalues are ensured by the Liu's conditions ([3]):

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, \tag{2.6}$$

$$\Delta_3 = (a_1a_2 - a_3)a_3 - {a_1}^2 a_4 = 0. \tag{2.7}$$

The conditions (2.6) are satisfied if

$$\begin{aligned} &-b_1+d_1 \geq 0, \\ &\alpha > \alpha^* = \frac{c_3}{l_3(-b_1+d_1+f_1)+b_3 l_1}, \\ &(-b_1+d_1+f_1)c_3+(f_1-d_1)b_3 < 0. \end{aligned} \tag{2.8}$$

The relation (2.7) can be expressed in the form

$$\left[a^{(1)}(\beta,\gamma)\beta + a^{(2)}\gamma\right]\alpha^2 + \left[b^{(1)}(\beta,\gamma)\beta + b^{(2)}(\gamma)\gamma\right]\alpha + c^{(1)}(\beta,\gamma)\beta + c^{(2)}\gamma^2 = 0,$$

where in the power of (2.8) there is

$$\begin{aligned} a^{(2)} &= (-b_1+d_1+f_1)[b_3(f_1-d_1) \\ &\quad + c_3(-b_1+d_1+f_1)][-b_3 l_1 - l_3(-b_1+d_1+f_1)] > 0, \\ c^{(2)} &= c_3 l_3[b_3(f_1-d_1)+c_3(-b_1+d_1+f_1)] < 0. \end{aligned}$$

Take γ at any positive level and denote it γ_0. Then $a^{(2)}\gamma_0 > 0$ and $c^{(2)}{\gamma_0}^2 < 0$. Denote

$$\begin{aligned} A(\beta) &= a^{(1)}(\beta,\gamma_0)\beta + a^{(2)}\gamma_0, \\ B(\beta) &= b^{(1)}(\beta,\gamma_0)\beta + b^{(2)}(\gamma_0)\gamma_0, \\ C(\beta) &= c^{(1)}(\beta,\gamma_0)\beta + c^{(2)}{\gamma_0}^2. \end{aligned}$$

Take β such small that $A(\beta) > 0$ and $C(\beta) < 0$. Denote this β as β_0. Then the equation $A(\beta_0)\alpha^2 + B(\beta_0)\alpha + C(\beta_0) = 0$ has two roots

$$\alpha_{1,2} = \frac{-B(\beta_0) \pm \sqrt{(B(\beta_0))^2 - 4A(\beta_0)C(\beta_0)}}{2A(\beta_0)} = \begin{cases} \alpha_1 < 0 \\ \alpha_2 > 0 \end{cases},$$

and the function $\Delta_3(\alpha) = A(\beta_0)\alpha^2 + B(\beta_0)\alpha + C(\beta_0)$ has the following form (Fig. 2.1):

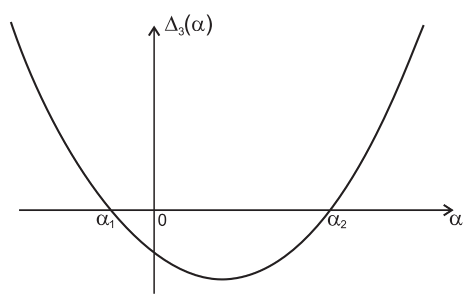

Fig. 2.1.

Denote $\alpha_2 = \alpha_0$. The triple $(\alpha_0, \beta_0, \gamma_0)$ is the critical triple of the model (2.3). Thus for given specific values of β_0, γ_0 it is always possible to find such a value α_0 that the condition (2.7) is satisfied. From Figure 1 we get in a small neighborhood of α_0:

$$\begin{aligned} &\text{If} \quad \alpha > \alpha_0 \quad \text{then} \quad \Delta_3(\alpha) = (a_1a_2 - a_3)a_3 - {a_1}^2 a_4 > 0. \\ &\text{If} \quad \alpha < \alpha_0 \quad \text{then} \quad \Delta_3(\alpha) = (a_1a_2 - a_3)a_3 - {a_1}^2 a_4 < 0. \\ &\text{If} \quad \alpha = \alpha_0 \quad \text{then} \quad \Delta_3(\alpha) = (a_1a_2 - a_3)a_3 - {a_1}^2 a_4 = 0. \end{aligned} \tag{2.9}$$

Consider a critical triple $(\alpha_0, \beta_0, \gamma_0)$ of the model (2.3). Further we shall investigate the behavior of Y_1, ρ_1, R_1 and L_{S1} around the equilibrium E_1^* with respect to parameter α, $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, $\varepsilon > 0$, and fixed parameters $\beta = \beta_0, \gamma = \gamma_0$.

Performed considerations enable us to answer the question about the stability of the equilibrium E_1^* of the model (2.3) in a small neighborhood of the critical value α_0.

Theorem 1. *Let the conditions* (2.8) *be satisfied and let* $(\alpha_0, \beta_0, \gamma_0)$ *be such a critical triple of the model* (2.3) *that* $A(\beta_0) > 0$ *and* $C(\beta_0) < 0$. *Then:*

1. *If* $\alpha_0 \le \alpha^*$ *then at every* $\alpha > \alpha^*$ *the equilibrium* E_1^* *is asymptotically stable.*
2. *If* $\alpha_0 > \alpha^*$ *then at every* $\alpha > \alpha_0$ *the equilibrium* E_1^* *is asymptotically stable and at every* $\alpha^* < \alpha < \alpha_0$ *the equilibrium* E_1^* *is unstable.*

Proof. The conditions (2.8) guarantee that $a_1 > 0, a_2 > 0, a_3 > 0$, and $a_4 > 0$ what together with (2.9) at $\alpha > \alpha^* \ge \alpha_0$ means that the Routh-Hurwitz necessary and sufficient conditions for all the roots of the equation (2.5) to have negative real parts ($\Delta_1(\alpha) > 0, \Delta_2(\alpha) > 0, \Delta_3(\alpha) > 0, \Delta_4(\alpha) > 0$) are satisfied. If $\alpha^* < \alpha < \alpha_0$ then $\Delta_3(\alpha) < 0$ what means that the Routh-Hurwitz conditions are not satisfied. □

To gain the bifurcation equation of the model (2.3) it is suitable to transform (2.3) to its partial normal form on invariant surface. After the shift of α_0 into the origin by relation $\alpha_1 = \alpha - \alpha_0$ the model (2.3) takes the form

$$\begin{aligned}
\dot{Y}_1 &= \alpha_0 \left[(b_1 - d_1 - f_1)Y_1 + f_2\rho_1 - b_3R_1\right] + (b_1 - d_1 - f_1)Y_1\alpha_1 \\
&\quad + f_2\rho_1\alpha_1 - b_3R_1\alpha_1 + \sum_{k=2}^{4} \alpha_0 a_k {Y_1}^k + \sum_{k=2}^{4} \alpha_1 a_k {Y_1}^k + O(|Y_1|^5) \\
\dot{\rho}_1 &= \beta_0 \left(f_1Y_1 - f_2\rho_1 - c_3R_1\right) \\
\dot{R}_1 &= \gamma_0 \left(l_1Y_1 - l_3R_1 - L_{S1}\right) \\
\dot{L}_{S1} &= (f_1 - d_1)Y_1 - f_2\rho_1 - c_3R_1.
\end{aligned} \tag{2.10}$$

Consider the matrix $\mathbf{M}$ which transfers the matrix $\mathbf{A}(\alpha_0, \beta_0, \gamma_0)$ into its Jordan form $\mathbf{J}$. Then the transformation $x = \mathbf{M}y$, $x = (Y_1, \rho_1, R_1, L_{S1})^T$, $y = (Y_2, \rho_2, R_2, L_{S2})^T$ takes the model (2.10) into the model

$$\begin{aligned}
\dot{Y}_2 &= \lambda_1 Y_2 \quad + F_1(Y_2, \rho_2, R_2, L_{S2}, \alpha_1) \\
\dot{\rho}_2 &= \lambda_2 \rho_2 \quad + F_2(Y_2, \rho_2, R_2, L_{S2}, \alpha_1) \\
\dot{R}_2 &= \lambda_3 R_2 \quad + F_3(Y_2, \rho_2, R_2, L_{S2}, \alpha_1) \\
\dot{L}_{S2} &= \lambda_4 L_{S2} + F_4(Y_2, \rho_2, R_2, L_{S2}, \alpha_1),
\end{aligned} \tag{2.11}$$

where $\rho_2 = \overline{Y}_2, F_2 = \overline{F}_1$, and F_3, F_4 are real (the symbol " - " means complex conjugate expression in the whole article; for the sake of simplicity we suppose that λ_3, λ_4 are real and not equal).

Theorem 2. *There exists a polynomial transformation*

$$\begin{aligned}
Y_2 &= Y_3 \quad + h_1(Y_3, \rho_3, \alpha_1) \\
\rho_2 &= \rho_3 \quad + h_2(Y_3, \rho_3, \alpha_1) \\
R_2 &= R_3 \ + h_3(Y_3, \rho_3, \alpha_1) \\
L_{S2} &= L_{S3} + h_4(Y_3, \rho_3, \alpha_1),
\end{aligned} \tag{2.12}$$

where $h_j(Y_3, \rho_3, \alpha_1),\ j = 1,2,3,4,$ *are nonlinear polynomials with constant coefficients of the kind*

$$h_j(Y_3, \rho_3, \alpha_1) = \sum_{m_1,m_2,m_3} \nu^{(m_1,m_2,m_3)} {Y_3}^{m_1} {\rho_3}^{m_2} {\alpha_1}^{m_3}, \quad j = 1,2,3,4, \quad h_2 = \overline{h}_1,$$

with the property

$$h_j(\sqrt{\alpha_1}Y_3, \sqrt{\alpha_1}\rho_3, \alpha_1) = \sum_{m_1,m_2,m_3} \nu^{(m_1,m_2,m_3)} (\sqrt{\alpha_1})^k {Y_3}^{m_1} {\rho_3}^{m_2}, \quad k \le 4,$$

which transforms the model (2.11) *into its partial normal form on invariant surface*

$$\begin{aligned}
\dot{Y_3} &= \lambda_1 Y_3 + \delta_1 Y_3 \alpha_1 + \delta_2 {Y_3}^2 \rho_3 + U^\circ(Y_3, \rho_3, R_3, L_{S3}, \alpha_1) + U^*(Y_3, \rho_3, R_3, L_{S3}, \alpha_1) \\
\dot{\rho_3} &= \lambda_2 \rho_3 + \overline{\delta}_1 \rho_3 \alpha_1 + \overline{\delta}_2 Y_3 {\rho_3}^2 + \overline{U^\circ} + \overline{U^*} \\
\dot{R_3} &= \lambda_3 R_3 + V^\circ(Y_3, \rho_3, R_3, L_{S3}, \alpha_1) + V^*(Y_3, \rho_3, R_3, L_{S3}, \alpha_1) \\
\dot{L_{S3}} &= \lambda_4 L_{S3} + W^\circ(Y_3, \rho_3, R_3, L_{S3}, \alpha_1) + W^*(Y_3, \rho_3, R_3, L_{S3}, \alpha_1),
\end{aligned} \tag{2.13}$$

where $U^\circ(Y_3, \rho_3, 0, 0, \alpha_1) = V^\circ(Y_3, \rho_3, 0, 0, \alpha_1) = W^\circ(Y_3, \rho_3, 0, 0, \alpha_1) = 0$ *and* $U^*(\sqrt{\alpha_1}Y_3, \sqrt{\alpha_1}\rho_3, \sqrt{\alpha_1}R_3, \sqrt{\alpha_1}L_{S3}, \alpha_1) = V^*(\sqrt{\alpha_1}Y_3, \sqrt{\alpha_1}\rho_3, \sqrt{\alpha_1}R_3, \sqrt{\alpha_1}L_{S3}, \alpha_1)$ $= W^*(\sqrt{\alpha_1}Y_3, \sqrt{\alpha_1}\rho_3, \sqrt{\alpha_1}R_3, \sqrt{\alpha_1}L_{S3}, \alpha_1) = O(\sqrt{\alpha_1})^5.$

The resonant terms δ_1 *and* δ_2 *in the model* (2.13) *are determined by the formulae*

$$\begin{aligned}
\delta_1 &= \frac{\partial^2 F_1}{\partial \alpha_1 \partial Y_2}, \\
\delta_2 &= \frac{1}{2\lambda_2} \frac{\partial^2 F_1}{\partial {Y_2}^2} \frac{\partial^2 F_1}{\partial Y_2 \partial \rho_2} + \frac{1}{6\lambda_1} \frac{\partial^2 F_1}{\partial {\rho_2}^2} \frac{\partial^2 F_2}{\partial {Y_2}^2} \\
&\quad + \frac{1}{\lambda_1} \frac{\partial^2 F_1}{\partial Y_2 \partial \rho_2} \frac{\partial^2 F_2}{\partial Y_2 \partial \rho_2} - \frac{1}{\lambda_3} \frac{\partial^2 F_1}{\partial Y_2 \partial R_2} \frac{\partial^2 F_3}{\partial Y_2 \partial \rho_2} + \frac{1}{2(2\lambda_1 - \lambda_3)} \frac{\partial^2 F_1}{\partial \rho_2 \partial R_2} \frac{\partial^2 F_3}{\partial {Y_2}^2} \\
&\quad - \frac{1}{\lambda_4} \frac{\partial^2 F_1}{\partial Y_2 \partial Ls_2} \frac{\partial^2 F_4}{\partial Y_2 \partial \rho_2} + \frac{1}{2(2\lambda_1 - \lambda_4)} \frac{\partial^2 F_1}{\partial \rho_2 \partial Ls_2} \frac{\partial^2 F_4}{\partial {Y_2}^2} + \frac{1}{2} \frac{\partial^3 F_1}{\partial {Y_2}^2 \partial \rho_2},
\end{aligned}$$

where all derivatives in δ_1 *and* δ_2 *are calculated at* E_1^* *and* $\alpha_1 = 0$.

Proof. Differentiating (2.12) in the power of (2.11) and (2.13) we get the equations for the determination of the individual terms of the polynomials $h_j,\ j = 1,2,3,4,$ and the resonant terms δ_1, δ_2 by standard "step by step" procedure. As the whole record of this procedure is rather long we are omitting it. □

The model (2.13) takes in polar coordinates $Y_3 = re^{i\varphi}, \rho_3 = re^{-i\varphi}$ the form

$$\begin{aligned}
\dot{r} &= r(ar^2 + b\alpha_1) + \widetilde{U}^\circ(r, \varphi, R_3, L_{S3}, \alpha_1) + \widetilde{U}^*(r, \varphi, R_3, L_{S3}, \alpha_1) \\
\dot{\varphi} &= \omega + c\alpha_1 + dr^2 + \frac{1}{r}\left[\Phi^\circ(r, \varphi, R_3, L_{S3}, \alpha_1) + \Phi^*(r, \varphi, R_3, L_{S3}, \alpha_1)\right] \\
\dot{R_3} &= \lambda_3 R_3 + \widetilde{V}_1^\circ(r, \varphi, R_3, L_{S3}, \alpha_1) + \widetilde{V}_1^*(r, \varphi, R_3, L_{S3}, \alpha_1) \\
\dot{L_{S3}} &= \lambda_4 L_{S3} + \widetilde{W}_1^\circ(r, \varphi, R_3, L_{S3}, \alpha_1) + \widetilde{W}_1^*(r, \varphi, R_3, L_{S3}, \alpha_1),
\end{aligned} \tag{2.14}$$

where $a = \operatorname{Re}\delta_2,\ b = \operatorname{Re}\delta_1$.

The behavior of solutions of the model (2.14) around its equilibrium for small parameters α_1 depends on the signs of the constants a, b. It is known that to every constant solution of the bifurcation equation $ar^2 + b\alpha_1 = 0$ a periodic solution of (2.14) corresponds (see for example [1]).

The following theorem ([4]) gives a sufficient condition for the negativeness of the coefficient b.

Theorem 3. *Let the assumptions of Theorem 1 be satisfied. Let in addition $\alpha_0 > \alpha^*$ and the critical value β_0 be such that $c_3 - \beta_0 f_2 l_3 > 0$. Then the value of the coefficient b is negative.*

Analyzing the model (2.14) and taking into account all transformations which have been done to get (2.14) we can formulate on the base of Poincaré-Andronov-Hopf bifurcation theorem (see for example [7]) the following statement.

Theorem 4. *Let the assumptions of Theorem 1 be satisfied. Let in addition $\alpha_0 > \alpha^*$ and the critical value β_0 be such that $c_3 - \beta_0 f_2 l_3 > 0$. Then:*

1. *If $a > 0$ then the equilibrium E_1^* of the model (2.3) is unstable also at the critical triple $(\alpha_0, \beta_0, \gamma_0)$ and to every $\alpha > \alpha_0$ there exists an unstable limit cycle.*
2. *If $a < 0$ then the equilibrium E_1^* of the model (2.3) is asymptotically stable also at the critical triple $(\alpha_0, \beta_0, \gamma_0)$ and to every $\alpha^* < \alpha < \alpha_0$ there exists a stable limit cycle.*

2.1. Numerical example. Take the functions I, S, T, C, D, F, L from the model (1.1) in the form

$$
\begin{aligned}
I &= 1.8 + 0.6\sqrt{Y} - 0.3R, & C &= 0.024 + 0.2R,\\
S &= -0.8 + 0.01(Y^D)^2 + 0.3R, & D &= 2.7 - 0.3Y,\\
Y^D &= 0.7Y + 0.2, & F &= -0.6 - 0.05Y + 0.03\rho,\\
T &= -0.2 + 0.3Y, & L &= 2 + 0.8Y - 0.2R.
\end{aligned}
$$

Then (1.1) has the following form

$$
\begin{aligned}
\dot{Y} &= \alpha\left[0.6\sqrt{Y} - 0.01(0.7Y + 0.2)^2 - 0.35Y + 0.03\rho - 0.6R + 4.7\right]\\
\dot{\rho} &= -\beta\left(-0.05Y + 0.03\rho + 0.2R - 0.576\right)\\
\dot{R} &= \gamma\left(0.8Y - 0.2R - L_S + 2\right)\\
\dot{L_S} &= -0.25Y - 0.03\rho - 0.2R + 3.276.
\end{aligned}
\tag{2.15}
$$

The equilibrium of (2.15) is

$$
E^* = (Y^* = 9, \rho^* = 1.2541667, R^* = 4.941875, L_S^* = 8.211625),
$$

where the values of Y and L_S are taken in 10^2 milliards. After the transformation of E^* into the origin and Taylor expansion the model (2.15) has the form

$$
\begin{aligned}
\dot{Y_1} &= \alpha(-0.341Y_1 + 0.03\rho_1 - 0.6R_1)\\
&\quad + \alpha\left[-\frac{691}{90000}{Y_1}^2 + \frac{1}{6480}{Y_1}^3 - \frac{1}{93312}{Y_1}^4 + O(|Y_1|^5)\right]\\
\dot{\rho_1} &= \beta\left(0.05Y_1 - 0.03\rho_1 - 0.2R_1\right)\\
\dot{R_1} &= \gamma\left(0.8Y_1 - 0.2R_1 - L_{S1}\right)\\
\dot{L_{S1}} &= -0.25Y_1 - 0.03\rho_1 - 0.2R_1.
\end{aligned}
\tag{2.16}
$$

The matrix of linear approximation of this model is

$$\mathbf{A}(\alpha,\beta,\gamma)=\begin{pmatrix} -0.341\alpha & 0.03\alpha & -0.6\alpha & 0 \\ 0.05\beta & -0.03\beta & -0.2\beta & 0 \\ 0.8\gamma & 0 & -0.2\gamma & -\gamma \\ -0.25 & -0.03 & -0.2 & 0 \end{pmatrix}.$$

Take $\beta_0 = 1, \gamma_0 = 1$. Then $\alpha_0 = \frac{5(26610277753+462357\sqrt{46690885481})}{933779562749} \doteq 0.6774$. The eigenvalues of the matrix $\mathbf{A}(\alpha_0, \beta_0, \gamma_0)$ are

$$\lambda_1 \doteq 0.3886\, i, \quad \lambda_2 \doteq -0.3886\, i, \quad \lambda_3 \doteq -0.3748, \quad \lambda_4 \doteq -0.0862.$$

The formulae for the calculation of the resonant constants δ_1 and δ_2, which are introduced in Theorem 2, give the values

$$\delta_1 \doteq -0.3180 - 0.3537\, i, \quad \delta_2 \doteq 0.0020 - 0.0024\, i.$$

The bifurcation coefficients are

$$a = \operatorname{Re}\delta_2 \doteq 0.0020, \quad b = \operatorname{Re}\delta_1 \doteq -0.3180,$$

and the partial normal form of the model (2.16) on invariant surface for critical triple $(\alpha_0, \beta_0, \gamma_0) = (0.6774, 1, 1)$ in polar coordinates is

$$\begin{aligned}
\dot{r} =&\ r(0.0020r^2 - 0.3180\alpha_1) + \widetilde{U}^{\circ}(r,\varphi,R_3,L_{S3},\alpha_1) + \widetilde{U}^{*}(r,\varphi,R_3,L_{S3},\alpha_1)\\
\dot{\varphi} =&\ 0.3886 - 0.3537\alpha_1 - 0.0024r^2 + \frac{1}{r}\left[\Phi^{\circ} + \Phi^{*}\right]\\
\dot{R}_3 =&\ -0.3748R_3 \ \ + \widetilde{V}_1^{\circ}(r,\varphi,R_3,L_{S3},\alpha_1) + \widetilde{V}_1^{*}(r,\varphi,R_3,L_{S3},\alpha_1)\\
\dot{L}_{S3} =&\ -0.0862L_{S3} + \widetilde{W}_1^{\circ}(r,\varphi,R_3,L_{S3},\alpha_1) + \widetilde{W}_1^{*}(r,\varphi,R_3,L_{S3},\alpha_1).
\end{aligned}$$

The value of α^* from (2.8) is $\alpha^* \doteq 0.3648$. On the base of the values a and b we get the following result: To every value of parameter $\alpha > \alpha_0 \doteq 0.6774$ the equilibrium of the model (2.15) is asymptotically stable and there exists an unstable limit cycle. To every value of parameter $\alpha \in (0.3648, 0.6774)$ this equilibrium is unstable.

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