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# EXISTENCE AND LOCATION RESULT FOR THE BENDING OF A SINGLE ELASTIC BEAM

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**Abstract.** In this paper it is obtained an existence and location result for the fourth order fully nonlinear equation

$$u^{(iv)} = f\left(t, u, u', u'', u'''\right), \quad 0 < t < 1,$$

with the Lidstone boundary conditions

$$u\left(0\right) = u''\left(0\right) = u\left(1\right) = u''\left(1\right) = 0,$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function verifying a Nagumo-type condition. We remark that $f$ must only verify some assumptions of bound type and no monotonicity restrictions are considered, as it is usual. The existence of at least a solution lying between a pair of well ordered lower and upper solutions is obtained using *a priori* estimates, lower and upper solutions method and degree theory.

**Key words.** Fourth order BVP, lower and upper solutions, Nagumo-type condition, a *priori* estimate, odd mapping theorem, degree theory

**AMS subject classifications.** 34B15, 34B18, 34L30

**1. Introduction.** The bending of a single elastic beam with both endpoints simply supported can be studied by the fully nonlinear differential equation

$$u^{(iv)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \qquad 0 < t < 1, \tag{1.1}$$

where $f : I \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, with the Lidstone boundary conditions

$$u\left(0\right) = u''\left(0\right) = u\left(1\right) = u''\left(1\right) = 0. \tag{1.2}$$

These fourth order boundary value problems have been studied by many authors either in a point of view of a beam application (see [7, 8] and the references therein) or referred to suspension bridges (see [9], the survey paper [5] and the references therein). In short it is applied a variational approach in the cases where the nonlinearity depends only on $u$ or $u''$ ([7, 8, 13, 14]), a topological method ( [1, 2, 15, 17]) or both ([4]). However, in all the above papers there are no dependence on odd-order derivatives.

In the present paper, lower and upper solutions technique together with *a priori* bounds are used to obtain an existence and location result, following the arguments suggested by [3] for second order, [11, 12, 16] for higher order and applying the odd mapping theorem ([6, 10]). Let us point out that in fourth order problems with Lidstone boundary conditions, (1.1)–(1.2), lower and upper solutions can not be considered in an independent way, that is they must be considered as a pair (see Definitions 2.4, 3.2

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and the Counter-example in last section). Usually, it is assumed that the nonlinearity verifies some monotonicity or, in a more general case, some conditions of monotone type ([12, 16]). Our existence and location theorems (see THEOREM 3.1 and THEOREM 3.3) improves the above results because it is only assumed that $f$ satisfies some bound-type conditions, more precisely, for $\alpha$ and $\beta$ lower and upper solutions of (1.1)–(1.2),

$$f(t,\beta(t),\beta'(t),\beta''(t),\beta'''(t)) \leq f(t,x_0,x_1,\beta''(t),\beta'''(t)),$$

$$f\left(t,\alpha\left(t\right),\alpha'\left(t\right),\alpha''\left(t\right),\alpha'''\left(t\right)\right) \geq f\left(t,x_0,x_1,\alpha''\left(t\right),\alpha'''\left(t\right)\right)$$

hold for $(t,x_0,x_1) \in [0,1] \times R^2$, such that $\alpha(t) \leq x_0 \leq \beta(t)$ and $\alpha'(t) \leq x_1 \leq \beta'(t)$. These assumptions are weaker than the previous ones, as it can be seen in the example of last section.

To prove the existence part of THEOREM 3.1 it is used a Nagumo-type growth condition, to obtain an *a priori* estimation on the third derivative and an open bounded set where the topological degree is well defined. The technique applied allows us to locate the solution and some derivatives on adequate strips defined by well ordered lower and upper solutions and the corresponding derivatives. So it can also be used to prove the existence of positive solutions for problem (1.1)–(1.2) if it will be assumed in THEOREM 3.1 that $\alpha(t) \geq 0$, for every $t \in [0,1]$. In this sense the existence part of [15] is also improved.

**2. Definitions and preliminary result.** To obtain an *a priori* estimate on $u'''$ it must be defined a Nagumo-type growth condition that will be an important tool for the definition of a set where the Leray-Schauder degree can be evaluated and non null.

DEFINITION 2.1. Given a subset $E \subset [0,1] \times \mathbb{R}^4$, a function $f \in C([0,1],\mathbb{R})$ satisfies a Nagumo-type condition in $E$ if there exists a real function $h_E \in C\left(\mathbb{R}_0^+, [a,+\infty[\right)$, for some $a > 0$, such that

$$|f\left(t,x_0,x_1,x_2,x_3\right)| \leq h_E\left(|x_3|\right), \quad \forall\left(t,x_0,x_1,x_2,x_3\right) \in E, \tag{2.1}$$

with

$$\int_0^{+\infty} \frac{s}{h_E\left(s\right)}\,\mathrm{d}s = +\infty. \tag{2.2}$$

LEMMA 2.2 ([16] Lemma 1). *Let the functions $\gamma_i$, $\Gamma_i \in C([0,1],R)$ be such that $\gamma_i(t) \leq \Gamma_i(t)$, for each $i = 0,1,2$ and $t \in [0,1]$, and define the set*

$$E = \left\{(t,x_0,x_1,x_2,x_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i\left(t\right) \leq x_i \leq \Gamma_i\left(t\right),\ i = 0,1,2\right\}.$$

*Assume there exists a continuous function $\varphi : R_0^+ \to [a,+\infty)$, for some $a > 0$, such that*

$$\int_\eta^{+\infty} \frac{s}{\varphi(s)}\,\mathrm{d}s > \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(t)$$

*where $\eta \geq 0$ is given by $\eta = \max\left\{\Gamma_2(0) - \gamma_2(1), \Gamma_2(1) - \gamma_2(0)\right\}$. Then there is $r > 0$ such that, for every continuous function $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfying a Nagumo-type condition and every solution $u(t)$ of problem (1.1)–(1.2) verifying $\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t)$, for $i = 0,1,2$ and $t \in [0,1]$, satisfies $\|u'''\|_\infty \leq r$.*

REMARK 2.3. Observe that $r$ depends only on the functions $h_E, \gamma_2$ and $\Gamma_2$ and it does not depend on the boundary conditions.

Definitions of well ordered lower and upper solutions for problem (1.1)–(1.2) must be done as a couple of functions and can not be defined by independent way.

Definition 2.4. The functions $\alpha, \beta \in C^4\left(]0,1[\right) \cap C^3\left([0,1]\right)$ verifying

$$\alpha(t) \leq \beta(t), \quad \alpha'(t) \leq \beta'(t), \quad \alpha''(t) < \beta''(t), \qquad \forall t \in [0,1], \tag{2.3}$$

define a pair of lower and upper solutions of problem (1.1)–(1.2) if the following conditions are verified:

$$\begin{aligned} \text{(i)} \quad & \alpha^{(iv)}(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)), \\ & \beta^{(iv)}(t) \leq f(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t)) \end{aligned} \tag{2.4}$$

$$\begin{aligned} \text{(ii)} \quad & \alpha(0) \leq 0,\ \alpha''(0) \leq 0,\ \alpha''(1) \leq 0, \\ & \beta(0) \geq 0,\ \beta''(0) \geq 0,\ \beta''(1) \geq 0, \end{aligned} \tag{2.5}$$

$$\text{(iii)}\ \alpha'(0) - \beta'(0) \leq \min\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\}. \tag{2.6}$$

Remark 2.5. Condition (iii) can not be removed. (See Counter-example).

**3. Existence and location results.** The existence and location result obtained in this section provides not only the existence of solution but define also some strips where the solution and its derivatives are defined.

Theorem 3.1. *Suppose that there exists a pair of lower and upper solutions of* (1.1)–(1.2), $\alpha(t)$ *and* $\beta(t)$, *respectively. Let* $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function such that* $f$ *satisfies the Nagumo-type condition in*

$$E_1 = \left\{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t),\ i = 0,1,2\right\}.$$

*Moreover if*

$$f(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t)) \leq f(t, x_0, x_1, \beta''(t), \beta'''(t)), \tag{3.1}$$

*and*

$$f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)) \geq f(t, x_0, x_1, \alpha''(t), \alpha'''(t)), \tag{3.2}$$

*hold for* $(t, x_2, x_3) \in [0,1] \times \mathbb{R}^2$, $\alpha(t) \leq x_0 \leq \beta(t)$ *and* $\alpha'(t) \leq x_1 \leq \beta'(t)$, *then there is at least a solution* $u(t) \in C^4([0,1])$ *of problem* (1.1)–(1.2) *satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t),\ \alpha'(t) \leq u'(t) \leq \beta'(t),\ \alpha''(t) \leq u''(t) \leq \beta''(t), \forall t \in [0,1].$$

*Proof.* Consider the continuous truncations

$$\delta_i(t, x_i) = \begin{cases} \alpha^{(i)}(t), & x_i < \alpha^{(i)}(t) \\ x_i, & \beta^{(i)}(t) \geq x_i \geq \alpha^{(i)}(t) \\ \beta^{(i)}(t), & x_i > \beta^{(i)}(t) \end{cases}, \quad i = 0,1,2,$$

the function $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(t,x) = \frac{\beta^{(iv)}(t)\left[\delta_2(t,x) - \alpha''(t)\right] - \alpha^{(iv)}(t)\left[\delta_2(t,x) - \beta''(t)\right]}{\beta''(t) - \alpha''(t)},$$

and, for $\lambda \in [0,1]$, the homotopic problem

$$\begin{aligned} u^{(iv)}(t) = {} & \lambda f\left(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)\right) \\ & + (1-\lambda)\gamma(t, u''(t)) + u''(t) - \delta_2(t, u''(t)), \end{aligned} \tag{3.3}$$

with boundary conditions

$$u^{(i)}(0) = (1-\lambda)\frac{\beta^{(i)}(0) + \alpha^{(i)}(0)}{2}, \ u^{(i)}(1) = (1-\lambda)\frac{\beta^{(i)}(1) + \alpha^{(i)}(1)}{2}, \tag{3.4}$$

for $i = 0, 2$.

**Step 1.** *Every solution $u(t)$ of problem (3.3)–(3.4) satisfies*

$$\alpha^{(i)}(t) \le u^{(i)}(t) \le \beta^{(i)}(t), \quad \forall t \in [0,1],$$

*for $i = 0, 1, 2$ independently of $\lambda \in [0,1]$.*

Assume, by contradiction, that the above inequalities do not hold for $i = 2$. So there exist $\lambda \in [0,1]$, $t \in [0,1]$ and a solution $u$ of (3.3)–(3.4) such that $u''(t) > \beta''(t)$ or $\alpha''(t) > u''(t)$. In the first case define

$$u''(t_1) - \beta''(t_1) := \max_{t \in [0,1]} \left[u''(t) - \beta''(t)\right] > 0.$$

By (3.4) and Definition 2.4

$$u''(0) \le (1-\lambda)\frac{\beta''(0)}{2} < \beta''(0)$$

for every $\lambda \in [0,1]$ and so $t_1 \neq 0$. Analogously it can be proved that $t_1 \neq 1$. Then, $t_1 \in \left]0,1\right[$, $u'''(t_1) = \beta'''(t_1)$ and $u^{(iv)}(t_1) \le \beta^{(iv)}(t_1)$. Then by (3.1), the following contradiction holds for $\lambda \in [0,1]$:

$$\begin{aligned} \beta^{(iv)}(t_1) \ge {} & u^{(iv)}(t_1) \\ = {} & \lambda f\left(t_1, \delta_0(t_1, u(t_1)), \delta_1(t_1, u'(t_1)), \beta''(t_1), \beta'''(t_1)\right) \\ & + (1-\lambda)\gamma(t_1, u''(t_1)) + u''(t_1) - \beta''(t_1) \\ \ge {} & \lambda f\left(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1), \beta'''(t_1)\right) \\ & + (1-\lambda)\beta^{(iv)}(t_1) + u''(t_1) - \beta''(t_1) \\ \ge {} & \lambda\beta^{(iv)}(t_1) + (1-\lambda)\beta^{(iv)}(t_1) + u''(t_1) - \beta''(t_1) > \beta^{(iv)}(t_1). \end{aligned}$$

The case $u''(t) < \alpha''(t)$, for all $t \in [0,1]$ yields to a similar contradiction and therefore

$$\alpha''(t) \le u''(t) \le \beta''(t), \quad \forall t \in [0,1]. \tag{3.5}$$

By (2.6) and (3.4) it can be obtained

$$\begin{aligned} u'(0) &= u(1) - u(0) - \int_0^1 \int_0^t u''(s)\ \mathrm{d}s\ dt \\ &\geq u(1) - u(0) - \int_0^1 \int_0^t \beta''(s)\ \mathrm{d}s\ dt \\ &= \beta'(0) - \beta(1) + \beta(0) + u(1) - u(0) \\ &= \beta'(0) + \frac{1+\lambda}{2}\left[\beta(0) - \beta(1)\right] + \frac{1-\lambda}{2}\left[\alpha(1) - \alpha(0)\right] \\ &\geq \beta'(0) + \min\left\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\right\} \geq \alpha'(0). \end{aligned}$$

Analogously $u'(0) \leq \beta'(0)$ and so

$$\alpha'(0) \leq u'(0) \leq \beta'(0).$$

As, by (3.5), $(\beta' - u')(t)$ is a nondecreasing function then

$$\beta'(t) - u'(t) \geq \beta'(0) - u'(0) \geq 0,\ \forall t \in [0,1],$$

and $\beta'(t) \geq u'(t)$ for every $t \in [0,1]$. By similar arguments

$$\beta(t) - u(t) \geq \beta(0) - u(0) = \frac{1+\lambda}{2}\beta(0) - \frac{1-\lambda}{2}\alpha(0) \geq 0,$$

i.e. $\beta(t) \geq u(t)$ for $t \in [0,1]$.

The inequalities $u'(t) \geq \alpha'(t)$ and $u(t) \geq \alpha(t)$, for all $t \in [0,1]$, can be proved in analogously way.

**Step 2.** *There exists $r > 0$ such that every solution $u(t)$ of problem* (3.3)–(3.4) *verifies*

$$|u'''(t)| < r, \quad \forall t \in [0,1],$$

*independently of* $\lambda \in [0,1]$.

Let $u(t)$ be a solution of (3.3)–(3.4). Then by Step 1

$$u^{(iv)}(t) = \lambda f(t, u(t), u'(t), u''(t), u'''(t)) + (1-\lambda)\gamma(t, u''(t)).$$

Consider, for $\lambda \in [0,1]$, the auxiliary function $F_\lambda : E_1 \to \mathbb{R}$ given by

$$F_\lambda(t, x_0, x_1, x_2, x_3) = \lambda f(t, x_0, x_1, x_2, x_3) + (1-\lambda)\gamma(t, x_2).$$

As $f$ verifies (2.1) in $E_1$ then

$$\begin{aligned} |F_\lambda(t, x_0, x_1, x_2, x_3)| &\leq |f(t, x_0, x_1, x_2, x_3)| + C \\ &\leq h_{E_1}(|x_3|) + C, \end{aligned}$$

with $C$ a real positive number such that

$$C \geq \max_{t\in[0,1]} \left\{ \left|\alpha^{(iv)}(t)\right| + \left|\beta^{(iv)}(t)\right| \right\}.$$

Defining $\overline{h}_{E_1} : \mathbb{R}_0^+ \to [a, +\infty[$ given by $\overline{h}_{E_1}(t) = C + h_{E_1}(t)$, $F_\lambda$ verifies (2.1) with $E$ and $h_E$ replaced by $E_1$ and $\overline{h}_E$, respectively. Condition (2.2) holds since

$$\int_0^{+\infty} \frac{s}{\overline{h}_{E_1}(s)} \mathrm{d}s = \int_0^{+\infty} \frac{s}{h_{E_1}(s) + C} \mathrm{d}s \geq \frac{1}{1 + \frac{C}{a}} \int_0^{+\infty} \frac{s}{h_{E_1}(s)} \mathrm{d}s = +\infty.$$

Then by Lemma 2.2 there is $r > 0$ such that

$$|u'''(t)| < r, \quad \forall t \in [0,1].$$

Remark that $r$ is independent of $\lambda$ since $h_{E_1}$ does not depend on $\lambda$ .

**Step 3.** *For $\lambda = 1$ problem* (3.3)–(3.4) *has at least a solution $u_1(t)$ which is solution of problem* (1.1)–(1.2).

Define the operators

$$\mathcal{L} : C^4([0,1]) \subset C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$$

by

$$\mathcal{L}u = \left(u^{(iv)}, u(0), u''(0), u(1), u''(1)\right)$$

and, for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4$ by

$$\begin{aligned}\mathcal{N}_\lambda u = (&\lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)) \\ &+ (1-\lambda)\gamma(t, u''(t)) + u''(t) - \delta_2(t, u''(t)), A_{0,\lambda}, A''_{0,\lambda}, A_{1,\lambda}, A''_{1,\lambda}\big),\end{aligned}$$

where

$$A_{0,\lambda}^{(i)} = (1-\lambda)\frac{\beta^{(i)}(0) + \alpha^{(i)}(0)}{2}, \; A_{1,\lambda}^{(i)} = (1-\lambda)\frac{\beta^{(i)}(1) + \alpha^{(i)}(1)}{2},$$

for $i = 0, 2$.

As $\mathcal{L}^{-1}$ is compact it can be defined the completely continuous operator

$$\mathcal{T}_\lambda : \left(C^3([0,1]), \mathbb{R}\right) \to \left(C^3([0,1]), \mathbb{R}\right)$$

by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

Consider the real numbers $r_i > 0$, $i = 0, 1, 2$, such that

$$r_i > \max_{t \in [0,1]} \left\{ \left|\alpha^{(i)}(t)\right|, \left|\beta^{(i)}(t)\right| \right\}.$$

For $r$ given by Step 2 define the set

$$\Omega = \left\{ x \in C^3([0,1]) : \left\|x^{(i)}\right\|_\infty < r_i, \; i = 0, 1, 2, \; \|x'''\|_\infty < r \right\}.$$

Remark that, by Steps 1 and 2, the degree $d\,(I-\mathcal{T}_\lambda,\Omega,0)$ is well defined for every $\lambda\in[0,1]$. To evaluate $d\,(I-\mathcal{T}_0,\Omega,0)$ it is considered the equation $x=\mathcal{T}_0\,(x)$ which is equivalent to the problem

$$\left\{\begin{array}{c} u^{(iv)}\,(t)=\gamma\,(t,u''\,(t))+u''\,(t)-\delta_2\,(t,u''\,(t))\,, \\ \\ u\,(0)=A_{0,0},\quad u''\,(0)=A''_{0,0},\quad u\,(1)=A_{1,0},\quad u''\,(1)=A''_{1,0}. \end{array}\right. \tag{3.6}$$

Defining new functions

$$\overline{u}\,(t)=u\,(t)-\frac{\alpha\,(t)+\beta\,(t)}{2} \tag{3.7}$$

and

$$\overline{\delta}_2\,(t,x_2)=\delta_2\left(t,x_2+\frac{\alpha''\,(t)+\beta''\,(t)}{2}\right)-\frac{\alpha''\,(t)+\beta''\,(t)}{2},$$

that is

$$\overline{\delta}_2\,(t,x_2)=\left\{\begin{array}{ll} \operatorname{sgn}(x_2)\dfrac{\beta''\,(t)-\alpha''\,(t)}{2} & \text{if } |x_2|>\dfrac{\beta''\,(t)-\alpha''\,(t)}{2} \\ x_2 & \text{if } |x_2|\le\dfrac{\beta''\,(t)-\alpha''\,(t)}{2} \end{array}\right.,$$

then

$$\delta_2\,(t,u'')=\overline{\delta}_2\,(t,\overline{u}'')+\frac{\alpha''\,(t)+\beta''\,(t)}{2}$$

and

$$\gamma\,(t,u''\,(t))=\frac{\beta^{(iv)}\,(t)-\alpha^{(iv)}\,(t)}{\beta''\,(t)-\alpha''\,(t)}\;\overline{\delta}_2\,(t,\overline{u}''\,(t))+\frac{\beta^{(iv)}\,(t)+\alpha^{(iv)}\,(t)}{2}.$$

Applying the change of variable given by (3.7) in problem (3.6) it is obtained the equivalent problem composed by

$$\overline{u}^{(iv)}\,(t)=\frac{\beta^{(iv)}\,(t)-\alpha^{(iv)}\,(t)}{\beta''\,(t)-\alpha''\,(t)}\;\overline{\delta}_2\,(t,\overline{u}''\,(t))+\overline{u}''\,(t)-\overline{\delta}_2\,(t,\overline{u}''\,(t)) \tag{3.8}$$

with the boundary conditions

$$\overline{u}\,(0)=\overline{u}''\,(0)=\overline{u}\,(1)=\overline{u}''\,(1)=0. \tag{3.9}$$

Therefore equation $x=\mathcal{T}_0\,(x)$ is also equivalent to problem (3.8)–(3.9) and by the odd mapping theorem

$$d\,(I-\mathcal{T}_0,\Omega,0)\neq 0.$$

By degree theory the equation $x=\mathcal{T}_0\,(x)$ has at least a solution and by the invariance under homotopy

$$d\,(I-\mathcal{T}_0,\Omega,0)=d\,(I-\mathcal{T}_1,\Omega,0)\neq 0.$$

So equation $x = \mathcal{T}_1(x)$ and the equivalent problem

$$\begin{aligned} u^{(iv)}(t) &= f\left(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)\right) \\ &\quad + u''(t) - \delta_2(t, u''(t)), \end{aligned}$$

with the boundary conditions (1.2) has at least a solution $u_1(t)$ in $\Omega$.

By Step 1 this solution $u_1(t)$ is also a solution of the initial problem (1.1)–(1.2). □

If data on lower and upper solutions are considered on the beam right endpoint then a new definition must be assumed, with the corresponding first derivatives in reversed order.

Definition 3.2. The functions $\alpha, \beta \in C^4(]0,1[) \cap C^3([0,1])$ such that

$$\alpha(t) \le \beta(t), \quad \beta'(t) \le \alpha'(t), \quad \alpha''(t) < \beta''(t), \quad \forall t \in [0,1], \tag{3.10}$$

define a pair of lower and upper solutions of problem (1.1)–(1.2) if (2.4) and the following conditions are verified:

$$\alpha(1) \le 0,\ \alpha''(0) \le 0,\ \alpha''(1) \le 0,$$

$$\beta(1) \ge 0,\ \beta''(0) \ge 0,\ \beta''(1) \ge 0,$$

$$\alpha'(1) - \beta'(1) \ge \max\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\}.$$

With these lower and upper solutions a new existence and location result holds.

Theorem 3.3. *Suppose that there exists a pair of lower and upper solutions of* (1.1)–(1.2), $\alpha(t)$ *and* $\beta(t)$ *as in* Definition 3.2. *Let* $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function such that* $f$ *verifies the Nagumo-type condition in*

$$\begin{aligned} E_2 = \{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \le x_0 \le \beta(t), \quad \beta'(t) \le x_1 \le \alpha'(t), \\ \alpha''(t) \le x_2 \le \beta''(t)\}. \end{aligned}$$

*Moreover if* (3.1) *and* (3.2) *hold for* $(t, x_0, x_1) \in [0,1] \times \mathbb{R}^2$, $\alpha(t) \le x_0 \le \beta(t)$ *and* $\beta'(t) \le x_1 \le \alpha'(t)$, *then there is at least a solution* $u(t) \in C^4([0,1])$ *of problem* (1.1)–(1.2) *satisfying*

$$\alpha(t) \le u(t) \le \beta(t),\ \beta'(t) \le u'(t) \le \alpha'(t),\ \alpha''(t) \le u''(t) \le \beta''(t), \forall t \in [0,1].$$

Example:Consider the fourth order boundary value problem

$$\begin{cases} u^{(iv)} = \mathrm{e}^{-sgn(u)\ u}\ \ u'', \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.11}$$

Functions $\alpha, \beta : [0,1] \to R$ given by

$$\alpha(t) := -t^2 - t,\ \beta(t) := t^2 + t$$

define a pair of lower and upper solutions of (3.11) and although the boundary conditions of Definitions 2.4 and 3.2 are satisfied, only (2.6) holds.

Notice that

$$f\left(t, x_0, x_1, x_2, x_3\right) = \mathrm{e}^{-\operatorname{sgn}(x_0)\ x_0}\ \ x_2$$

does not verify the monotone type assumption used in [16],

$$f\left(t, \alpha\left(t\right), \alpha'\left(t\right), x_2, x_3\right) \geq f\left(t, x_0, x_1, x_2, x_3\right) \geq f\left(t, \beta\left(t\right), \beta'\left(t\right), x_2, x_3\right),$$

for $(t, x_2, x_3) \in [0,1] \times \mathbb{R}^2$ and $\alpha(t) \leq x_0 \leq \beta(t)$ , $\alpha'(t) \leq x_1 \leq \beta'(t)$, but it satisfies (3.2) and (3.1).

Since the Nagumo-type condition is verified in

$$E = \big\{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : -t^2 - t \leq x_0 \leq t^2 + t, \\ -2t - 1 \leq x_1 \leq 2t + 1, \quad -2 \leq x_2 \leq 2\big\},$$

then, by Theorem 3.1, there exists a solution $u(t)$ of (3.11) such that,

$$-t^2 - t \leq u(t) \leq t^2 + t, \quad -2t - 1 \leq u'(t) \leq 2t + 1, \quad -2 \leq u''(t) \leq 2,$$

for every $t \in [0,1]$.

Counter-example: To prove that (2.6) can not be removed, consider the fourth order problem

$$\begin{cases} u^{(iv)} = -\left(u'\right)^2 + u'' + 2u''', \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.12}$$

The functions

$$\alpha(t) := \frac{t(1-3t)}{6}, \quad \beta(t) := \frac{t(1+t)}{6}.$$

satisfy assumptions (2.4) and (2.5) but (2.6) is not verified since

$$\alpha'(0) - \beta'(0) = 0 > \min\left\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\right\} = -\frac{1}{3}.$$

Problem (3.12) has only the trivial solution $u(t) \equiv 0$ and

$$0 = u(t) < \alpha(t) < \beta(t), \ 0 = u'(t) < \alpha'(t) < \beta'(t), \ \forall t \in \left]0, \frac{1}{6}\right[,$$

that is the localization given by Theorem 3.1 does not hold.

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