Proceedings of Equadiff-11
2005, pp. 303–310
ISBN 978-80-227-2624-5

THE BEAM OPERATOR AND THE FUČÍK SPECTRUM*

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Abstract. The aim of this paper is mainly to introduce a new variational approach in order to obtain a robust and global algorithm which is suitable for the exploration of unknown Fučík spectrum structure in the case of a beam operator. Qualitative results of numerical experiments that involve unknown branches of the Fučík spectrum are essential for better understanding of the behaviour of PDE models such as suspension bridge models.

Key words. The Fučík spectrum, suspension bridge, jumping nonlinearity, beam operator, variational methods.

AMS subject classifications. 35B10, 35L70, 65N25.

1. Introduction.

Let $L : \mathrm{dom}(L) \subset L^2(\Omega) \to L^2(\Omega)$ be a linear self-adjoint operator, $\Omega \subset \mathbb{R}^N$ being open and bounded set. The set

$$\Sigma(L) := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\} \tag{1.1}$$

is called the Fučík spectrum, where $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. This set was firstly introduced by Fučík (see [10]) in order to extend existence results for semi-linear boundary value problems. Nowadays, several recent papers such as [1], [2] or [9] deal with the structure of the Fučík spectrum $\Sigma(L)$ in the case of a general operator L. For instance in [1], authors give a description of the Fučík spectrum of L away from its essential spectrum, they give local and global results describing the Fučík spectrum of L and also existence results for semi-linear equations using degree computations between the Fučík curves. All the same there are still many open problems, especially in applying these general results in particular cases of partial differential operators such as wave or beam operators.

The Fučík spectrum plays an important role in several mathematical models. Let us mention the following one-dimensional nonlinear model of a suspension bridge

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t), & (x,t) \in (0,\pi)\times\mathbb{R}, \\ u(0,t) = u\,(\pi,t) = u_{xx}(0,t) = u_{xx}\,(\pi,t) = 0, & t \in \mathbb{R}, \\ u(x,t) = u(x,t+T), & (x,t) \in (0,\pi)\times\mathbb{R}. \end{cases} \tag{1.2}$$

This problem can be interpreted as a normalized model of the vertical motion of a suspension bridge (see e.g. Drábek, Holubová, Matas and Nečesal [7]). The Fučík spectrum of the related beam operator indicates points of resonance for the model (1.2). But the Fučík spectrum for such beam operator is not explicitely known (see [8] for more careful analysis). Thus we design an algorithm in order to explore the Fučík curves numerically.

*This work is a joint work with Gabriela Holubová and was supported by the Grant Agency of Czech Republic, project # GA201/03/0671, and the Ministry of Education of the Czech Republic, Research Plan # MSM4977751301.

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2. The Fučík spectrum for a beam operator. Let us briefly recall in this section what is known about the Fučík spectrum for the following problem

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + \alpha u^{+}(x,t) - \beta u^{-}(x,t) = 0, & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, & t \in \mathbb{R}, \\ u(x,t) = u(x,t+T), & (x,t) \in (0,\pi) \times \mathbb{R}. \end{cases} \tag{2.1}$$

Let us set

$$\omega := 2\pi/T, \quad \Omega^{\omega} := (0,\pi) \times (0, 2\pi/\omega), \quad H^{\omega} := L^2(\Omega^{\omega})$$

with a standard L^2-norm $\|\cdot\|$ and standard inner product $\langle \cdot,\cdot \rangle$. Let us define for $k \in \mathbb{N}$ and $l \in \mathbb{Z}$

$$\lambda^{\omega}_{k,l} := l^2\omega^2 - k^4, \quad \varphi^{\omega}_{k,l}(x,t) := \begin{cases} 2\sqrt{\omega}/\pi \sin kx \ \ \sin l\omega t & \text{for } k \in \mathbb{N}, \ \ l \in \mathbb{N}, \\ \sqrt{\omega}/\pi \sin kx & \text{for } k \in \mathbb{N}, \ \ l = 0, \\ 2\sqrt{\omega}/\pi \sin kx \ \ \cos l\omega t & \text{for } k \in \mathbb{N}, \ -l \in \mathbb{N}. \end{cases}$$

The set $\{\varphi^{\omega}_{k,l} : \ k \in \mathbb{N}, \ l \in \mathbb{Z}\}$ forms an orthonormal basis in H^{ω}. The abstract realization of the beam differential operator $u \mapsto -(u_{tt} + u_{xxxx})$ with the periodic and the Navier boundary conditions from (2.1) is the linear operator $L^{\omega} : \mathrm{dom}\,(L^{\omega}) \subset H^{\omega} \to H^{\omega}$ defined by

$$\begin{aligned} \mathrm{dom}\,(L^{\omega}) &:= \left\{ u \in H : \sum_{k=1}^{+\infty} \sum_{l=-\infty}^{+\infty} \left|\lambda^{\omega}_{k,l}\right|^2 \left|u^{\omega}_{k,l}\right|^2 < +\infty \right\}, \\ L^{\omega}u(x,t) &:= \sum_{k=1}^{+\infty} \sum_{l=-\infty}^{+\infty} \lambda^{\omega}_{k,l}\, u^{\omega}_{k,l}\, \varphi^{\omega}_{k,l}(x,t), \end{aligned}$$

where $u^{\omega}_{k,l} := \langle u, \varphi^{\omega}_{k,l} \rangle$. Then L^{ω} is densely defined, closed and self-adjoint operator. Moreover, L^{ω} has a pure point spectrum $\sigma(L^{\omega})$ made of eigenvalues $\{\lambda^{\omega}_{k,l} : k \in \mathbb{N}, l \in \mathbb{Z}\}$. Due to orthogonality of basis $\{\varphi^{\omega}_{k,l}\}$ in H^{ω}, it is possible to locate some inadmissible areas of $\mathbb{R}^2$ that have a zero intersection with the Fučík spectrum $\Sigma(L^{\omega})$. More precisely, the problem (2.1) has only a trivial solution for any

$$(\alpha,\beta) \in (-\infty, \lambda^{\omega}_{1,0}) \times (\lambda^{\omega}_{1,0}, +\infty) \cup (\lambda^{\omega}_{1,0}, +\infty) \times (-\infty, \lambda^{\omega}_{1,0}) \cup (\lambda_i, \lambda_j) \times (\lambda_i, \lambda_j),$$

where λ_i and λ_j are two successive eigenvalues of L^{ω} (see Fig. 2.1 and [20] for more details). It is convenient to form eigenvalues $\{\lambda^{\omega}_{k,l} : k \in \mathbb{N}, l \in \mathbb{Z}\}$ into an infinite-dimensional matrix Λ in the following way

$$\Lambda := \left[\begin{array}{c|cccccc} \lambda^{\omega}_{1,0} & \lambda^{\omega}_{1,1} & \lambda^{\omega}_{1,2} & \lambda^{\omega}_{1,3} & \lambda^{\omega}_{1,4} & \cdots \\ \hline \lambda^{\omega}_{2,0} & \lambda^{\omega}_{2,1} & \lambda^{\omega}_{2,2} & \lambda^{\omega}_{2,3} & \lambda^{\omega}_{2,4} & \cdots \\ \lambda^{\omega}_{3,0} & \lambda^{\omega}_{3,1} & \lambda^{\omega}_{3,2} & \lambda^{\omega}_{3,3} & \lambda^{\omega}_{3,4} & \\ \lambda^{\omega}_{4,0} & \lambda^{\omega}_{4,1} & \lambda^{\omega}_{4,2} & \lambda^{\omega}_{4,3} & \lambda^{\omega}_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array} \right].$$

If we take into account only such solutions of the problem (2.1), which are of the form $u(x,t) = y(t)\sin x$, then the problem (2.1) can be reduced to the following problem

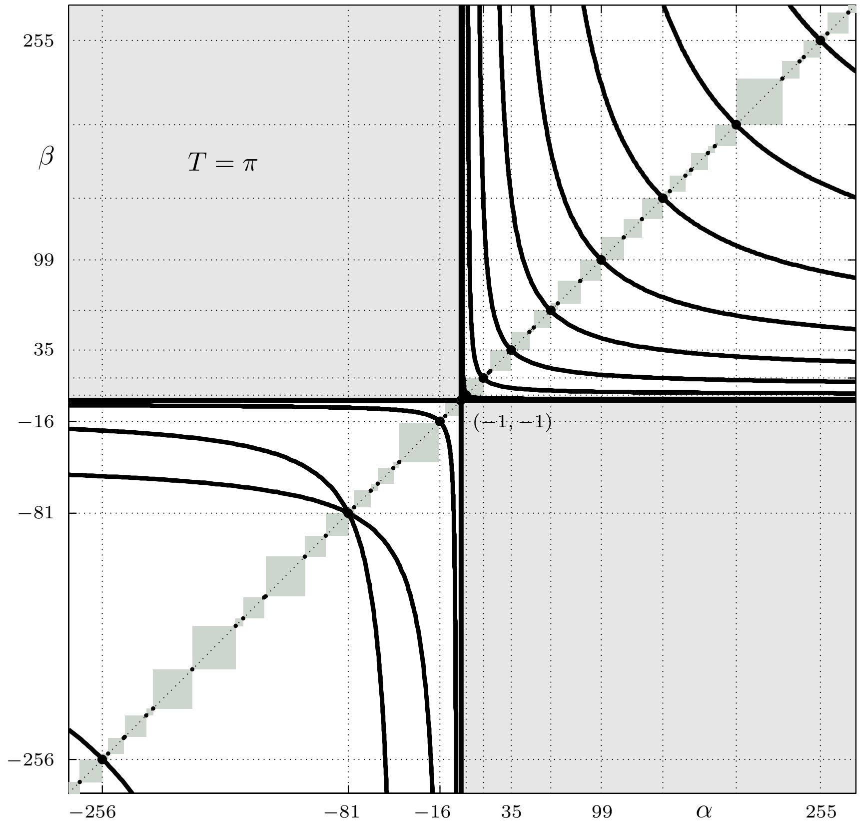

FIG. 2.1. *Known parts of the Fučík spectrum* $\Sigma(L^\omega)$ *of the beam operator.*

for the periodic ordinary differential operator

$$\begin{cases} y''(t) + (\alpha+1)y^+(t) - (\beta+1)y^-(t) = 0, & t \in \mathbb{R}, \\ y(t) = y(t+T), & t \in \mathbb{R}. \end{cases} \tag{2.2}$$

The Fučík spectrum for such periodic problem (2.2) is given by explicit analytic formulas (see [6]), the particular Fučík curves emanate from the points on the diagonal $\alpha = \beta$ determined by the eigenvalues which are located in the first row of the matrix Λ (see FIG. 2.1).

On the other hand, if we focus only on time independent solutions of the problem (2.1) $u(x,t) = y(x)$, then the problem (2.1) reads as the following Navier boundary value problem containing the fourth order ordinary differential operator

$$\begin{cases} y^{\mathrm{IV}}(x) + \alpha y^+(x) - \beta y^-(x) = 0, & x \in (0,\pi), \\ y(0) = y''(0) = y(\pi) = y''(\pi) = 0. \end{cases} \tag{2.3}$$

The corresponding Fučík spectrum of (2.3) is not given by explicit analytic formulas, but the existence of particular Fučík curves is proved by Krejčí (see [16]). In other words, each eigenvalue in the first column of the matrix Λ gives arise to one or two Fučík curves, which can be explored using a standard continuation method combined with an one-dimensional shooting method (see FIG. 2.1 and [21] for details).

Clearly, we expect that all other eigenvalues, which are located not only in the first row or column of the matrix Λ, give arise to other nontrivial Fučík curves. Our expectation is hardly supported by existence results for a general self-adjoint operator by Ben-Naoum, Fabry and Smets (see [1]). In our particular case of the beam operator, their results give us the local existence of Fučík curves close to the diagonal $\alpha = \beta$ in the case of simple eigenvalues.

As for the numerical methods for the partial differential Fučík spectrum problems, there exists only limited group of sporadic numerical results and experiments. One way is to combine for instance Newton's method and Mountain Pass Algorithm to localize the higher Fučík curves (see e.g. results in [13] for the Laplace operator). Alternative approach is based on a continuation shooting method presented in [19] or [20], which means a suitable approach how to explore the Fučík curves for the wave operator $u \mapsto -(u_{tt} - u_{xx})$. On the other hand, in the case of the beam operator L^ω, a continuation shooting method is completely inapplicable due to instability of solutions of the initial boundary value problem related to (2.1) with respect to small perturbations of initial data and parameters α and β. Thus, the goal of the following two sections is to formulate the main ideas of our new approach that is completely different from previous methods and that will be suitable for exploring the Fučík curves including their asymptotes in the case of the beam operator L^ω.

3. Variational approach. Let $\mu \in \mathbb{R} \setminus \sigma(L)$, where $\sigma(L)$ denotes the spectrum of L. Then using the following transformation $\mathcal{T}_\mu$

$$\mathcal{T}_\mu = \mathcal{T}_\mu(\alpha, \beta, u) = (m, \lambda, v), \qquad \mathcal{T}_\mu^{-1} = \mathcal{T}_\mu^{-1}(m, \lambda, v) = (\alpha, \beta, u),$$

$$\mathcal{T}_\mu : \begin{cases} m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}, \\ v = (\mu I - L)u, \end{cases} \qquad \mathcal{T}_\mu^{-1} : \begin{cases} \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \\ \beta = \mu - \dfrac{1}{\lambda(1-m)}, \\ u = (\mu I - L)^{-1} v, \end{cases}$$

it is possible to write the Fučík spectrum problem $Lu = \alpha u^+ - \beta u^-$ as the nonlinear homogeneous eigenpair problem

$$(\mu I - L)^{-1} v = \lambda(v + m|v|). \tag{3.1}$$

Moreover, let us define the corresponding Rayleigh quotient $J : L^2(\Omega) \to \mathbb{R}$ as

$$J(v) = \frac{F(v)}{G(v)}, \quad \text{where} \quad F(v) = \frac{1}{2}\int_\Omega (\mu I - L)^{-1} v \cdot v \, \mathrm{d}x, \qquad G(v) = \frac{1}{2}\int_\Omega v^2 + m|v|v \, \mathrm{d}x.$$

Due to the homogenity of the problem (3.1), a pair (λ, v) is an eigenpair of (3.1), if and only if v is a critical point of J and $\lambda = J(v)$ is the corresponding critical value.

Let us consider the Neumann operator $L^{\mathtt{N}} : \mathrm{dom}(L^{\mathtt{N}}) \subset C([0,\pi]) \to C([0,\pi])$ by

$$L^{\mathtt{N}} y := -y'', \qquad \mathrm{dom}(L^{\mathtt{N}}) := \left\{ C^2([0,\pi]) : \; y'(0) = y'(\pi) = 0 \right\}.$$

Let us set up e.g. $\mu = 2$. Then taking all fixed $m \in [-1, 1]$, the minimization and the maximization processes of functional $J(v)$ generate parts of the second and the first nontrivial Fučík curves of $L^{\mathtt{N}}$, respectively (see Fig. 3.1). Thus, this approach involve only those parts of Fučík curves for which $(\alpha - \mu)(\beta - \mu) > 0$. This disadvantage can be overcome if we add e.g. another parameter δ as in the following section.

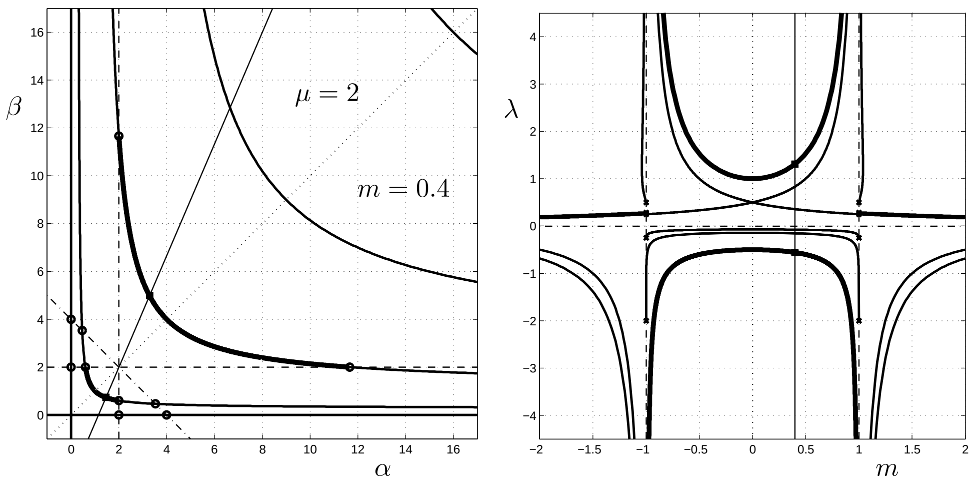

FIG. 3.1. *The first four nontrivial Fučík curves of* $\Sigma(L^{\mathrm{N}})$ *transformed by* $\mathcal{T}_\mu$.

4. Revisited variational approach. Let $\mu \in \mathbb{R} \setminus \sigma(L)$ and $\delta \in \mathbb{R}$ be such that $(\mu+\delta) \notin \sigma(L)$. If we take into account the following generalized transformation

$$\mathcal{T}_{\mu,\delta} = \mathcal{T}_{\mu,\delta}(\alpha,\beta,u) = (m,\tilde{\lambda},v), \qquad \mathcal{T}_{\mu,\delta}^{-1} = \mathcal{T}_{\mu,\delta}^{-1}(m,\tilde{\lambda},v) = (\alpha,\beta,u),$$

$$\mathcal{T}_{\mu,\delta} : \begin{cases} m = \dfrac{\beta-\alpha}{\beta+\alpha-2\mu}, \\[2mm] \tilde{\lambda} = \dfrac{2\mu-\alpha-\beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu-\alpha-\beta)}, \\[2mm] v = (\mu I - L)u, \end{cases} \qquad \mathcal{T}_{\mu,\delta}^{-1} : \begin{cases} \alpha = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1+m)}, \\[2mm] \beta = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1-m)}, \\[2mm] u = (\mu I - L)^{-1} v, \end{cases}$$

then the Fučík spectrum problem $Lu = \alpha u^+ - \beta u^-$ reads as the nonlinear problem

$$((\mu+\delta)I - L)^{-1} v = \tilde{\lambda}(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|). \tag{4.1}$$

Due to homogenity of such an eigenpair problem, the critical points of the corresponding Rayleigh quotient $J : L^2(\Omega) \to \mathbb{R}$

$$J(v) = \frac{F(v)}{G(v)}, \qquad F(v) = \frac{1}{2}\int_\Omega ((\mu+\delta)I - L)^{-1} v \cdot v \,\mathrm{d}x,$$

$$G(v) = \frac{1}{2}\int_\Omega v^2 + m(I - \delta[(\mu+\delta)I - L]^{-1})|v| v \,\mathrm{d}x,$$

together with their critical values $\lambda = J(v)$ are in one to one correspondence with eigenpairs (λ, v) of (4.1).

Let us take $\mu = 1/6$ and $\delta = 4$. Then all nontrivial Fučík curves of $\sigma(L^{\mathrm{N}})$ are located in a strip $\{(m,\lambda) \in [-1,1] \times \mathbb{R}\}$ (see FIG. 4.1). As a consequence of this fact, the whole second and the whole third nontrivial Fučík curves of $\Sigma(L^{\mathrm{N}})$ including their asymptotes (for $m = \pm 1$) can be obtained by the maximization and the minimization of the functional $J(v)$, respectively.

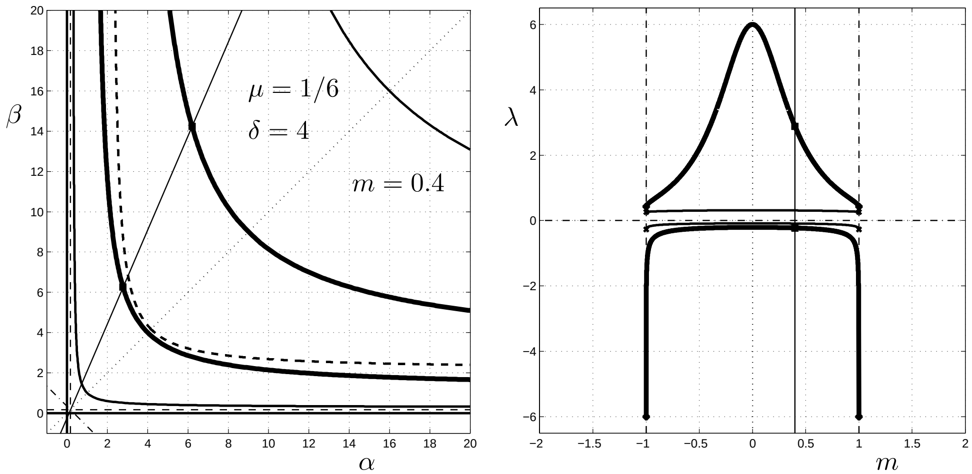

FIG. 4.1. *The first four nontrivial Fučík curves of* $\Sigma(L^{\mathrm{N}})$ *transformed by* $\mathcal{T}_{\mu,\delta}$.

5. Example. In order to demonstrate the applicability of our variational approach also in the case of partial differential operators, let us consider the following Neumann problem for the wave operator $u \mapsto -(u_{tt} - u_{xx})$

$$\begin{cases} u_{tt} - u_{xx} + \alpha u^+ - \beta u^- = 0, & (x,t) \in (0,2) \times (0,1), \\ u_x(0,t) = u_x(X,t) = 0, & t \in [0,1], \\ u_t(x,0) = u_t(x,T) = 0, & x \in [0,2]. \end{cases}$$

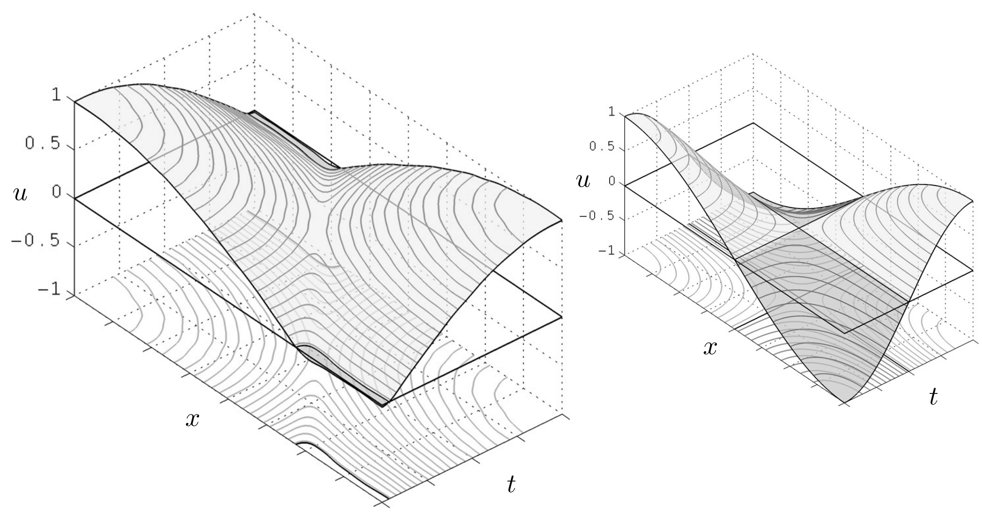

FIG. 5.1. *The generalized eigenfunction for* $\beta = +\infty$ *and the starting eigenfunction for* $\alpha = \beta$.

Let us set up $\mu = 1$ and $\delta = 0$. Starting with the eigenfunction $\cos \pi x/2 \cos \pi t$, the minimization process of the corresponding functional $J(v)$ for $m = 1$ leads to $\lambda = 0.8513$ with $\|J'(v)\|_2 = 2.3 \cdot 10^{-5}$. Then we recover the generalized eigenfunction with $\alpha = 1.5873$ and $\beta = +\infty$ using the inverse transformation $\mathcal{T}^{-1}_{\mu,\delta}$ (see FIG. 5.1).

6. Conclusion and further research. Our variational approach provides us a robust and global algorithm for exploring the generalized eigenfunctions and particular Fučík curves including their asymptotic behaviour. No continuation technique is required and presented variational approach is suitable also in the case of the beam operator L^ω (see [14] how to effectively implement the inverse operator $(\mu I - L^\omega)^{-1}$). Finally, let us note that using our variational approach, we explore the qualitatively different behaviour of the Fučík curves and their corresponding generalized eigenfunctions in the case of the beam operator L^ω in contrast with known results concerning the Fučík spectrum for ordinary differential operators.

7. Acknowledgement. The author acknowledges the support of the Grant Agency of Czech Republic, project # GA201/03/0671, and the Ministry of Education of the Czech Republic, Research Plan # MSM4977751301, which made it possible to discuss the topic of this paper in a wider mathematical community.

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